S. Kajesh
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ASSIGNMENT-II.

MATHEMATICS-II

1. (A). form the partial D.E by the elimination of arbitrary functions $\phi(x+y+z, x^2+y^2+z^2)=0$

Given that $\phi(x+y+z, x^2+y^2+z^2)=0 \longrightarrow (1)$

let $x+y+z=u$, & $x^2+y^2+z^2=v$, then the eqn (1) becomes $\phi(u,v)=0$

$$u=\phi(v)$$

$$x+y+z=\phi(x^2+y^2+z^2) \longrightarrow (2)$$

eq(2) contains only one arbitrary function ϕ, then we get a 1st order PDE

diff. eq(2) partially wrt x & y, we get

$$\Rightarrow \frac{\partial}{\partial x}(x+y+z) = \frac{\partial}{\partial x}\phi(x^2+y^2+z^2)$$

y as constant

$$1+0+\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$1+p = \phi'(x^2+y^2+z^2)(2x+2zp) \longrightarrow (3)$$

$$\Rightarrow \frac{\partial}{\partial y}(x+y+z) = \frac{\partial}{\partial y}\phi(x^2+y^2+z^2)$$

x as constant

$$0+1+\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2).\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$1+q = \phi'(x^2+y^2+z^2)(2y+2zq) \longrightarrow (4)$$

By eliminating ϕ' from eq(3) & (4), we get

$$\frac{(3)}{(4)} \Rightarrow \frac{1+p}{1+q} = \frac{\phi'(x^2+\cancel{y^2}+z^2)(2x+2zp)}{\phi'(x^2+\cancel{y^2}+z^2)(2y+2zq)}$$

$$\frac{1+p}{1+q} = \frac{2(x+zp)}{2(y+zq)}$$

$$(1+p)(y+zq) = (1+q)(x+zp)$$

$$y+zq+py+p\cancel{zq} = x+zp+qx+\cancel{qzp}$$

$$qx+zp+x-y-zq-py=0$$

$$p(z-y)+q(x-z)=y-x$$

which is formation of 1st order PDE.

1 (B). Solve $x^2(y-z)P + y^2(z-x)Q = z^2(x-y)$

G.T $x^2(y-z)P + y^2(z-x)Q = z^2(x-y) \longrightarrow ①$

Here $P = x^2(y-z)$, $Q = y^2(z-x)$ $R = z^2(x-y)$

consider the lagrange's Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

The problem can not be solved by method of grouping we have use method of multipliers.

Taking multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we have

$$E.R = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x^2(y-z) + \frac{1}{y}y^2(z-x) + \frac{1}{z}(z^2(x-y))}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{zx} - \cancel{yz}}$$

$$E.R = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

$$= \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \times E.R = 0$$

Again integrating on B.S

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0$$

$$\log x + \log y + \log z = \log c_1$$

$$\log(xyz) = \log c_1$$

$$\log(xyz) = \log c_1 \qquad [\because \log a + \log b + \log c = \log(a.b.c)]$$

$xyz = c_1$ is an independent Solution

Again taking multipliers $(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2})$, we have

$$E.R = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{x^2}x^2(y-z) + \frac{1}{y^2}y^2(z-x) + \frac{1}{z^2}z^2(x-y)}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$$E \cdot R = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\because \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0 \times E \cdot R = 0$$

Integrating on B.S we get

$$\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = \int 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = C_2$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -C_2 = C_2', \text{ where } C_2' = -C_2$$

be another independent solution

The complete sol. of eq① is

$$F(C_1, C_2') = 0$$

i.e. $F\left(x \cdot y \cdot z, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$

Where F is any arbitrary function.

2. (A). prove that $\nabla(r^n) = n r^{n-2} \bar{r}$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$r^2 = x^2+y^2+z^2$ then $\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$L.H.S = \nabla r^n$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \Sigma \bar{i}\, n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \Sigma \bar{i}\, n.\, r^{n-1} \frac{x}{r}$$

$$= \Sigma \bar{i}\, n\, r^{n-1} r^{-1} x$$

$= n r^{n-2} \Sigma \bar{i} x$

$= n r^{n-2} (x\bar{i} + y\bar{j} + z\bar{k})$

$= n r^{n-2} \bar{r}$

$=$ R.H.S

2. (B) Prove that $\text{grad}(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times \text{curl}\,\bar{a} + \bar{a}\times\text{curl}\,\bar{b}$

consider $\bar{a}\times \text{curl}\,\bar{b} = \bar{a}\times(\nabla\times\bar{b})$

$$= \Sigma\, \bar{a}\times\left(\bar{i}\times\frac{\partial \bar{b}}{\partial x}\right)$$

$\left[\because \nabla\times = \Sigma\, \bar{i}\times\frac{\partial}{\partial x} \quad \nabla\times\bar{f} = \Sigma\,\bar{i}\times\frac{\partial \bar{f}}{\partial x}\right]$

$[\because \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}]$

$$\bar{a}\times\text{curl}\,\bar{b} = \Sigma\, \bar{a}\times\left(\bar{i}\times\frac{\partial \bar{b}}{\partial x}\right)$$

$\left[\because \bar{a}\cdot\nabla = \Sigma(\bar{a}\cdot\bar{i})\frac{\partial}{\partial x} \quad (\bar{a}\cdot\nabla)\phi = \Sigma(\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x} \quad (\bar{a}\cdot\nabla)\bar{f} = \Sigma(\bar{a}\cdot\bar{i})\frac{\partial \bar{f}}{\partial x}\right]$

$$= \Sigma\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right)\bar{i} - (\bar{a}\cdot\bar{i})\frac{\partial \bar{b}}{\partial x}$$

$$= \Sigma\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right)\bar{i} - \Sigma(\bar{a}\cdot\bar{i})\frac{\partial \bar{b}}{\partial x}$$

$$= \Sigma\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right)\bar{i} - (\bar{a}\cdot\nabla)\bar{b} \longrightarrow (1)$$

Similarly $\bar{b}\times\text{curl}\,\bar{a} = \Sigma\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right)\bar{i} - (\bar{b}\cdot\nabla)\bar{a} \longrightarrow (2)$

(1)+(2) $\Rightarrow$ $\bar{a}\times\text{curl}\,\bar{b} + \bar{b}\times\text{curl}\,\bar{a} = \Sigma\underbrace{\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right)}\bar{i} - (\bar{a}\cdot\nabla)\bar{b} + \Sigma\underbrace{\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right)}\bar{i} - (\bar{b}\cdot\nabla)\bar{a}$

$$= \Sigma\,\bar{i}\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x} + \bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} - (\bar{b}\cdot\nabla)\bar{a}$$

$$\bar{a}\times\text{curl}\,\bar{b} + \bar{b}\times\text{curl}\,\bar{a} = \Sigma\,\bar{i}\,\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b}) - (\bar{a}\cdot\nabla)\bar{b} - (\bar{b}\cdot\nabla)\bar{a}$$

$$(\bar{a}\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)\bar{a} + \bar{a}\times\text{curl}\,\bar{b} + \bar{b}\times\text{curl}\,\bar{a} = \Sigma\,\bar{i}\,\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b})$$

$\left[\because \Sigma\,\bar{i}\,\frac{\partial\phi}{\partial x} = \nabla\phi\right]$

$$= \nabla(\bar{a}\cdot\bar{b})$$

$$(\bar{a}\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\times\text{curl}\,\bar{b}) + \bar{b}\times\text{curl}\,\bar{a} = \text{grad}(\bar{a}\cdot\bar{b})$$

3. Verify stokes theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$, where S is the circular disc $x^2+y^2 \leq 1, z=0$

Given $\bar{F} = -y^3\bar{i} + x^3\bar{j}$, the boundary c of S is a circle in the xy-plane; $x^2+y^2=1$, $z=0$, we use parametric co-ordinates

$x = \cos\theta$, $y = \sin\theta$, $z=0$; $0 \leq \theta \leq 2\pi$, $dx = -\sin\theta d\theta$
$dy = \cos\theta d\theta$

∵ By using stoke's theorem

$$\oint_c \bar{F}\cdot d\bar{r} = \int_S \text{curl } \bar{F}\cdot\bar{n}\, ds$$

L.H.S

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c -y^3 dx + x^3 dy$$

$$= \int_{\theta=0}^{2\pi} -\sin^3\theta(-\sin\theta d\theta) + \cos^3\theta(\cos\theta d\theta)$$

$$= \int_{\theta=0}^{2\pi} (\sin^4\theta + \cos^4\theta)\, d\theta$$

$$= 2\int_0^{\pi} (\sin^4\theta + \cos^4\theta) d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} (\sin^4\theta + \cos^4\theta) d\theta$$

$$= 4\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 4\left[\frac{3\pi}{16} + \frac{3\pi}{16}\right]$$

$$= 4\left[\frac{6\pi}{16}\right]$$

$$= \frac{3\pi}{2}$$

R.H.S

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$$

$$\Rightarrow \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2+3y^2)$$

$$(\nabla\times\bar{F})\cdot\bar{n} = \bar{k}(3x^2+3y^2)\cdot\bar{k} = 3(x^2+y^2)$$

$$\because \int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds = \int_S 3(x^2+y^2)\, dx\, dy$$

put $x = r\cos\theta$, $y = r\sin\theta$, then $dx\,dy = r\,dr\,d\theta$

Hence $r: 0 \to 1$; $\theta: 0 \to 2\pi$

$$\int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} 3r^2 \, r\, dr\, d\theta$$

$$= 3 \left(\frac{r^4}{4}\right)_0^1 (\theta)_0^{2\pi}$$

$$= \frac{3}{\cancel{4}_2} (\cancel{2}\pi)$$

$$= \frac{3\pi}{2}$$

L.H.S = R.H.S

$\therefore$ Stokes theorem is verified.