Axiomatic Definition of Boolean Algebra:

- Boolean algebra is an algebraic structure defined by a set of elements, B, together with two binary operators, + and $\cdot$ provided that the following postulates are satisfied:

1. (a) The structure is closed with respect to the operator +.
   (b) The structure is closed with respect to the operator $\cdot$.
2. (a) The element 0 is an identity element w.r.t +; that is $x+0 = 0+x = x$.
   (b) The element 1 is an identity element w.r.t. $\cdot$; that is $x \cdot 1 = 1 \cdot x = x$.
3. (a) The structure is commutative w.r.t +; that is $x+y = y+x$.
   (b) The $\cdot$; that is $x \cdot y = y \cdot x$.
4. (a) The operator $\cdot$ is distributive over +; that is $x \cdot (y+z) = (x \cdot y) + (x \cdot z)$.
   (b) The operator + $\cdot$ that is $x + (y \cdot z) = (x+y) \cdot (x+z)$.
5. For every element $x \in B$, there exists an element $x' \in B$ (called the complement of x) such that (a) $x + x' = 1$ and b) $x \cdot x' = 0$.
6. There exist at least two elements $x, y \in B$ such that $x \neq y$.

Postulates and Theorems of Boolean Algebra

1. Postulate 2	(a) $x+0=x$	(b) $x\cdot 1=x$
2. Postulate 5	(a) $x+x'=1$	(b) $x\cdot x'=0$
3. Theorem 1	(a) $x+x=x$	(b) $x\cdot x=x$
4. Theorem 2	(a) $x+1=1$	(b) $x\cdot 0=0$
5. complement	$(x')'=x.$	
6. commutative	(a) $x+y=y+x$	(b) $xy=yx$
7. Associative	(a) $x+(y+z)=(x+y)+z$	(b) $x(yz)=(xy)z$
8. distributive	(a) $x(y+z)=xy+xz$	(b) $x+yz=(x+y)(x+z)$
9. De Morgan	(a) $(x+y)'=x'y'$	(b) $(xy)'=x'+y'$
10. Absorption	(a) $x+xy=x$	(b) $x(x+y)=x$.

Boolean Algebra:-

→ George Boole introduced a systematic treatment of logic and developed for this purpose an algebraic system now called as Boolean Algebra.

- Shanon introduced a two-valued Boolean algebra called switching algebra.
- In Boolean, $1+1=1$ it means that which is a unique.
- The symbol which represent an arbitrary elements of an boolean algebra is known as variable.

Basic Definitions:

The operation of ordinary algebra ~~is~~ ($+ = OR$, $\cdot = AND$) is applied for the logical circuits is called Boolean algebra.

Definition 1:-

A set B of elements (a, b, c ...) with an equivalence relation (denoted by =), two binary operations, one of them denoted by '+' and the other denoted by '.' and a complement is a boolean algebra if and only if

i) Associativity: The + and . operations are associative

$$(a+b)+c = a+(b+c) = a+b+c.$$

$$(ab)\,c = a(bc) = abc.$$

ii) Commutativity: The + and . operations are commutative

$$a+b = b+a.$$

$$ab = ba.$$

iii) Distributivity: The two operations are distributive over each other.

$$a+bc = (a+b)(a+c).$$

$$a(b+c) = ab+ac.$$

iv) Identity:- $a+0=a$

$a \cdot 1 = a$

v) Complement:- $a+a'=1$

$a \cdot a' = 0$

Basic Theorems and Properties of Boolean Algebra:

1. Duality:- The principle of duality theorem says that, starting with a boolean relation, we can drive another boolean relation by

i) Changing each OR sign to AND sign.

ii) changing each AND sign to an OR sign.

iii) Complement $0 \rightarrow 1$ and $1 \longrightarrow 0$.

Example:- Dual of $A+\bar{A}=1$ is $\bar{A} \cdot A = 0$.

literal : x (or) $\bar{x}$.

Example: Find the dual of $v+\bar{\omega}[\bar{x}+y(\bar{v}+z)]$.

$$= v \cdot [\bar{\omega} + (\bar{x} \cdot y + (\bar{v} \cdot z))].$$

Proof of distributivity:-

$$A+BC = (A+B)(A+C)$$

$$= A \cdot A + A \cdot C + AB + BC.$$

$$= A + A[B+C] + BC$$

$$= A[1+B+C] + BC$$

$$= A \cdot 1 + BC$$

z

- Basic Theorems :

$x+0=x$ $\quad x\cdot 1=x$

$x+x'=1$ $\quad x\cdot x'$

1) $A+A=A$

Eg: $0+0=0$

$1+1=1$

Proof:- $A+A = (A+A)\cdot 1$

$= (A+A)\cdot(A+\bar{A})$ $\quad (a+b)\cdot(a+c)$ [∵ By using complement]

$a+(b\cdot c)$ [∵ By using distributivity]

$= A + A\cdot A'$

$= A + 0$

$= A$

$\therefore \boxed{A+A=A}$

2) $A\cdot A = A$

Eg:- $0\cdot 0 = 0$

$1\cdot 1 = 1$

Proof:- $A\cdot A + 0$

$= A\cdot A + A\cdot A'$ $\quad (a\cdot b)+(a\cdot c)$ [∵ By using complement]

$= A(A+A')$ $\quad a(b+c)$

$= A(1)$

$= A$

$\therefore \boxed{A\cdot A = A}$

3) $A+1=1$

Eg: ~~A+0~~

$0+1=1$

$1+1=1$

Proof:- $A+1 = (A+1)\cdot 1$

$= (A+1)(A+\bar{A})$ [∵ By using complement]

$= A+(1\cdot\bar{A})$ [∵ By using distributive]

$= A+\bar{A}$ $\quad 1\cdot a' = a'$

$A+(B\cdot C)=(A+B)\cdot(A+C)$.

$= 1$

$\therefore \boxed{A+1=1}$

4) $A \cdot 0 = 0$

Eg:- $0 \cdot 0 = 0$

$1 \cdot 0 = 0$

Proof :- By using duality.

$A + 1 = 1$

$A \cdot 0 = 0$ ✓

$\therefore \boxed{A \cdot 0 = 0}$

5) $(\bar{\bar{A}}) = A$ (or) $(A')' = A$

Eg:- $\bar{\bar{0}} = \bar{1} = 0$

$\bar{\bar{1}} = \bar{0} = 1$

6) $A + AB = A$

Proof :- $A + AB = A \cdot 1 + AB$ [$\because$ By using duality]

$= A[1 + \cancel{A}B]$ [$A \cdot 1$ = identity]

$= A[1] = A \cdot 1$

$= A$

$\therefore \boxed{A + AB = A}$

7) $A[A+B] = A$

Proof :- $A[A+B] = A \cdot A + A \cdot B$ [$\because A \cdot A = A$ By 2^{nd} theorem].

$= A + AB$

$= A[1+B]$

$= A.$

$\therefore \boxed{A[A+B] = A.}$

8) $A + \bar{A}B = A + B$

Proof :- $A + \bar{A}B = A + AB + \bar{A}B$ [$\because$ By using 6] $A = A + AB$

$= A + B[A + \bar{A}]$

$= A + B \cdot 1$

$= A + B.$

$\therefore \boxed{A + \bar{A}B = A + B.}$

(or)

Proof :- $A + \bar{A}B =$

By using distributive law,

$(A+\bar{A})(A+B) = A + \bar{A}B.$

$(1)(A+B) = A + \bar{A}B$

$A + \bar{A}B = A + B.$

$\therefore \boxed{A + \bar{A}B = A + B.}$

9) $A \cdot (\bar{A} + B) = AB.$

Proof :- $A \cdot (\bar{A} + B) = (A + AB) \cdot (\bar{A} + B)$ [$\because$ By 6].

$= A\bar{A} + AB + 0 + A \cdot BB$

$= 0 + AB + AB$

$= AB.$

10) Demorgan's Theorem :- DeMorgan suggested two theorems that form an imp. part of Boolean algebra. In the eq. form, they are

i) $\overline{AB} = \bar{A} + \bar{B}$ ⇒ The complement of a product = sum of complements.

A	B	$\overline{AB}$	$\bar{A}+\bar{B}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

ii) $\overline{A+B} = \bar{A} \cdot \bar{B}$ ⇒ The complement of a sum is equal to the product of the complements.

A	B	$\overline{A+B}$	$\bar{A}\bar{B}$
0	0	1	1
0	1	0	0
1	0	0	0

→ Simplify the given expressions by using boolean rules.

1) $ABCD + ABC = ABC$

$ABCD + ABC = ABC[D+1]$

$= ABC[1]$

$= ABC.$

$\therefore ABCD + ABC = ABC.$

2) $A \cdot \bar{A}C = 0$

$(A \cdot \bar{A}) \cdot C$

$= 0 \cdot C$

$= 0.$

3) $ABCD + A\bar{B}CD = ACD$

Sol:- $= ACD[B+\bar{B}]$

$= ACD(1)$

$= ACD.$

4) $A + \bar{A}B + A\bar{B} = A+B.$

$= A[1+\bar{B}] + \bar{A}B.$

$= A[1] + \bar{A}B.$

$= A + \bar{A}B$ [∵ by ⑧]

$= A+B$

5) $xy + xyz + xy\bar{z} + \bar{x}yz = xy + yz.$

$= xy[1+z+\bar{z}] + \bar{x}yz$

$= xy(1] + \bar{x}yz$

$= xy + \bar{x}yz$

$= y[x+\bar{x}z).$ [∵ By using ⑧]

$= y[x+z]$

$= xy + yz.$

6) $\bar{A}BC\bar{D} + BC\bar{D} + B\bar{C}\bar{D} + B\bar{C}D.$

$= BC[\bar{A}\bar{D} + \bar{D}] + B\bar{C}[\bar{D} + D].$

$= BC[\bar{D}(\bar{A}+1)] + B\bar{C}(1).$

$= BC\bar{D} + B\bar{C}$

$= B[\bar{C} + C\bar{D}] = B[\bar{C} + \bar{D}].$

7) $\overline{\bar{A}B + \bar{A} + AB} = 0$.

$\Rightarrow \bar{A}B + \bar{A} + AB = (\bar{A}B + AB) + \bar{A}$

$= 1 + \bar{A}$

$= 1$

$A = 0 ; \bar{A} = 1$
$A = 1 , \bar{A} = 0$.

$\overline{\bar{A}B + \bar{A} + AB} = \bar{1} = 0$.

8) $\overline{\bar{A}B + A + AB} = 0$.

$A + \bar{A} = 1$

$\Rightarrow \overline{\bar{A}B + A + AB} = \overline{(\bar{A}B + AB) + A}$

$= \overline{1 + A}$

$= \bar{1}$

$= 0$.

9) $\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC$.

$= \bar{A}\bar{C}[\bar{B} + B] + \bar{A}BC$.

$= \bar{A}\bar{C}[1] + \bar{A}BC$

$= \bar{A}\bar{C} + \bar{A}BC$

$= \bar{A}[\bar{C} + BC]$ [∵ By using (8)]

$= \bar{A}[\bar{C} + B]$.

10) $AB + ABC + AB(D+E) = AB$.

$= AB[1 + C + D + E]$

$= AB[1]$

$= AB$.

** **NOTE**:- If a function consists of two variables A, B. we can express them as 2^{2^n}. (n = no. of variables).

$\Rightarrow 2^{2^n} = 2^{2^2} = 2^4 = 16$ forms.

$[AB, \bar{A}B, A\bar{B}, \bar{A}\bar{B}, A+B, \bar{A}+B, A+\bar{B}, \bar{A}+\bar{B}, \bar{A}\cdot 0, \bar{B}\cdot 0, \bar{A}\cdot 1]$.

→ Boolean Functions :-

- We use Boolean expressions to describe Boolean functions as:

$$f(A,B,C) = (A+\bar{B})C \text{ (or) } f = (A+\bar{B})C$$

Literals

$$\text{(or) } f(A,B,C,D) = \underbrace{A + B\bar{C} + \bar{A}CD}_{\text{Product terms}} = SOP$$

- Each occurence of a variable in either a complemented (or) an uncomplemented form is called a literal.

$$f(A,B,C,D) = \underbrace{(B+\bar{D}) \cdot (A+\bar{B}+C) \cdot (\bar{A}+C)}_{\text{Sum terms}} = POS$$

→ Canonical and Standard Forms :-

- The canonical forms are the special cases of SOP and POS forms. These are also known as standard SOP and POS forms.

i) Standard SOP form (or) Minterm Canonical Form :

- For example in expression $AB+AB\bar{C}$ the first product term do not contain literal 'C'.
- If each term in SOP form contains all the literals then the SOP form is known as standard (or) Canonical SOP form.
- Each individual term in the standard SOP form is called minterm e.g:- AB.

E.g:- standard SOP form : $\underbrace{A\bar{B}C + ABC + \bar{A}BC}$.

Each product term consists of all literals in either complemented form (or) uncomplemented form.

ii) Standard POS form (or) Maxterm Canonical form:

- If each term in pos form contains all the literals then the POS form is known as standard (or) Canonical POS form.

$$E.g:-\ f(A,B,C) = (A+B+C)\cdot(A+\bar{B}+C)\cdot(\bar{A}+B+C)$$ → std POS form.

→ **POS and SOP simplification:**

- This is nothing but a converting expressions in std SOP (or) POS forms.

→ **Steps to convert SOP/POS to std SOP/POS form:**

step 1:- Find the missing literal in each product term/sum term if any.

step 2:- (a) AND each product term having missing literal and form by ORing the literal and its complement (SOP)

(b) Viceversa for POS.

step 3:- Expand the terms

Step 4:- Reduce the terms by rules.

SOP simplification:

Probl: Convert the given expression in standard SOP form

$$f(A,B,C) = AC+AB+BC.$$

Sol:- $f(A,B,C) = AC+AB+BC.$

step 1:- $AC+AB+BC$

→ A miss

→ C miss

→ B miss.

step 2:- $AC\cdot 1 + AB\cdot 1 + BC\cdot 1$

$$= AC\cdot[B+\bar{B}] + AB\cdot[C+\bar{C}] + BC[A+\bar{A}].$$

step 3:-

$= ACB + AC\bar{B} + A\bar{B}C + AB\bar{C} + A\bar{B}C + \bar{A}BC$

Step 4: $= ABC + A\bar{B}C + AB\bar{C} + \bar{A}BC.$

$\therefore f(A,B,C) = ABC + A\bar{B}C + AB\bar{C} + \bar{A}BC.$

Prob:-

$f(A,B,C) = A + ABC$

$\quad \hookrightarrow$ B, C are miss

$= A \cdot 1 + ABC$

$= A \cdot [B+\bar{B}]\,[C+\bar{C}] + ABC.$

$= A[BC + B\bar{C} + \bar{B}C + \bar{B}\bar{C}] + ABC.$

$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C.$

$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C}.$

$\therefore f(A,B,C) = ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C}.$

POS Simplification:-

Prob:- Convert the given expression in std POS form

$f(A,B,C) = (A+B)\,(B+C)\,(A+C).$

Sol:- step 1:- $(A+B)\,(B+C)\,(A+C)$

$\hookrightarrow$ B miss

$\rightarrow$ A miss

$\rightarrow$ C miss

step 2:- $(A+B+0)\,(B+C+0)\,(A+C+0)$

$= (A+B+C\cdot\bar{C})\,(B+C+A\cdot\bar{A})\,(A+C+B\cdot\bar{B}).$

step 3:- $= (A+B+C)(A+B+\bar{C})\,(B+C+A)\,(B+C+\bar{A})\,(A+B+C)\,(A+\bar{B}+C).$

step 4:- $= (A+B+C)(A+B+\bar{C})\,(\bar{A}+B+C)\,(A+\bar{B}+C).$

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Prob: $f(A,B,C) = A \cdot (A+B+C)$.

step 1: $A \cdot (A+B+C)$

$\hookrightarrow$ B, C, miss

$= (A+0+0)\,(A+B+C)$

$= (A+B\cdot\bar{B}+C\cdot\bar{C})\,(A+B+C)$.

$= (A+B\cdot\bar{B}+C)\,(A+B\cdot\bar{B}+\bar{C})\,(A+B+C)$.

$= (A+C+B)\,(A+C+\bar{B})\,(A+\bar{C}+B)\,(A+\bar{C}+\bar{B})\,(A+B+C)$.

$= (A+B+C)\,(A+\bar{B}+C)\,(A+\bar{B}+\bar{C})\,(A+B+\bar{C})$.

Minterms and Maxterms:-

- Minterms and Maxterms for three Binary Variables.

Variables x y z	Minterms $m_i (\Sigma)$ Term	Designation	Maxterms $M_i (\Pi)$ Term	Designation
0 0 0	$x'y'z'$	m_0	$x+y+z$	M_0
0 0 1	$x'y'z$	m_1	$x+y+z'$	M_1
0 1 0	$x'yz'$	m_2	$x+y'+z$	M_2
0 1 1	$x'yz$	m_3	$x+y'+z'$	M_3
1 0 0	$xy'z'$	m_4	$x'+y+z$	M_4
1 0 1	$xy'z$	m_5	$x'+y+z'$	M_5
1 1 0	xyz'	m_6	$x'+y'+z$	M_6
1 1 1	xyz	m_7	$x'+y'+z'$	M_7

Prob: Convert the given expression into standard SOP form (or) Minterm Canonical form (or) Canonical form.

1) $f = AB + BC + AC.$

Identify missing literals in each product term.

$$f = \underset{\substack{\downarrow \\ C \\ \text{miss}}}{AB} + \underset{\substack{\downarrow \\ A \\ \text{miss}}}{BC} + \underset{\substack{\downarrow \\ B \\ \text{miss}}}{AC}$$

$= AB\cdot 1 + 1\cdot BC + A\cdot 1\cdot C.$

$= AB(C+\bar{C}) + (A+\bar{A})BC + A(B+\bar{B})\cdot C.$

$= ABC + ABC' + ABC + \bar{A}BC + ABC + A\bar{B}C.$

$= ABC + \bar{A}BC + A\bar{B}C + AB\bar{C}.$ $[\because A+A=A]$

$= m_7 + m_3 + m_5 + m_6$

$f = \Sigma m(3,5,6,7)$

Prob:- Convert the given expression into standard POS form (or) Max term form (or) Canonical form.

$f(A,B,C) = (A+B)(B+C)$

Identify missing literals in each sum term

$$f = \underset{\substack{\downarrow \\ C\ \text{miss}}}{(A+B)}\ \underset{\substack{\downarrow \\ A\ \text{miss}}}{(B+C)}$$

$= (A+B+0)\ (B+C+0)$

$= (A+B+C\cdot C')\ (A\cdot A'+B+C)$

$= (A+B+C)(A+B+C')\ (A+B+C)\ (A'+B+C)$

$= (A+B+C)\ (A+B+C')\ (A'+B+C)$

$= M_0 \cdot M_1 \cdot M_4.$

$f = \Pi M(0,1,4).$

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Prob:- Convert the given expression into standard SOP form and standard POS form $f = A+BC+AC$.

Sol:- $f = A+BC+AC$.

Identify the missing literals in each product ~~form~~ term.

$$f = \underset{\downarrow\ B,C}{A} + \underset{\downarrow\ A}{BC} + \underset{\downarrow\ B}{AC}.$$

$$f = A.1.1. + BC.1 + AC.1$$

$$= A(B+B')(C+C') + B.C.(A+A') + AC.(B+B')$$

$$= A(BC+BC'+B'C+B'C') + ABC+A'BC + ABC+AB'C.$$

$$= ABC+ABC'+AB'C+AB'C'+ABC+A'BC+ABC+AB'C.$$

$$= ABC + AB'C' + AB'C + A'BC + AB'C' + ABC'$$

$$= m_7 + m_4 + m_5 + m_3 + m_6.$$

$$f = \Sigma m(3,4,5,6,7). \rightarrow \text{standard SOP form.}$$

Standard POS form

$$f = A+BC+AC$$

$$f = A.(B+C).(A+C)$$ [∵ using duality theorem].

Identify the missing literals in each sum term.

$$f = \underset{\downarrow\ B,C}{A}\quad \underset{\downarrow\ A,}{(B+C)}\ \underset{\downarrow\ B.}{(A+C)}$$

$$= (A+0+0)\ (B+C+0)\ (A+0+C)$$

$$= (A+B.B'+C.C')\ (B+C+A.A')\ (A+B.B'+C).$$

$$= (A+B.B'+C)\ (A+B.B'+C')\ (B+C+A)\ (B+C+A')\ (A+B+C)\ (A+B'+C).$$

$$= (A+C+B)\ (A+C+B')\ (A+B+C')\ (A+B'+C')\ (A+B+C)\ (A'+B+C)\ (A+B+C)\ (A+B'+C).$$

$$= (A+B+C)\ (A'+B+C)\ (A+B'+C)(A+B+C')\ (A+B'+C')$$

$f = M_0 \cdot M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdot$

$f = \pi M(0,1,2,3,4) \cdot$