* Solutions to 1st order differential equations [Exact & reducible to exact]

Exact differential equ:

let $M(x,y)dx + N(x,y)dy = 0$ be a 1st order and 1st degree diff equ, where M and N are real valued functions for some x, y.

Then the equ $Mdx + Ndy = 0$ is said to be exact diff equ, if $\exists$ a function $f(x,y) \ni \frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$

Ex: let $\underbrace{(2xy)}_{M}dx + \underbrace{(x^2)}_{N}dy = 0$ be an exact diff equ. If $\exists$ a function $f(x,y) \ni$

$\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$

$$\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + y\,2x = $$

$$x^2dy + 2xy\,dx = $$

given equ is $2xydx + x^2dy = 0$

$$d(x^2y) = 0$$

$$x^2y = \int 0\,dx$$

$$\underbrace{x^2y}_{f(x,y)} = C$$

$$\left[\because \int 0\,dx = C,\quad \int_a^b 0\,dx = 0\right]$$

$f(x,y) = x^2 y$

$\frac{\partial f}{\partial x} = y(2x) = M$

$\frac{\partial f}{\partial y} = x^2(1) = x^2 = N$

condition for exactness :

If $M(x,y)$ and $N(x,y)$ are two real valued functions which have continuous partial derivatives, then a necessary and sufficient condition for D.E $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Procedure to solve an exact differential equation:

① let the given diff equ is of form $M\,dx + N\,dy = 0$. check the condition for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

② If the given diff equ is exact then integrating the term in M w.r.to 'x'. treating y as constant.

③ Integrate only those terms of N which do not contain x, with respect to y

④ The sum of ② & ③ equated to an arbitrary constant 'C' will be the required G.S of the given diff equ.

i.e $\int M\,dx$ + $\int$ (the terms of N free from x) $dy = C$
y as constant

* Solve the D.E $(hx+by+f)dy + (ax+hy+g)dx = 0$

Given DE is $(ax+hy+g)dx + (hx+by+f)dy = 0$ —— ①

comparing equ ① with $M\,dx + N\,dy = 0$

Here $M = ax+hy+g$, $N = hx+by+f$

$\frac{\partial M}{\partial y} = 0+h(1)+0$ & $\frac{\partial N}{\partial x} = h(1)+0+0$

$\frac{\partial M}{\partial y} = h$ $\qquad$ $\frac{\partial N}{\partial x} = h$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Methods to find integrating factors:

Type ①: Integrating factor can be found by inspection method:

In this method the following differentials can be frequently used

① $d(xy) = x\,dy + y\,dx$

② $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

③ $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

④ $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{1}{\frac{y}{x}} \cdot \frac{x\,dy - y\,dx}{x^2}$

$= \frac{x\,dy - y\,dx}{xy}$

⑤ $d\left(\log \frac{x}{y}\right) = \frac{1}{\frac{x}{y}} \cdot \frac{y\,dx - x\,dy}{y^2} = \frac{y\,dx - x\,dy}{xy}$

⑥ $d(\log(xy)) = \frac{1}{xy}(x\,dy + y\,dx)$

⑦ $d(\log(x^2+y^2)) = \frac{1}{x^2+y^2}(2x\,dx + 2y\,dy)$

⑧ $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{y\,dx - x\,dy}{y^2}$

$= \frac{y\,dx - x\,dy}{x^2+y^2}$

⑨ $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{x\,dy - y\,dx}{x^2}$

$= \frac{x\,dy - y\,dx}{x^2+y^2}$

⑩ $d\left(\frac{e^x}{y}\right) = \frac{y\,e^x\,dx - e^x\,dy}{y^2}$

Type ②:

If $M\,dx + N\,dy = 0$ is a homogeneous diff equ and $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor [I.F] of $\underbrace{M\,dx + N\,dy = 0}$

Type ③: If $M\,dx + N\,dy = 0$ is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an I F of $M\,dx + N\,dy = 0$

Type ④: For the diff equ $M\,dx + N\,dy = 0$, if $\exists$ $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then the I F is given by $e^{\int f(x)\,dx}$

Type ⑤: For the diff equ $M\,dx + N\,dy = 0$, if $\exists$ $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = g(y)$, then the I F is given by $e^{\int g(y)\,dy}$

Higher order linear differential equations

Linear differential equations with constant coefficients:

An equ of the form $\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}}{dx^{n-1}}y + P_2 \frac{d^{n-2}}{dx^{n-2}}y + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = Q(x) \longrightarrow (1)$

where $P_1, P_2 - - - P_{n-1}, P_n$ are real constants and $Q(x)$ is continuous function of x is called Linear diff equ of higher order 'n' with constant coefficients

let us denote $\frac{d}{dx}, \frac{d^2}{dx^2}, - — \frac{d^{n-1}}{dx^{n-1}}, \frac{d^n}{dx^n}$ with $D, D^2, - — D^{n-1}, D^n$ so that

$$Dy = \frac{d}{dx}y, \quad D^2y = \frac{d^2}{dx^2}.y, \quad - - - - \quad D^{n-1}y = \frac{d^{n-1}}{dx^{n-1}}y, \quad D^ny = \frac{d^n}{dx^n}y$$

The operator form of equ (1) is

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + - — + P_{n-1} D + P_n)y = Q(x)$$

compare above equ with $f(D)y = Q(x)$

$$y = y_c = c_1 e^{-ax} + c_2 e^{ax}$$, where c_1, c_2 are arbitrary constants.

* Solve $\frac{d^2y}{dx^2} + y = 0$

G.T $\frac{d^2y}{dx^2} + y = 0$ ——— (1)

The operator form of eqn (1) is

$$D^2 y + y = 0 \quad \left[\because \frac{d^2}{dx^2} = D^2\right]$$

$$(D^2+1)\, y = 0$$

compare above eqn with $f(D)\,y = 0$ $[\because Q(x) = 0]$

Here $f(D) = D^2+1$

A.E is $f(m) = 0$

i.e $m^2+1 = 0$

$$m^2 + i^2 = 0 \quad [\because a^2+b^2 = (a+ib)(a-ib)]$$

$$(m+i)(m-i) = 0$$

$$m = \pm i = 0 \pm i = m_1 \pm i m_2 \text{ say}$$

Roots are complex and conjugate

$$C.F\,(y_c) = e^{0x}(c_1 \cos 1x + c_2 \sin 1x) \quad [\because e^{0x} = e^0 = 1]$$

$$C.F\,(y_c) = c_1 \cos 1x + c_2 \sin 1x$$

The G.S of eqn (1) is $y = y_c$ $[\because y_p = 0]$

$y = c_1 \cos 1x + c_2 \sin 1x$, where c_1, c_2 are arbitrary constants.

S.NO	Roots of A.E $f(m)=0$	complementary function (y_c)	G.S of $f(D)y=0$ is $y=y_c$ [$\because y_p=0$]
1	$m=m_1,m_2,\dots m_n$ (i.e Roots are real & distinct)	$c_1e^{m_1x}+c_2e^{m_2x}+\dots+c_ne^{m_nx}$	$y=c_1e^{m_1x}+c_2e^{m_2x}+\dots+c_ne^{m_nx}$ where $c_1,c_2,\dots c_n$ are 'n' arbitrary constants.
2	$m=\underline{m_1,m_1},m_3,m_4\dots m_n$ (i.e Two roots are real and equal and remaining roots are distinct)	$(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+c_4e^{m_4x}+\dots+c_ne^{m_nx}$	$y=(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+c_4e^{m_4x}+\dots+c_ne^{m_nx}$
3	$m=m_1,m_1,m_1,m_4,m_5\dots m_n$	$(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+c_5e^{m_5x}+\dots+c_ne^{m_nx}$	$y=(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+c_5e^{m_5x}+\dots+c_ne^{m_nx}$
4	$m=m_1\pm im_2,m_3,m_4\dots m_n$ (i.e Two roots are complex and conjugate and remaining roots are distinct)	$e^{m_1x}(c_1\cos m_2x+c_2\sin m_2x)+c_3e^{m_3x}+c_4e^{m_4x}+\dots+c_ne^{m_nx}$	$y=e^{m_1x}(c_1\cos m_2x+c_2\sin m_2x)+c_3e^{m_3x}+c_4e^{m_4x}+\dots+c_ne^{m_nx}$

S.No	Roots of A.E $f(m)=0$	C.F (y_c)	G.S of $f(D)y=0$ is $y=y_c$
5	$m = m_1 \pm i m_2$ (Twice), m_5, m_6, --- m_n	$e^{m_1 x}\left((c_1+c_2 x)\cos m_2 x + (c_3+c_4 x)\sin m_2 x\right) + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$	$y = e^{m_1 x}\left((c_1+c_2 x)\cos m_2 x + (c_3+c_4 x)\sin m_2 x\right) + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$
6	$m = m_1 \pm \sqrt{m_2}$, m_3, m_4 --- m_n (i.e Two roots are irrational roots and remaining roots are distinct)	$e^{m_1 x}\left(c_1 \cosh\sqrt{m_2}\,x + c_2 \sinh\sqrt{m_2}\,x\right) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$	$y = e^{m_1 x}\left(c_1 \cosh\sqrt{m_2}\,x + c_2 \sinh\sqrt{m_2}\,x\right) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$
7	$m = m_1 \pm \sqrt{m_2}$ (Twice), m_5, m_6, --- m_n	$e^{m_1 x}\left((c_1+c_2 x)\cosh\sqrt{m_2}\,x + (c_3+c_4 x)\sinh\sqrt{m_2}\,x\right) + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$	$y = e^{m_1 x}\left((c_1+c_2 x)\cosh\sqrt{m_2}\,x + (c_3+c_4 x)\sinh\sqrt{m_2}\,x\right) + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$

Suppose

$m = 1, 5, 7 \Rightarrow y_c = c_1 e^{1x} + c_2 e^{5x} + c_3 e^{7x}$

$m = \underline{1, 1}, -2, 3 \Rightarrow y_c = (c_1 + c_2 x) e^{1x} + c_3 e^{-2x} + c_4 e^{3x}$

$m = 1, \underbrace{2, 2, 2} \Rightarrow y_c = c_1 e^{m_1 x} + (c_2 + c_3 x + c_4 x^2) e^{2x}$

(or)

$(c_1 + c_2 x + c_3 x^2) e^{2x} + c_4 e^{1x}$

$m = \pm i\sqrt{3} \Rightarrow$

$y_c = e^{0x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

$m = \underset{m_1 \pm i m_2}{\overset{\text{real}}{\underline{1}} \pm i3}, 2, 5 \Rightarrow y_c = e^{1x}(c_1 \cos 3x + c_2 \sin 3x) + c_3 e^{2x} + c_4 e^{5x}$

$m = \underset{m_1 \pm i m_2}{\underline{1} \pm i3} \text{ (Twice)}, 2, 5 \Rightarrow y_c = e^{1x}((c_1 + c_2 x) \cos 3x + (c_3 + c_4 x) \sin 3x) + c_5 e^{2x} + c_6 e^{5x}$

$m = 2 \pm \sqrt{3}, -1, 4 \Rightarrow y_c = e^{2x}(c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x) + c_3 e^{-1x} + c_4 e^{4x}$

$m = 2 \pm \sqrt{3} \text{ (Twice)}, -1, 4 \Rightarrow y_c = e^{2x}((c_1 + c_2 x) \cosh\sqrt{3}x + (c_3 + c_4 x) \sinh\sqrt{3}x) + c_5 e^{-1x} + c_6 e^{4x}$

Type I : P.I of $f(D)y = Q(x)$, when $Q(x) = e^{ax}$, where 'a' is a constant.

$$\therefore \text{P.I } (y_p) = \frac{1}{f(D)} Q(x) = \frac{1}{f(D)} e^{ax}$$

$$\left[\because y_p = \frac{1}{D+2} e^{3x} \right]$$

$e^{3x} \to Q(x)$, $D+2 \to f(D)$

$\frac{1}{5} e^{3x}$

case (i) : If $f(a) \neq 0$, then $\text{P.I } (y_p) = \frac{1}{f(a)} e^{ax}$

$$\left[\because y_p = \frac{1}{(D-2)^k} e^{2x} \right.$$

D by 2 ans DN = 0

$$\left. = \frac{x^k}{k!} e^{2x} \right]$$

case (ii) : If $f(a) = 0$. Then $(D-a)$ is a factor of $f(D)$.

If 'a' is root repeated k times for $f(a) = 0$, then $f(D) = (D-a)^k \phi(D)$, where $\phi(a) \neq 0$. Then we have

$$\text{P.I } y_p = \frac{1}{f(D)} Q(x) = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{(D-a)^k \phi(D)} e^{ax}$$

$$= \frac{1}{\phi(a)} \left(\frac{1}{(D-a)^k} e^{ax} \right)$$

$$= \frac{1}{\phi(a)} \left(\frac{x^k}{k!} e^{ax} \right)$$

$$\left[\because y_p = \frac{1}{\underbrace{(D-3)^2}_{=0} \underbrace{(D+2)}_{\neq 0}} e^{3x} \right.$$

$$= \frac{1}{(3+2)} \frac{1}{(D-3)^2} e^{3x}$$

$$\left. = \frac{1}{5} \left(\frac{x^2}{2!} e^{3x} \right) \right]$$

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Note:

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$a^2-b^2 = (a+b)(a-b)$$
$$a^2+b^2 = (a+ib)(a-ib)$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

* H.W solve $(D^3-5D^2+8D-4)y = e^{2x}$

* solve $(D^2-3D+2)y = \cosh x$

G.T $(D^2-3D+2)y = \cosh x$ —— (1)

compare equ (1) with $f(D)y = Q(x)$

Here $f(D) = D^2-3D+2$ & $Q(x) = \cosh x$

$$= \frac{e^x + e^{-x}}{2}$$

A.E is $f(m)=0$

$\therefore$ e $m^2-3m+2=0$

$1 \times 2 = 2$ ✗

$-1 \times -2 = 2$ ✓

Note: P.I of $f(D)y = Q(x)$, when $\frac{1}{f(D)}$ is expressed as partial fractions

let $f(D) = (D-\alpha_1)(D-\alpha_2) - - - (D-\alpha_n)$

then $P.I = \frac{1}{f(D)} Q(x)$

$$= \left(\frac{1}{(D-\alpha_1)(D-\alpha_2) - - (D-\alpha_n)}\right) Q(x)$$

$$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + - - - + \frac{A_n}{D-\alpha_n}\right) Q(x)$$

$$\left[\because \frac{1}{D-a} Q(x) = e^{ax} \int e^{-ax} Q(x)\,dx\right]$$

$$= A_1 \frac{1}{D-\alpha_1} Q(x) + A_2 \frac{1}{D-\alpha_2} Q(x) + - - - + A_n \frac{1}{D-\alpha_n} Q(x)$$

$$= A_1 \left[e^{\alpha_1 x} \int e^{-\alpha_1 x} Q(x)\,dx\right] + A_2\, e^{\alpha_2 x} \int e^{-\alpha_2 x} Q(x)\,dx + - - - + A_n\, e^{\alpha_n x} \int e^{-\alpha_n x} Q(x)\,dx$$

Type II : P.I of $f(D)y = Q(x)$, when $Q(x) = \sin bx$ (or) $\cos bx$, where b is a constant

let $f(D)y = \sin bx$, then P.I (y_p) $= \frac{1}{f(D)} \sin bx$. let $f(D) = \phi(D^2)$, then

$y_p = \frac{1}{\phi(D^2)} \sin bx$

case(i) : If $\phi(-b^2) \neq 0$, then $y_p = \frac{1}{f(D)} Q(x)$

$= \frac{1}{\phi(D^2)} \sin bx$

$= \frac{1}{\phi(-b^2)} \sin bx$, provided $\phi(-b^2) \neq 0$

if we replace only D^2 by $-b^2$ and then rationalise the DN

similarly $\frac{1}{f(D)} \cos bx = \frac{1}{\phi(D^2)} \cos bx$

$= \frac{1}{\phi(-b^2)} \cos bx$, provided $\phi(-b^2) \neq 0$

$f(D) = D^2 - 4 \quad Q(x) = \sin 2x$

then $y_p = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2 - 4} \sin 2x$

only D^2 by -2^2 sub

DN $\neq 0$

$= \frac{1}{-2^2 - 4} \sin 2x$

$= \frac{-1}{8} \sin 2x$

<u>case(ii)</u>: If $\phi(-b^2) = 0$, then D^2+b^2 is a factor of $\phi(D^2)$ & hence it is a factor of $f(D)$. Let $f(D) = (D^2+b^2)\,F(D^2)$, where $F(-b^2) \neq 0$

Then $P.I\,(y_p) = \dfrac{1}{f(D)} Q(x)$

$$= \frac{1}{(D^2+b^2)\,F(D^2)} \sin bx$$

$$= \frac{1}{F(-b^2)}\left(\frac{1}{D^2+b^2} \sin bx\right), \quad F(-b^2) \neq 0$$

only D^2 by $-b^2$ sub DN=0 using below formula

$$= \frac{1}{F(-b^2)}\left(\frac{-x}{2b} \cos bx\right)$$

similarly $y_p = \dfrac{1}{f(D)} \cos bx = \dfrac{1}{(D^2+b^2)\,F(D^2)} \cos bx$

$$= \frac{1}{F(-b^2)}\left(\frac{1}{D^2+b^2} \cos bx\right), \quad F(-b^2) \neq 0$$

only D^2 by $-b^2$ sub DN=0 using below formula

$$= \frac{1}{F(-b^2)} \cdot \left(\frac{x}{2b} \sin bx\right)$$

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Note:

① $\frac{1}{D^2+b^2}\sin bx = \frac{-x}{2b}\cos bx$

② $\frac{1}{D^2+b^2}\cos bx = \frac{x}{2b}\sin bx$

only D^2 by $-b^2$ sub $DN=0$ using these two formulas.

* ③ $2\sin A\cos B = \sin(A+B)+\sin(A-B)$

④ $2\cos A\sin B = \sin(A+B)-\sin(A-B)$

⑤ $2\cos A\cos B = \cos(A+B)+\cos(A-B)$

⑥ $2\sin A\sin B = \cos(A-B)-\cos(A+B)$

⑦ $\cos 2\theta = \underline{2\cos^2\theta-1} = \underline{1-2\sin^2\theta} = \cos^2\theta-\sin^2\theta$

⑧ $\sin 2\theta = 2\sin\theta\cos\theta$

⑨ $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

⑩ $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

* solve $(D^2+4)y = e^x + \sin 2x + \cos 2x$

G.T $(D^2+4)y = e^x + \cos 2x + \sin 2x$ —— (1)

compare equ (1) with $f(D)y = Q(x)$

Here $f(D) = D^2+4$ & $Q(x) = e^x + \cos 2x + \sin 2x$

A.E is $f(m) = 0$

i.e $m^2+4=0$

$m^2+2^2=0 \Rightarrow (m+i2)(m-i2)=0 \quad [\because a^2+b^2=(a+ib)(a-ib)]$

$m = \pm i2 = 0 \pm i2$

roots are complex & conjugate

$C.F(y_c) = e^{0x}(c_1\cos 2x + c_2 \sin 2x) \quad [\because e^{0x}=1]$

$y_c = c_1\cos 2x + c_2\sin 2x$

& $y_p = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2+4} e^x + \cos 2x + \sin 2x$

$y_p = \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}\cos 2x + \frac{1}{D^2+4}\sin 2x$

↓ TYPE-I ↓ Each and every D by 1 & $DN \neq 0$

↓ TYPE-II ↓ only D^2 by -2^2 Sub $DN = 0$

↓ TYPE-II

$= \frac{1}{1^2+4}e^x + \frac{x}{2(2)}\sin 2x + \frac{-x}{2(2)}\cos 2x$

$y_p = \frac{e^x}{5} + \frac{x}{4}\sin 2x - \frac{x}{4}\cos 2x$

* **Type-III :** P.I of $f(D)y = Q(x)$, when $Q(x) = x^k$, where k is +ve integer (ie $k = 1, 2, 3, \dots$)

$$P.I\,(y_p) = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} x^k$$

$f(D) = D^2 + 3D$ $Q(x) = x^2$

$$y_p = \frac{1}{D^2 + 3D} x^2 = \frac{1}{3D\left(\underbrace{\frac{D^2}{3D}}_{\phi(D)} + 1\right)} x^2 = \frac{1}{3D}\left(1 + \frac{D}{3}\right)^{-1} x^2$$

To evaluate P.I, reduce $\frac{1}{f(D)}$ to the form $\frac{1}{1 \pm \phi(D)}$ by taking the lowest degree term from $f(D)$. Now we write $\frac{1}{f(D)}$ as $(1 \pm \phi(D))^{-1}$ and expand it in ascending powers of D using Binomial theorem up to the term containing D^k. Then operate x^k with the terms of the expansion of $(1 \pm \phi(D))^{-1}$

i.e $y_p = \frac{1}{1 \pm \phi(D)} x^k = (1 \pm \phi(D))^{-1} x^k$ $\left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\right.$

If $f(D)$ is resolvable into partial factors, then split up $\frac{1}{f(D)}$ into partial fractions and proceed.

we frequently use the following rules

$$\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$$

$$\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$$

$$\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$$

$$\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$$

$$\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$$

$$\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$x = -D, \ n = -1$$

The G.S of eqn (1) is $y = y_c + y_p$

$y = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + x^3 - 3x^2 + 6$, where c_1, c_2 are arbitrary constants

* Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

G.T $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ —— (1)

The operator form of eqn (1) is

$D^2y + Dy = x^2 + 2x + 4$

$(D^2 + D)y = x^2 + 2x + 4$

compare above eqn with $f(D)y = Q(x)$

Here $f(D) = D^2 + D$ & $Q(x) = x^2 + 2x + 4$

A.E is $f(m) = 0$

i.e $m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

roots are real & distinct

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$C.F\ (y_c) = c_1 e^{0x} + c_2 e^{-1x} \quad [\because e^{0x} = 1]$

$(y_c) = c_1 + c_2 e^{-1x}$

$P.I\ (y_p) = \frac{1}{f(D)} R(x)$

$= \frac{1}{D^2+D}\ x^2+2x+4$

$\Downarrow$ Type-III [i.e TT, lowest degree term common from $f(D)$]

$= \frac{1}{D(\frac{D^2}{D}+1)}\ x^2+2x+4$

$D = \frac{d}{dx}$

$\frac{1}{D} = \frac{1}{\frac{d}{dx}} = \int$

$= \frac{1}{D(1+D)}\ x^2+2x+4$

$= \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$

$= \frac{1}{D}(1-D+D^2-\cancel{D^3}+\cancel{D^4}-\dots)(x^2+2x+4)$ [$\because$ Neglecting $D^3, D^4, \dots$ i.e $D^3x^2 = D^4x^2 = \dots = 0$]

$= \frac{1}{D}(1-D+D^2)\underbrace{(x^2+2x+4)}$

$y_p = \frac{1}{D}(x^2+\cancel{2x}+4-(2x+\cancel{2}+0)+(\cancel{2}+0+0)) = \frac{1}{D}(x^2+4) = \int(x^2+4)\,dx = \frac{x^3}{3}+4x$

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$$y_p = \frac{-1}{6}\left[x + \frac{x^3}{3} + \frac{1}{6}\left(\underbrace{0+2x}-\underbrace{(1)-x^2}\right) + \frac{1}{36}\left(0+2x-2(0+2)\right) - \frac{1}{216}(0+2)\right]$$

$$y_p = -\frac{1}{6}\left[x + \frac{x^3}{3} + \frac{1}{6}(2x-1-x^2) + \frac{1}{36}(2x-4) - \frac{2}{216}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \left(1+\frac{2}{6}+\frac{2}{36}\right)x - \left(\frac{1}{6}+\frac{4}{36}+\frac{2}{216}\right)\right]$$

$$y_p = -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \left(\frac{25}{18}\right)x - \frac{31}{108}\right]$$

The G.S of given eq^n is $y = y_c + y_p$

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18} - \frac{31}{108}\right]$$, where c_1, c_2 & c_3 are arbitrary constants.

* solve $(D^3+2D^2+D)y = \underbrace{e^{2x}} + \underbrace{x^2+x} + \underbrace{\sin 2x}$

G.T $(D^3+2D^2+D)y = e^{2x} + x^2 + x + \sin 2x$ — ①

comparing eq^n ① with $f(D)y = Q(x)$

Here $f(D) = D^3+2D^2+D$ & $Q(x) = e^{2x} + x^2 + x + \sin 2x$

A.E is $f(m) = 0$

i.e $m^3 + 2m^2 + m = 0$

$m(m^2 + 2m + 1) = 0$

$m(m+1)^2 = 0$

$m = 0$ or $(m+1)^2 = 0$

$m = 0$ or $m = -1, -1$

$\therefore m = 0, -1, -1$

$C.F(y_c) = c_1 e^{0x} + (c_2 + c_3 x) e^{-1x}$

$y_c = c_1 + (c_2 + c_3 x) e^{-1x}$ $[\because e^{0x} = 1]$

$P.I(y_p) = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + x^2 + x + \sin 2x$

$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} (x^2 + x) + \frac{1}{D^3 + 2D^2 + D} \sin 2x$

y_{p_1} + y_{p_2} + y_{p_3} say

$\downarrow$ Type I $\quad$ $\downarrow$ Type - III $\quad$ $\downarrow$ Type - II consider $y_{p_1} = \dfrac{1}{D^3+2D^2+D} e^{2x}$

$\Downarrow$ Type-I

$\Downarrow$

$(a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$(D^2+2D)^3 = (D^2)^6+(2D)^3+3(D^2)^2(2D)+3D^2(2D)^2$

Each and every D by 2 sub $D \neq 0$

$y_{p_1} = \dfrac{1}{8+2(4)+2} e^{2x} = \dfrac{e^{2x}}{18}$

consider $y_{p_2} = \dfrac{1}{D^3+2D^2+\underline{D}} (x^2+x)$

$\Downarrow$ Type-III (Take lowest degree term common from $f(D)$)

$= \dfrac{1}{D\left(\underbrace{\dfrac{D^3+2D^2}{D}}+1\right)} x^2+x$

$= \dfrac{1}{D}\left(1+\underbrace{(D^2+2D)}\right)^{-1} (x^2+x)$ $\left[\because (1+\underbrace{D})^{-1} = 1-D+D^2-D^3+D^4- - - -\right]$

$= \dfrac{1}{D}\left(1-(D^2+2D)+(D^2+2D)^2-(D^2+2D)^3+ - - -\right)(x^2+x)$

$= \dfrac{1}{D}\left(1-(D^2+2D)+(\underline{D^4}+4D^2+4D^3)-(\underline{D^6}+8D^3+\underline{6D^5}+\underline{12D^4})+ - -\right)(x^2+x)$

$\{\because$ neglecting $D^3, D^4, D^5 - - - -$ $)$

$$y_{p_2} = \left(\frac{1}{D} - (D+2) + (4D+4D^2) - 8D^2\right)(x^2+x)$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - (2x + 1 + 2x^2 + 2x) + (4(2x+1) + 4(2+0)) - 8(2+0)$$

$$\int (x^2+x)\,dx = \frac{x^3}{3} + \frac{x^2}{2}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - 4x - 1 - 2x^2 + 8x + 4 + 8 - 16$$

$$= \frac{x^3}{3} + \left(\frac{1}{2} - 2\right)x^2 + 4x - 5$$

$$y_{p_2} = \frac{x^3}{3} - \frac{3}{2}x^2 + 4x - 5$$

consider $y_{p_3} = \dfrac{1}{D^3+2D^2+D} \sin 2x$

$\downarrow$ TYPE-II

rply D^2 by -2^2 sub $DN \neq 0$

$$= \frac{1}{(-2^2)D + 2(-2^2) + D} \sin 2x$$

$$= \frac{1}{-3D-8} \sin 2x = -\left[\frac{1}{3D+8} \sin 2x\right]$$

$$y_{p_3} = -\left[\frac{1}{3D+8} \frac{3D-8}{3D-8} \sin 2x \right]$$

$$= -\left\{ \frac{3D-8}{9D^2-64} \sin 2x \right]$$

$\downarrow$ mly D^2 by -2^2 sub Dr $\neq 0$

$$= -\left\{ \frac{3D-8}{9(-2^2)-64} \sin 2x \right]$$

$$= -\left[\frac{3D-8}{-100} \sin 2x \right]$$

$$= \frac{1}{100} \left[3D \sin 2x - 8 \sin 2x \right)$$

$$= \frac{1}{100} \left(3(2\cos 2x) - 8 \sin 2x \right)$$

$$y_{p_3} = \frac{1}{100} \left(6\cos 2x - 8 \sin 2x \right)$$

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Sub $y_{p_1}, y_{p_2}, y_{p_3}$ values in y_p

$$y_p = \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3}{2}x^2 + 4x - 5 + \frac{1}{100}(6\cos 2x - 8\sin 2x)$$

The G.S of eqn ① is $y = y_c + y_p$

$$y = c_1 + (c_2 + c_3 x)e^{-1x} + \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3}{2}x^2 + 4x - 5 + \frac{1}{100}(6\cos 2x - 8\sin 2x),$$ where c_1, c_2 & c_3 are arbitrary constants.

* Solve $y''' + 2y'' - y' - 2y = 1 - 4x^3$

$$y_p = \frac{1}{D^3 + 2D^2 - D - 2}(1 - 4x^3) \quad \text{Type-III}$$

$$= \frac{1}{-2\left(\frac{D^3 + 2D^2 - D}{-2} + 1\right)}(1 - 4x^3)$$

$$= \frac{-1}{2}\left[1 - \left(\frac{D^3 + 2D^2 - D}{2}\right)\right]^{-1}(1 - 4x^3)$$

$$= -\frac{1}{2}\left(1 + \frac{D^3 + 2D^2 - D}{2} + \left(\frac{D^3 + 2D^2 - D}{2}\right)^2 + \left(\frac{D^3 + 2D^2 - D}{2}\right)^3 + \cdots\right)(1 - 4x^3)$$

Ans $y = c_1 e^{-1x} + c_2 e^{-2x} + c_3 e^{1x} - \frac{1}{2}(-4x^3 + 6x^2 - 30x + 16)$

* solve $(D^2+5D+4)y = x^2$

Ans $y = c_1 e^{-1x} + c_2 e^{-4x} + \frac{1}{32}(8x^2 - 20x + 21)$

$\Rightarrow$ $D^2(D^2+4)y = 96x^2 + \sin 2x - k$

Ans $y = (c_1 + c_2 x) e^{0x} + e^{0x}(c_3 \cos 2x + c_4 \sin 2x) + 2x^4 - 6x^2 + 3 + \frac{x}{16}\cos 2x - \frac{k}{8}x^2$

Type-IV : P.I of $f(D)y = Q(x)$, when $Q(x) = e^{ax}.V$, where a is a constant and V is a function of x

To find P.I of $f(D)y = Q(x)$, where $Q(x) = e^{ax}.V$

Here V is a function of x (i.e $\sin bx$ or $\cos bx$ or x^k or a polynomial in x)

In this case $P.I(y_p) = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} e^{ax}.V$$

working rule:

To find P.I for $e^{ax}\cdot v$, take out e^{ax} to its left of $f(D)$ and replace every 'D' with $(D+a)$ so that $f(D)$ becomes $f(D+a)$ and now operate $\frac{1}{f(D+a)}$ with 'v' alone by the previous method

i.e $$y_p = e^{ax}\left(\frac{1}{f(D+a)} v\right)$$

* Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}(1+x)$

G.T $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}(1+x)$ —— (1)

The operator form of equ (1) is

$$D^2y - 7Dy + 6y = e^{2x}(1+x)$$

$$(D^2 - 7D + 6)\, y = e^{2x}(1+x)$$

compare above equ with $f(D)y = Q(x)$

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$$y = c_1 e^{-1x} + c_2 e^{1x} + c_3 \cos x + c_4 \sin x - \frac{1}{10} \cos x \,(2\cosh x) - (2x^4 + x + 47)$$

$$y = c_1 e^{-1x} + c_2 e^{+1x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} \cos x \cosh x - (2x^4 + x + 47)$$, where c_1, c_2, c_3 & c_4 are arbitrary constants.

* **Type-V: P.I of $f(D)y = Q(x)$, when $Q(x) = x \cdot V$, where V is any function of x which is either e^{ax} or $\sin bx$ or $\cos bx$:**

$$P.I\,(y_p) = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} x \cdot V$$

$$y_p = \left(x - \frac{f'(D)}{f(D)}\right) \frac{1}{f(D)} \cdot V$$

* Method of variation of parameters :

To solve $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ → (1) by the method of variation of parameters, follow these steps

(1) Reduce the given eqn to the standard form, if necessary.

(2) Find the solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ and let the solution be $C.F(y_c) = c_1y_1 + c_2y_2$.

Find y_1' & y_2'

(3) Find $W(y_1, y_2) = y_1y_2' - y_2y_1'$

(4) Take $P.I(y_p) = v_1y_1 + v_2y_2$, where $v_1 = -\int \frac{Ry_2}{W}dx$ & $v_2 = \int \frac{Ry_1}{W}dx$

(5) The G.S of eqn (1) $y = y_c + y_p$

* Apply the method of variation of parameters, to solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

G.T $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ —— ①

compare eqn ① with standard form (i.e $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$)

Here $P = 0$, $Q = 1$, $R = \operatorname{cosec} x$

The operator form of eqn ① is

$(D^2+1)y = \operatorname{cosec} x$ $\left[\because D^2 = \frac{d^2}{dx^2}\right]$

let $f(D) = D^2+1$

A.E is $f(m) = 0$

i.e $m^2+1 = 0$

$m^2 + 1^2 = 0$

$(m+i1)(m-i1) = 0$

$m = \pm i1 = 0 \pm i1$

roots are complex & conjugate

$C.F(y_c) = e^{0x}(c_1\cos 1x + c_2\sin 1x)$ $(\because e^{0x} = 1)$

$y_c = c_1\cos 1x + c_2\sin 1x = c_1y_1 + c_2y_2$ say

Here $y_1 = \cos x$ & $y_2 = \sin x$

$y_1' = -\sin x$ & $y_2' = \cos x$

Now $w(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= \cos x(\cos x) - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x$

$w = 1$

let $y_p = v_1 y_1 + v_2 y_2$ be the P.I of (1)

where $v_1 = -\int \frac{R y_2}{w} dx$ & $v_2 = \int \frac{R y_1}{w} dx$

$v_1 = -\int \frac{\operatorname{cosec} x \cdot \sin x}{1} dx$

$= -\int \frac{1}{\sin x} \cdot \sin x \; dx$

$v_1 = -x$

$v_2 = \int \frac{R y_1}{w} dx$

$= \int \frac{\operatorname{cosec} x \cdot \cos x}{1} dx$

$= \int \frac{1}{\sin x} \cos x \, dx$

$= \int \cot x \, dx$

$= \log|\sin x|$

$\therefore y_p = (-x)\cos x + (\log|\sin x|)\sin x$

The G.S of eqn (1) is $y = y_c + y_p$

$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log|\sin x|$

where c_1, c_2 are arbitrary constants

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