JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR

B. Tech II - I sem -CSE	**T**	**Tu**	**C**
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(15A05302) DISCRETE MATHEMATICS

II Year B.Tech.I Sem.

Course Objectives

Understand the methods of discrete mathematics such as proofs, counting principles, number theory, logic and set theory.Understand the concepts of graph theory, binomial theorem, and generating function in analysis of various computer science applications.

course Outcomes Able to apply mathematical concepts and logical reasoning to solve problems in different fields of Computer science and information technology.

Able to apply the concepts in courses like Computer Organization, DBMS, Analysis of Algorithms, Theoretical Computer Science, Cryptography, Artificial Intelligence

UNIT I:

Mathematical Logic:

Introduction, Connectives, Normal Forms, The theory of Inference for the Statement Calculus, The Predicate Calculus, Inference Theory of Predicate Calculus.

UNIT II:

SET Theory:

Basic concepts of Set Theory, Representation of Discrete structures, Relations and Ordering, Functions, Recursion.

UNIT III:

Algebraic Structures:

Algebraic Systems: Examples and General Properties, Semi groups and Monoids, Polish expressions and their compilation, Groups: Definitions and Examples, Subgroups and Homomorphism's, Group Codes.

Lattices and Boolean algebra:

Lattices and Partially Ordered sets, Boolean algebra.

UNIT IV:

An Introduction to Graph Theory:

Definitions and Examples, Sub graphs, complements, Graph Isomorphism, Vertex Degree: Euler Trails and Circuits, Planar Graphs, Hamilton Paths and Cycles, Graph Coloring and Chromatic Polynomials

Trees:

Definitions, Properties, Examples, Rooted Trees, Trees and Sorting, Weighted trees and Prefix Codes, Biconnected Components and Articulation Points

UNIT V:

Fundamental Principles of Counting:

The rules of Sum and Product, Permutations, Combinations: The Binomial Theorem, Combinations with Repetition

The Principle of Inclusion and Exclusion:

The Principle of Inclusion and Exclusion, Generalizations of Principle, Derangements: Nothing is in Its Right Place, Rook Polynomials, Arrangements with Forbidden Positions

Generating Functions:

Introductory Examples, Definitions and Examples: Calculation Techniques, Partitions of Integers, The Exponential Generating Functions, The Summation Operator.

TEXT BOOKS:

1. "Discrete Mathematical Structures with Applications to Computer Science", J.P. Tremblay and R. Manohar, McGraw Hill Education,2015.

2. "Discrete and Combinatorial Mathematics, an Applied Introduction", Ralph P. Grimaldi and B.V.Ramana, Pearson, 5th Edition, 2016.

REFERENCE BOOKS:

1. Graph Theory with Applications to Engineering by NARSINGH DEO, PHI.
2. Discrete Mathematics by R.K.Bishtand H.S. Dhami, Oxford Higher Education.
3. Discrete Mathematics theory and Applications by D.S.Malik and M.K.Sen, Cenegage Learning.
4. Elements of Discrete Mathematics, A computer Oriented approach by C L Liu and D P Mohapatra, MC GRAW HILL Education.
5. Discrete Mathematics for Computer scientists and Mathematicians by JOE L.Mott, Abraham Kandel and Theodore P.Baker, Pearson ,2nd Edition

UNIT II

SET Theory:

Basic concepts of Set Theory, Representation of Discrete structures, Relations and Ordering, Functions, Recursion.

2.0 INTRODUCTION

2.1 BASIC CONCEPTS OF SET THEORY

Set Definition: A collection of well defined objects is called as a set. The object comprising the set is called elements. We use capital letters to represent sets and small letters to represent elements.
The following are the examples of sets.

1. A battalion of Soldiers
2. The rivers in India
3. The vowels of alphabets

Capital Letters are used to represent the sets where as small letters are used to represent the elements.

If A is a set and a is a element belonging to the set a, we write it as a belongs to A i.e a **∈ A**

A functional Concept of set theory is the membership or belongs to a set is called members of the given set. A set is said to be well defined if it is possible to determine by means of certain rules, whether any given object is a member of the element.

Representation of sets:

Sets can be represented in two ways

1. All elements are listed, separated by commas and are enclosed by braces

Example:

A={ 1,2,3,4,5,6} , B={ a,b,c,d,e},

The repeated elements in sets are ignored.

2. second method it is defined by specifying a property that elements of the set have in common.

This can be represented as A: { x|P(x)}

Example: A={a,e,i,o,u} can be represented as A:{x|x is a vowels}

Finite set: If a set contains finite number of elements is called as Finite set and it can be represented as A: { 1,2,3 }

Infinite Set: If a set contains infinite number of elements is called as infinite set and it can be represented as A: { 1,2,3 , …. }

Null Set: If a set contains no elements is called as Null set and it can be represented as " $\varnothing$ " null.

A= {x|x is a multiple of 4, x is odd}

Singleton : if a set contains single element is called singleton set

Inclusion and Equality Sets

SUB SET:

A set contain within a set is called Sub set. The contained set is called subset and containing is a set.

It can be represented as

$A \subseteq B$, If $x \in A$ and $X \in B$

If A is not a subset of B i.e if at least one element of A does not belongs to B then it can be written as $A \not\subset B$.

i.e if a set contains n elements and the total number of subsets are 2^n.

SUPER SET

If a set A is a subset of B then it can be written as B is a super set of A and it can be represented as B $\supseteq$ A

PROPER SUBSET:

Any subset A is said to be proper subset of another set B is a subset of B, but there is at least one element of B which does not belongs to A i.e if A $\subseteq$ B but A B . It can be represented as A $\subset$ B.

EQUALITY SET:

Two sets A & B are called as Equality sets if and only if all the elements of A are in Set B and vice versa and can be represented as A=B.

UNIVERSAL SET:

Is one it includes every set under discussion is called Universal set and represented by E

2.2 RELATIONS and ORDERING

Introduction

The elements of a set may be related to one another. For example, in the set of natural numbers there is the 'less than' relation between the elements. The elements of one set may also be related to the elements another set.

2.2.1 Binary Relation

A binary relation between two sets A and B is a rule R which decides, for any elements, whether a is in relation R to b. If so, we write a R b. If a is not in relation R to b, then write as a /R b. We can also consider a R b as the ordered pair (a, b) in which case we can define a binary relation from A to B as a subset of A X B. This subset is denoted by the relation R.

Example: In general any set of ordered pairs defines a binary relation.

For example, the relation of father to his child is F = {(a, b) / a is the father of b} In this relation F, the first member is the name of the father and the second is the name of the child.The definition of relation permits any set of ordered pairs to define a relation.

For example, the set S given by

S = {(1, 2), (3, a), (b, a) ,(b, Joe)}

Definition

The **domain** D of a binary relation S is the set of all first elements of the ordered pairs in the relation.

i.e D(S)= {a / $ b for which **(a, b) Є S}**

The **range** R of a binary relation S is the set of all second elements of the ordered

pairs in the relation.(i.e) R(S) = {b / $ a for which (a, b) Є S}

For example

For the relation S = {(1, 2), (3, a), (b, a) ,(b, Joe)}

D(S) = {1, 3, b, b} and

R(S) = {2, a, a, Joe}

Let X and Y be any two sets. A subset of the Cartesian product X * Y defines a relation, say C. For any such relation C, we have D(C) Í X and R(C) Í Y, and the relation C is said to from X to Y.

If Y = X, then C is said to be a relation form X to X. In such case, c is called a relation in X . Thus any relation in X is a subset of X * X . The set X * X is called a *universal relation* in X, while the empty set which is also a subset of X * X is called a *void relation* inX.

For example

Let L denote the relation “less than or equal to” and D denote the relation “divides” where x D y means “ x divides y”. Both L and D are defined on the set {1, 2, 3, 4}

L = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4,4)}

D = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}

L Ç D = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4,4)} = D

2.2.3 Properties of Binary Relations:

Definition: A binary relation R in a set X is **reflexive if, for every x Є X, x R x, That is (x, x) Є** R, or R is reflexive in X **ó (x) (x Є X ® x R x).**

For example

1. The relation R is reflexive in the set of realnumbers.
2. The set inclusion is reflexive in the family of all subsets of a universal set.
3. The relation equality of set is also reflexive.
4. The relation is parallel in the set lines in a plane.
5. The relation of similarity in the set of triangles in a plane is reflexive.

Definition: A relation R in a set X is **symmetric** if for every x and y in X, whenever **x R y, then y R x.(i.e) R is symmetric in X ó (x) (y) (x Є X ^ y Є X ^ x R y ®** y R x}

For example

- The relation equality of set issymmetric.
- The relation of similarity in the set of triangles in a plane issymmetric.
- The relation of being a sister is not symmetric in the set of allpeople.
- However, in the set females it is symmetric.

Definition: A relation R in a set X is transitive if for every x and z in X, whenever **x R y and y R z , then x R z that is R is transitive in X ↔ y (x) (y)(z) (x Є X ^ y Є X ^ zЄ X** ∧ x R y ∧ y R z → x R z)

For example

- The relations **<, £, >, ³** and **=** are transitive in the set of real numbers
- The relations Í, Ì, Ê, É and equality are also transitive in the family ofsets.
- The relation of similarity in the set of triangles in a plane is transitive.

Definition: A relation R in a set X is **irreflexive if, for every x Є X , (x, x)ÏX.**

For example

- The relation **<**is irreflexive in the set of all realnumbers.
- The relation proper inclusion is irreflexive in the set of all nonempty subsets of a universalset.
- LetX={1,2,3}andS={(1,1),(1,2),(3,2),(2,3),(3,3)}isneitherirreflexivenor reflexive.

Definition:A relation R in a set x is **anti symmetric**if , for every x and y in X, whenever x R y and y R x, then x = y.

Symbolically,(x) (y) (x Є X ^ y Є X ^ x R y ^ y R x ® x = y)

For example

- The relations £ , ³ and **=** are antisymmetric
- The relation Í is anti symmetricin set ofsubsets.
- The relation **“divides” is anti symmetric in set of realnumbers.**
- **Consider the relation “is a son of” on the male children in a family.Evidently the relation** is not symmetric, transitive andreflexive.
- **The relation “ is a divisor of “ is reflexive and transitive but no**t symmetric on the set of naturalnumbers.
- **Consider the set H of all human beings. Let r be a relation “ is married to“**
  R is symmetric.
- Let I be the set of integers. R on I is defined as a R b if a **–** b is an even number.Ris an reflexive, symmetric and transitive.

2.2.4 Equivalence Relation

Definition: A relation R in a set A is called an **equivalence** relation if

- a R a for every i.e. R isreflexive
- **a R b => b R a for every a, b Є A i.e. R issymmetric**
- a R b and b R c => a R c for every **a, b, c Є A, i.e. R istransitive.**

For example

- The relation equality of numbers on set of realnumbers.
- The relation being parallel on a set of lines in aplane.

Problem1: Let us consider the set T of triangles in a plane. Let us definea relation **R in T as R= {(a, b) / (a, b Є T and a issimilar tob}** We have to show that relation R Is an equivalence relation

Solution :

- A triangle a is similar to itself. a Ra
- If the triangle a is similar to the triangle b, then triangle b is similar to the triangle a then a R b => b Ra
- If a is similar to b and b is similar to c, then a is similar to c (i.e) a R b and b R c => a R c.

Hence R is an equivalence relation.

Problem 2: Let x = {1, 2, 3, … 7} and R = {(x, y) / x **–** y is divisible by 3} Show that R is an equivalence relation.

Solution: For any a Є X, a- a is divisible by 3, Hence a R a, R is reflexive

For any a, b Є X, if a – b is divisible by 3, then b **–** a is also divisible by 3, R is symmetric.

For any a, b, c Є, if a R b and b R c, then a – b is divisible by 3 and b–c is divisible by 3. So that (a **–** b) + (b **–** c) is also divisible by 3, hence a **–** c is also divisible by 3. Thus R is transitive.

Hence R is equivalence.

Problem3 Let Z be the set of all integers. Let m be a fixed integer. Two integers a and b are said to be congruent modulo m if and only if m divides a-b, in which case we write a º(mod m). This relation is called the relation of congruence modulo m and we can show that is an equivalence relation.

Solution :

- a - a=0 and m divides a – a (i.e) **a R a, (a, a) Є R, R is** reflexive.
- a R b = m divides a-b

m divides b -
a b º a (mod
m) b Ra
that is R is symmetric.

- a R b and b R c => a ºb (mod m) and bº c (modm)
    - O m divides a – b and m dividesb-c
    - O a – b = km and b **– c = lm for some k ,l Єz**
    - O (a – b) + (b – c) = km +lm
    - O a – c = (k +l)m
    - O aº c (mod m)
    - O a Rc
    - O R is transitive

Hence the congruence relation is an equivalence relation.

2.2.5 Equivalence Classes:

Let R be an equivalence relation on a set A. For an**y a ЄA, the equivalence class** generated by a **is the set of all elements b Є A such a R b and is denoted [a].** It is also called the R **– equivalence class and denoted by a Є A.** i.e., **[a] = {b Є A / b R a}**

Let Z be the set of integer and R be the relation **called "congruence modulo 3"**
defined by R = {(x, y)/ xÎ Z Ù yÎZ Ù (x-y) is divisible by 3}

Then the equivalence classes are

[0] = {… -6, -3, 0, 3, 6, **…}**
[1] = {…, -5, -**2, 1, 4, 7, …}**
[2] = {…, -4, -**1, 2, 5, 8, …}**

Composition of binary relations:

Definition: Let R be a relation from X to Y and S be a relation from Y to Z. Then the relation R o S is given by R o S = {(x, z) / xÎX Ù z Î Z Ù y Î Y such that (x, y) Î R Ù (y, z) Î S)} is called the composite relation of R andS.

The operation of obtaining R o S is called the **composition of relations**.

Example: Let R = {(1, 2), (3, 4), (2, 2)} and

S = {(4, 2), (2, 5), (3, 1),(1,3)}

Then R o S = {(1, 5), (3, 2), (2, 5)} and S o R = {(4, 2), (3, 2), (1, 4)}

It is to be noted that R o S ≠ S o R.

Also Ro(S o T) = (R o S) o T = R o S o T

Note: We write R o R as R2; R o R o R as R3 and so on.

Definition

Let R be a relation from X to Y, a relation R from Y to X is called the **converse** of R, **where the ordered pairs of Ř are obtained by interchanging the numbers in each of the ordered** pairs of R. This means for x Î X and y Î **Y, that x R y ó y Ř x.**

Then the relation Ř is given by R = {(x, y) / (y, x) Î R} is called the converse

of R Example:

Let R = {(1, 2),(3, 4),(2, 2)}

Then Ř = {(2, 1),(4, 3),(2, 2)}

Note: If R is an equivalence rel**ation, then Ř is also an equivalence relation.**

Definition Let X be any finite set and R be a relation in X. The relation **R+ = R U R2 U R3…in X. is called the *transitive closure*** of R in X

Example: Let R = {(a, b), (b, c), (c, a)}.

Now R2 = R o R = {(a, c), (b, a), (c, b)}

R3 = R2 o R = {(a, a), (b, b), (c, c)}

R4 = R3 o R = {(a, b), (b, c), (c, a)} = R

R5= R3o R2 = R2 and so on.

Thus, R+ = R U R2 U R3 U R4 U…

= R U R2 U R3.

={(a, b),(b, c),(c, a),(a, c),(b, a),(c ,b),(a, b),(b, b),(c, c)}

We see that R+ is a transitive relation containing R. In fact, it is the smallest transitive relation containing R.

Partial Ordering Relations:

Definition

A binary relation R in a set P is called ***partial order relation*** or *partial ordering* in P iff R is reflexive, anti symmetric, and transitive.

A partial order relation is denoted by the symbol £., If £ is a partial ordering on P, then the ordered pair (P, £) is called a ***partially ordered set*** or a ***poset***.

- **Let R be the set of real numbers. The relation "less than or equal to "or**

  O ,is a partial ordering onR.
- Let X bea set and r(X) be its power set. The relation subset, Í on X is partialordering.
- **Let Sn be the set of divisors of n. The relation D means "divides" on Sn,**ispartial ordering on Sn.

In a Partially ordered set(P, £) , an element y Î P is Said to cover an element xÎ P if x <y And ifthere doesnot exist any element z Î P such that x £ z andz £ y; that is, y coversx Û (x < yÙ (x £ z £ y Þ X = z Ú z = y))

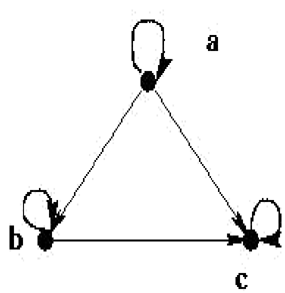
A partial order relation £ on a set P can be represented by means of a diagram known as aHasse diagram or partial order set diagram of (P, £). In such a diagram, each element is represented by a small circle or a dot. The circle for x Î P is drawn below the circle for y Î P if x <y, and a line is drawn between x and y if y covers x.

If x < y but y does not cover x, then x and y are not connected directly by a single line.However, they are connected through one or more elements of P.

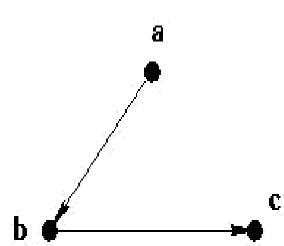
Hasse Diagram:

A Hasse diagram is a digraph for a poset which does not have loops and arcs implied by the transitivity.

Example 10: For the relation {< a, a >, < a, b >, < a, c >, < b, b >, < b, c >, < c, c >} on set {a, b,c}, the Hasse diagram has the arcs {< a, b >, < b, c >} as shown below.

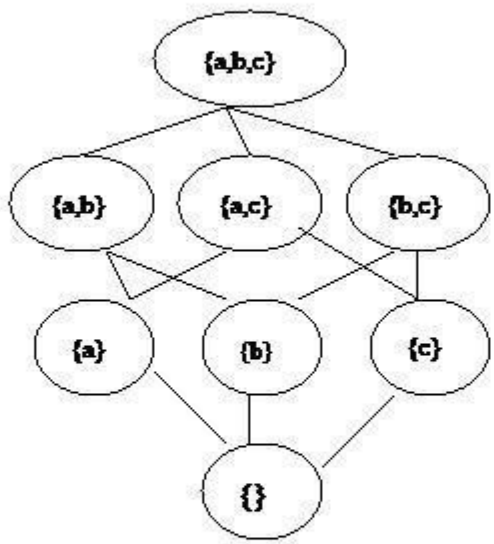


Digraph for Partial Order



Hasse Diagram

Ex: Let A be a given finite set and r(A) its power set. Let Í be the subset relation on the elements of r(A). Draw Hasse diagram of (r(A), Í) for A = {a, b, c}



3. Functions

3.1 Introduction

A function is a special type of relation. It may be considered as a relation in which each element of the domain belongs to only one ordered pair in the relation. Thus a function from A **to B is a subset of A X B having the property that for each a ЄA, there is one and only one b Є B** such that (a, b) ÎG.

Definition

Let A and B be any two sets. A relation f from A to B is called a function **if for every a Є A there is a unique b Є B such that (a, b) Є f .**

Note that the definition of function requires that a relation must satisfy two additional conditions in order to qualify as a function.

The first condition is that every a Є A must be related to some b Є B, (i.e) the domain of f must be A and not merely subset of A. The second requirement of uniqueness can be expressed **as (a, b) Є f ^ (b, c) Є f => b = c**

Intuitively, a function from a set A to a set B is a rule which assigns to every element of A, a unique element of B. **If a ЄA, then the unique element of B assigned to a under f is denoted by f** (a).The usual notation for a function f from A to B is f: A® B defi**ned by a ® f (a) where a Є A,** f(a) is called the image of **a** under f and **a** is called pre image of f(a).

- Let X = Y = **R** and f(x) = x2 + 2. Df = R and Rf ÍR.
- Let X be the set of all statements in logic and let Y = {True, False}. A

mapping f: X®Y is afunction.

- A program written in high level language is mapped into a machine language by a compiler. Similarly, the output from a compiler is a function of itsinput.
- Let X = Y = **R** and f(x) = x2 is a function from X ® Y,and g(x2) = x is not a function from X ®Y.

A mapping f: A ® B is called **one-to-one** (injective or 1 –1) if distinct elements of A are mapped into distinct elements of B. (i.e) f is one-to-one if

a1 = a2 => f (a1) = f(a2) or equivalently f(a1) ¹ f(a2) => a1 ¹ a2

For example, f: N ® N given by f(x) = x is 1-1 where N is the set of a natural numbers.

A mapping f: A® B is called **onto (surjective) if for every b Є B there is an a Є A such** that f (a) = B. i.e. if every element of B has a pre-image in A. Otherwise it is called **into**.

For example, f: Z®Z given by f(x) = x + 1 is an onto mapping. A mapping is both 1-1 and onto is called bijective

.

For example f: R®R given by f(x) = X + 1 is bijective.

Definition: A mapping f: R® b is called a **constant mapping** if, for all aÎA, f (a) = b, a fixed element.

For example f: Z®Z given by f(x) = 0, for all x ÎZ is a constant mapping.

Definition

A mapping f: A®A is called the **identity mapping** of A if f (a) = a, for all aÎA. Usually it is denoted by IA or simply I.

3.2 Composition of functions:

If f: A®B and g: B®C are two functions, then the composition of functions f and g, denoted by g o f, is the function is given by g o f : A®C and is given by

g o f = {(a, c) / a Є A ^ c Є C ^ $bÎ B ': f(a)= b ^ g(b) = c} and (g of)(a) = ((f(a))

Example 1: Consider the sets A = {1, 2, 3},B={a, b} and C = {x, y}. Let f: A® B be defined by f (1) = a ; f(2) = b and f(3)=b and Let g: B® C be defined by g(a) = x and g(b) = y

(i.e) f = {(1, a), (2, b), (3, b)} and g = {(a, x), (b, y)}. Then g o f: A®C is defined by

(g of) (1) = g (f(1)) = g(a) = x

(g o f) (2) = g (f(2)) = g(b) = y

(g o f) (3) = g (f(3)) = g(b) = y

i.e., g o f = {(1, x), (2, y),(3, y)}

If f: A® A and g: A®A, where A= {1, 2, 3}, are given by

f = {(1, 2), (2, 3), (3, 1)} and g = {(1, 3), (2, 2), (3, 1)} Then

g of = {(1, 2), (2, 1), (3, 3)}, fog= {(1, 1), (2, 3), (3, 2)}

f of = {(1, 3), (2, 1), (3, 2)} and gog= {(1, 1), (2, 2), (3, 3)}

Example 2: Let f(x) = x+2, g(x) = x **–** 2 and h(x) = 3x for x Î R, where R is the set of real numbers.

Then f o f = {(x, x+4)/xÎ R} f o g =

{(x, x)/ x Î X} g

o f = {(x, x)/ xÎ X}

g o g = {(x, x-4)/x Î X}

h o g = {(x,3x-6)/ x Î X} h o f
= {(x, 3x+6)/ x Î X}

3.3 Inverse functions:

Let f: A® B be a one-to-one and onto mapping. Then, its inverse, denoted by f -1 is given by f - 1 = {(b, a) / (a, b) Î f} Clearly f-1: B® A is one-to-one and onto.

Also we observe that f o f -1 = IB and f -1o f = IA. If f -1 exists then f is called **invertible**.

For example:Let f: R ®R be defined by f(x) = x + 2 Then f -1:
R® R is defined by f -1(x) = x - 2

Theorem: Let f: X ®Y and g: Y ® Z be two one to one and onto functions. Then gof is also one to one and onto function.

Proof

Let f:X ® Y g : Y ® Z be two one to one and onto functions. Let x1, x2 Î X

- g o f (x1) = g of(x2),
- g (f(x1)) =g(f(x2)),
- g(x1) = g(x2) since [f is1-1]

x1 = x2 since [g is 1-1}
so that gof is 1-1.

By the definition of composition, gof : X ® Z is a function.
We have to prove that every element of z Î Z an image element for some x Î X

undergof.
Since g is onto $ y ÎY ': g(y) = z and f is onto from X to Y,
$ x ÎX ': f(x) = y.
Now, gof (x) = g (f (x))
= g(y) [since f(x) = y]
= z [since g(y) = z]
which shows that gof is onto.

Theorem (g o f) -1 = f -1 o g-1

(i.e) the inverse of a composite function can be expressed in terms of the composition of the inverses in the reverse order.

Proof.

f: A ® B is one to one and onto. g: B ® C is one to one and onto.

gof: A ® C is also one to one and onto. Þ

(gof) -1: C ® A is one to one and onto.

Let a Î A, then there exists an element b Î b such that f (a) = b Þ a = f-1(b). Now b Î B Þ there exists an element c Î C such that g (b) = c Þ b = g - 1(c). Then (gof)(a) = g[f(a)] = g(b) = c Þ a = (gof) -1(c)**……..(1)**

(f -1 o g-1) (c) = f -1(g -1 (c)) = f -1(b) = a Þ a = (f -1 o g -**1)(c)** **….(2)** Combining (1) and (2), we have (gof) - 1 = f -1 o g -1

Theorem: If f: A ® B is an invertible mapping , then f o f -
1 = I B and f-1 o f = IA

Proof: f is invertible, then f -1 is defined by f(a) = b ó f-1(b) = a
where a Î A and bÎ B .
Now we have to prove that f of -1 = IB
. Let bÎ B and f -1(b) = a, a Î A
then fof-1(b) = f(f-1(b))
= f(a) = b
therefore f o f -1 (b) = b " b Î B => f o f -1 = IB Now
f -1 o f(a) = f -1 (f(a)) = f -1 (b) = a therefore f -1 o
f(a) = a " a Î A => f -1 o f = IA. Hence the theorem.

3.4 Recursive Functions:

The term "recursive function" is often used informally to describe any function that is defined with recursion. There are several formal counterparts to this informal definition, many of which only differ in trivial respects.

Kleene (1952) defines a "partial recursive function" of nonnegative integers to be any function f that isdefinedbyanoncontradictorysystemofequationswhoseleftandrightsidesarecomposedfrom (1) function symbols (for example, f, g, h, etc.), (2) variables for nonnegative integers (forexample, x, y, z, etc.), (3) the constant 0, and (4) the successor function $S(x) = x + 1$.

For example,

$f(x, 0)=0$ (1)

$f(x, S(y))=g(f(x, y), x)$ (2)

$g(x, 0)=x$ (3)

$g(x, S(y)) = S(g(x, y))$ (4)

Defines $f(x, y)$ to be the function xy that computes the product of $x$and$y$.

Note that the equations might not uniquely determine the value of $f$for every possible input, and in that sense the definition is "partial." If the system of equations determines the value of f for every input, then the definition is said to be "total." When the term "recursive function" is used alone, it is usually implicit that "total recursive function" is intended. Note that some authors use the term "general recursive function to mean partial recursive function, although others use it to mean "total recursivefunction."

The set of functions that can be defined recursively in this manner is known to be equivalent to the set of functions computed by Turing machines and by the lambda calculus.