UNIT-4

Sorting

Bringing Order to the World

Lecture Outline

- Iterative sorting algorithms (comparison based)
  - Selection Sort
  - Bubble Sort
  - Insertion Sort
- Recursive sorting algorithms (comparison based)
  - Merge Sort
  - Quick Sort

- Note: we only consider sorting data in **ascending order**

Why Study Sorting?

- When an input is sorted, many problems become easy (e.g. searching, min, max, *k*-th smallest)

- Sorting has a variety of interesting algorithmic solutions that embody many ideas
  - Comparison vs non-comparison based
  - Iterative
  - Recursive
  - Divide-and-conquer
  - Best/worst/average-case bounds
  - Randomized algorithms

Applications of Sorting

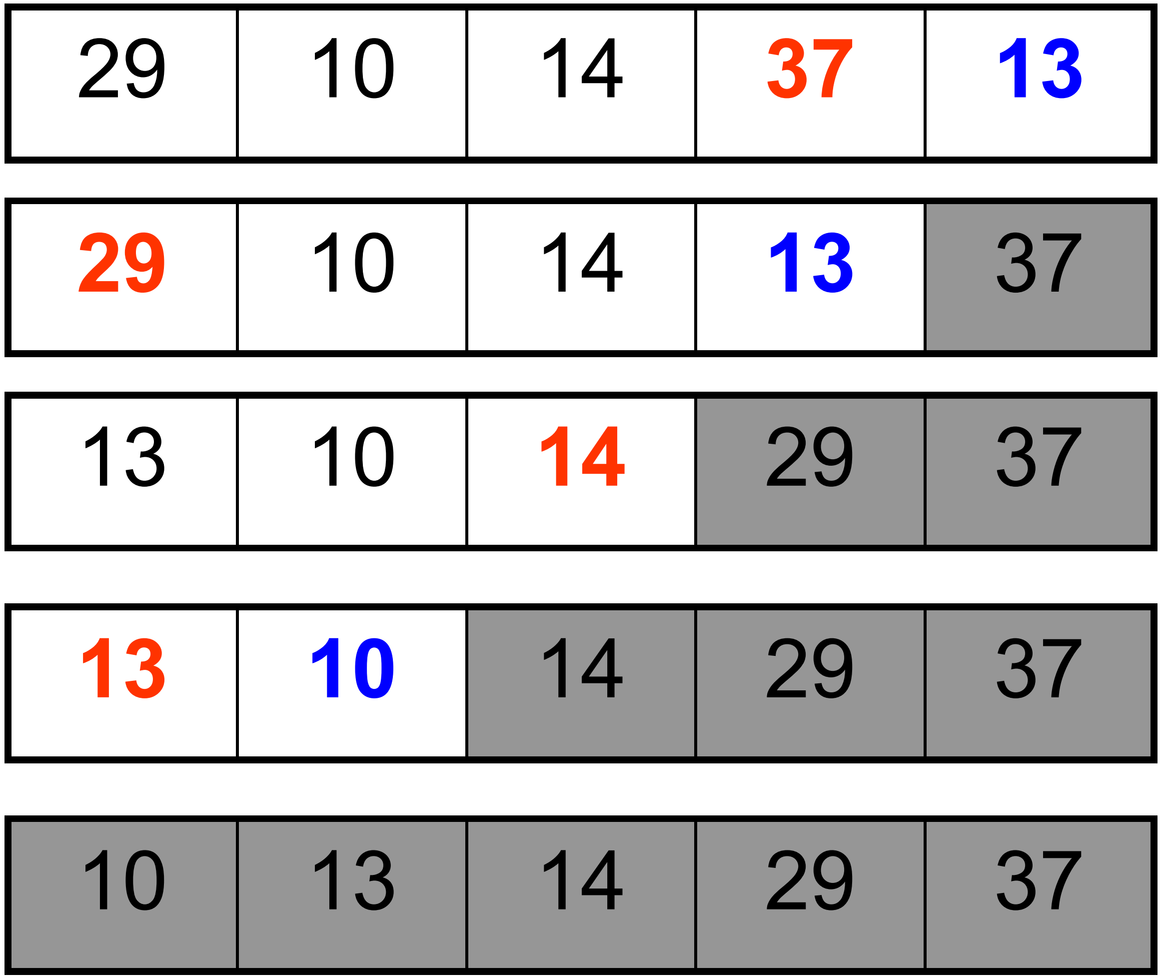
- Uniqueness testing
- Deleting duplicates
- Prioritizing events
- Frequency counting
- Reconstructing the original order
- Set intersection/union
- Finding a target pair *x*, *y* such that *x*+*y* = *z*
- Efficient searching

Selection Sort

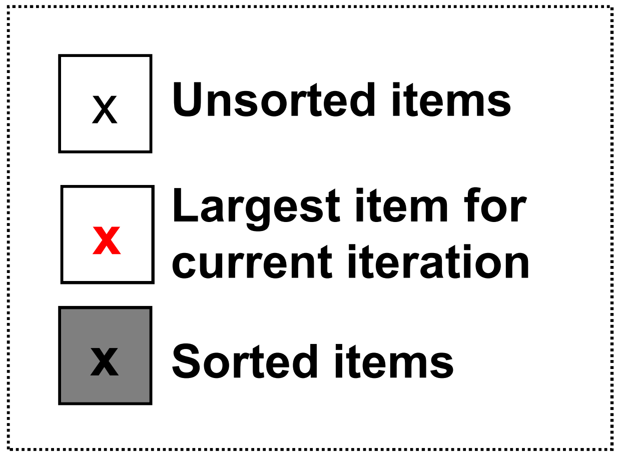
Selection Sort: Idea

- Given an array of *n* items
  1. Find the largest item *x*, in the range of [0…*n*−1]
  2. Swap *x* with the (*n*−1)th item
  3. Reduce *n* by 1 and go to Step 1

Selection Sort: Illustration



37 is the largest, swap it with the last element, i.e. 13.
Q: How to find the largest?



Sorted!

We can also find the smallest and put it the front instead
http://visualgo.net/sorting?create=29,10,14,37,13&mode=Selection

Selection Sort: Implementation

```
void selectionSort(int a[], int n) {
  for (int i = n-1; i >= 1; i--) {
    int maxIdx = i;
    for (int j = 0; j < i; j++)
      if (a[j] >= a[maxIdx])
        maxIdx = j;
    // swap routine is in STL <algorithm>
    swap(a[i], a[maxIdx]);
  }
}
```

Step 1: Search for maximum element

Step 2: Swap maximum element with the last item i

Selection Sort: Analysis

```
void selectionSort(int a[], int n) {
  for (int i = n-1; i >= 1; i--) {
    int maxIdx = i;
    for (int j = 0; j < i; j++)
      if (a[j] >= a[maxIdx])
        maxIdx = j;
    // swap routine is in STL <algorithm>
    swap(a[i], a[maxIdx]);
  }
}
```

Number of times executed
$n-1$ (int maxIdx = i;)
$n-1$ (for (int j = 0; j < i; j++))
$(n-1)+(n-2)+\ldots+1 = n(n-1)/2$ (if and maxIdx = j;)
$n-1$ (swap)
Total $= c_1(n-1) + c_2 * n * (n-1)/2 = \mathbf{O(n^2)}$

- c_1 and c_2 are cost of statements in outer and inner blocks

Bubble Sort

Bubble Sort: Idea

- Given an array of *n* items
  1. Compare pair of adjacent items
  2. Swap if the items are out of order
  3. Repeat until the end of array
     - The largest item will be at the last position
  4. Reduce *n* by 1 and go to Step 1

- Analogy
  - Large item is like “bubble” that floats to the end of the array

Bubble Sort: Illustration

(a) Pass 1

29	10	14	37	13
10	29	14	37	13
10	14	29	37	13
10	14	29	37	13
10	14	29	13	**37**

At the end of **Pass 1**, the largest item **37** is at the last position.

(b) Pass 2

10	14	29	13	**37**
10	14	29	13	**37**
10	14	29	13	**37**
10	14	13	**29**	**37**

At the end of **Pass 2**, the second largest item **29** is at the second last position.

x — **Sorted Item**

x — **Pair of items under comparison**

Bubble Sort: Implementation

```
void bubbleSort(int a[], int n) {
  for (int i = n-1; i >= 1; i--) {
    for (int j = 1; j <= i; j++) {
      if (a[j-1] > a[j])
        swap(a[j], a[j-1]);
    }
  }
}
```

Step 1: Compare adjacent pairs of numbers

Step 2: Swap if the items are out of order

29	10	14	37	13

http://visualgo.net/sorting?create=29,10,14,37,13&mode=Bubble

Bubble Sort: Analysis

- 1 iteration of the inner loop (test and swap) requires time bounded by a constant c
- Two nested loops
  - Outer loop: exactly n iterations
  - Inner loop:
    - when $i=0$, $(n-1)$ iterations
    - when $i=1$, $(n-2)$ iterations
    - …
    - when $i=(n-1)$, 0 iterations
- Total number of iterations = $0+1+\ldots+(n-1) = n(n-1)/2$
- Total time = $c\, n(n-1)/2 = \mathbf{O(n^2)}$

Bubble Sort: Early Termination

- Bubble Sort is inefficient with a $O(n^2)$ time complexity
- However, it has an interesting property
  - Given the following array, how many times will the inner loop swap a pair of item?

3	6	11	25	39

- Idea
  - If we go through the inner loop with no swapping
    ➔ the array is sorted
    ➔ can stop early!

Bubble Sort v2.0: Implementation

```
void bubbleSort2(int a[], int n) {
  for (int i = n-1; i >= 1; i--) {
    bool is_sorted = true;
    for (int j = 1; j <= i; j++) {
      if (a[j-1] > a[j]) {
        swap(a[j], a[j-1]);
        is_sorted = false;
      }
    } // end of inner loop
    if (is_sorted) return;
  }
}
```

Assume the array is sorted before the inner loop

Any swapping will invalidate the assumption

If the flag remains **true** after the inner loop ➔ sorted!

Bubble Sort v2.0: Analysis

- Worst-case
  - Input is in descending order
  - Running time remains the same: $O(n^2)$
- Best-case
  - Input is already in ascending order
  - The algorithm returns after a single outer iteration
  - Running time: $O(n)$

Insertion Sort

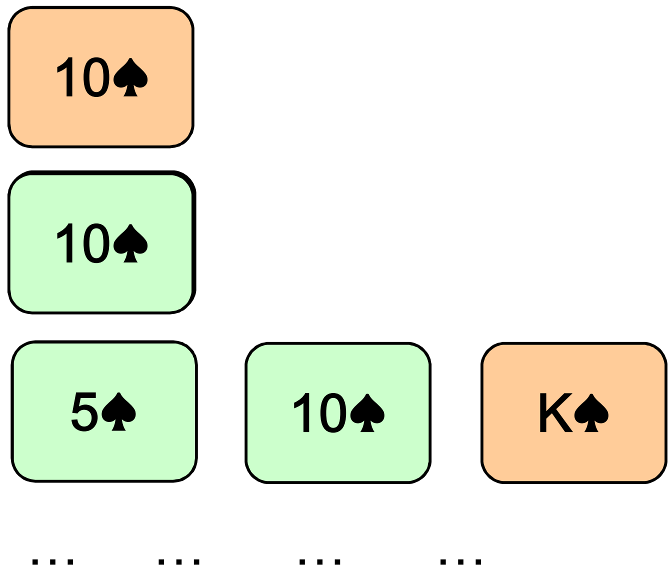
Insertion Sort: Idea

- Similar to how most people arrange a hand of poker cards
  - Start with one card in your hand
  - Pick the next card and insert it into its proper sorted order
  - Repeat previous step for all cards

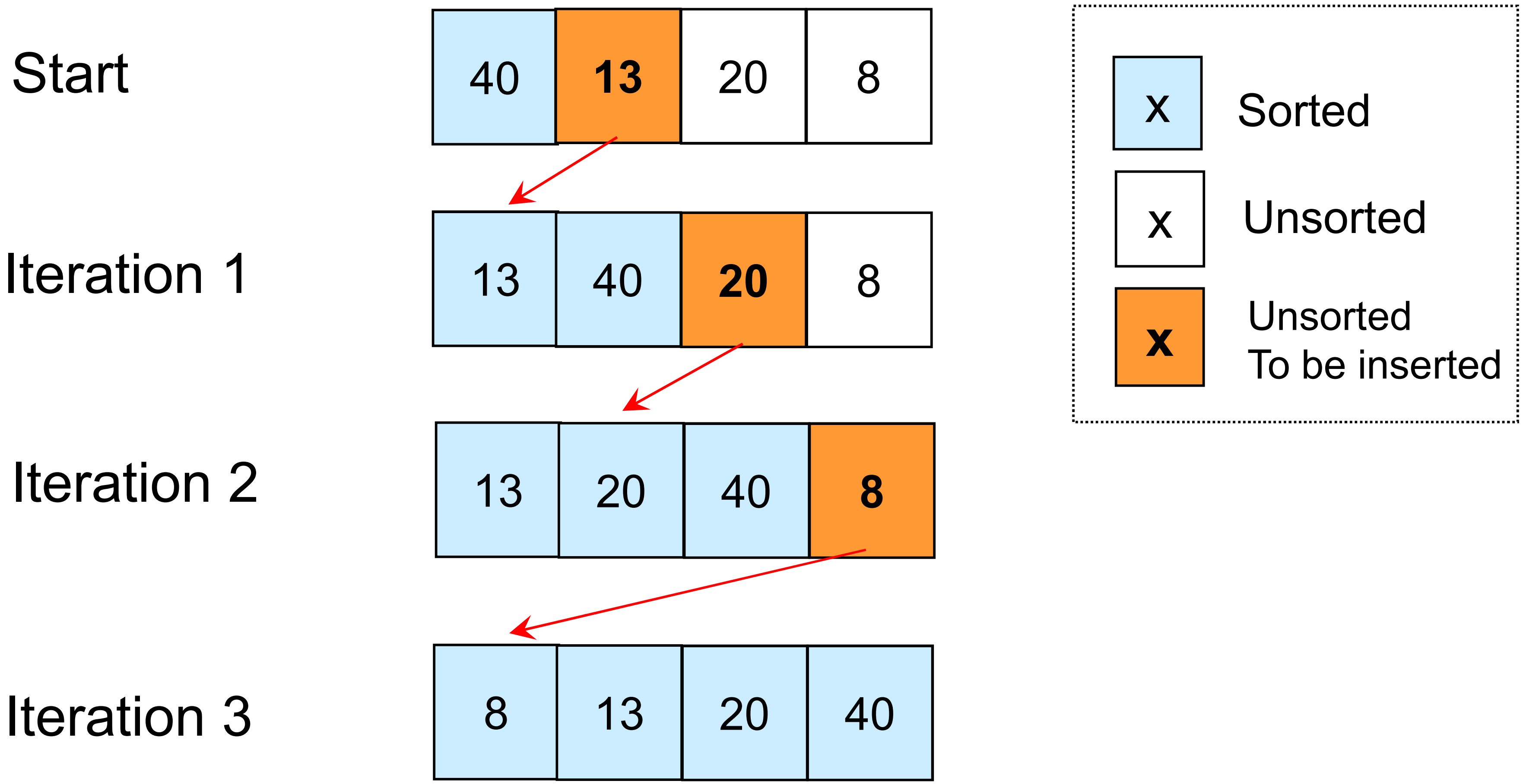
1st card: **10♠**

2nd card: **5♠**

3rd card: **K♠**



Insertion Sort: Illustration



http://visualgo.net/sorting?create=40,13,20,8&mode=Insertion

Insertion Sort: Implementation

```
void insertionSort(int a[], int n) {
  for (int i = 1; i < n; i++) {
    int next = a[i];
    int j;

    for (j = i-1; j >= 0 && a[j] > next; j--)
      a[j+1] = a[j];

    a[j+1] = next;
  }
}
```

next is the item to be inserted

Shift sorted items to make place for **next**

Insert **next** to the correct location

29	10	14	37	13

http://visualgo.net/sorting?create=29,10,14,37,13&mode=Insertion

Insertion Sort: Analysis

- Outer-loop executes ($n-1$) times
- Number of times inner-loop is executed depends on the input
  - **Best-case**: the array is already sorted and (a[j] > next) is always false
    - No shifting of data is necessary
  - **Worst-case**: the array is reversely sorted and (a[j] > next) is always true
    - Insertion always occur at the front
- Therefore, the **best-case** time is **O(n)**
- And the **worst-case** time is **O(n^2)**

Merge Sort

Merge Sort: Idea

- Suppose we only know how to merge two sorted sets of elements into one
  - Merge {1, 5, 9} with {2, 11} → {1, 2, 5, 9, 11}
- Question
  - Where do we get the two sorted sets in the first place?
- Idea (use **merge** to sort *n* items)
  - Merge each pair of elements into sets of 2
  - Merge each pair of sets of 2 into sets of 4
  - Repeat previous step for sets of 4 …
  - Final step: merge 2 sets of *n*/2 elements to obtain a fully sorted set

Divide-and-Conquer Method

- A powerful problem solving technique
- Divide-and-conquer method solves problem in the following steps
  - **Divide** step
    - Divide the large problem into smaller problems
    - Recursively solve the smaller problems
  - **Conquer** step
    - Combine the results of the smaller problems to produce the result of the larger problem

Divide and Conquer: Merge Sort

- Merge Sort is a divide-and-conquer sorting algorithm
- Divide step
  - Divide the array into two (equal) halves
  - Recursively sort the two halves
- Conquer step
  - Merge the two halves to form a sorted array

Merge Sort: Illustration

7	2	6	3	8	4	5

Divide into two halves

7	2	6	3

8	4	5

Recursively sort the halves

2	3	6	7

4	5	8

Merge them

2	3	4	5	6	7	8

- Question
  - How should we sort the halves in the 2nd step?

Merge Sort: Implementation

```
void mergeSort(int a[], int low, int high) {
  if (low < high) {
    int mid = (low+high) / 2;

    mergeSort(a, low   , mid );
    mergeSort(a, mid+1, high);

    merge(a, low, mid, high);
  }
}
```

Merge sort on **a[low...high]**

Divide a[ ] into two halves and **recursively** sort them

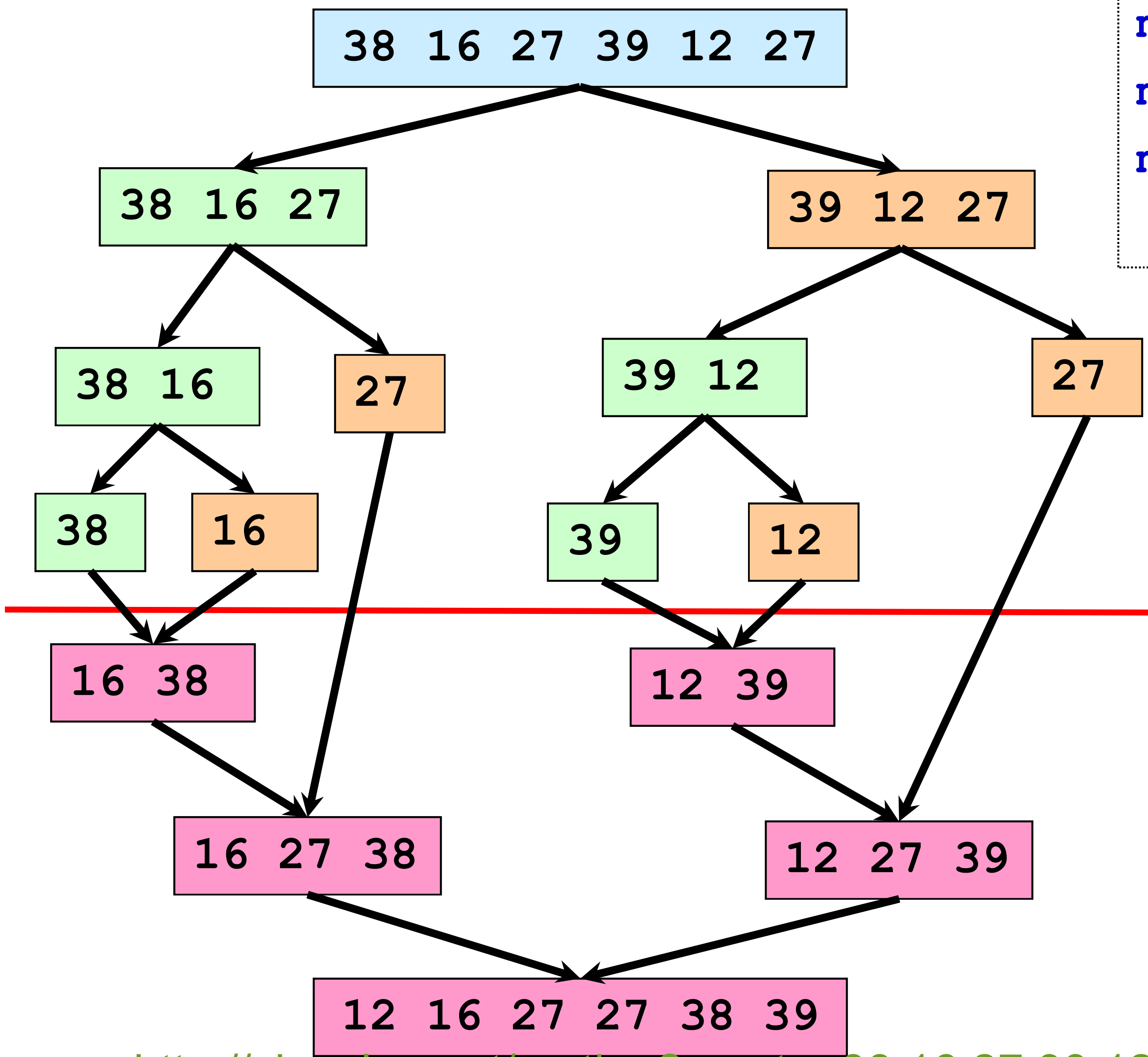
Conquer: merge the two sorted halves

Function to merge **a[low…mid]** and **a[mid+1…high]** into **a[low…high]**

- Note
  - **mergeSort()** is a recursive function
  - **low >= high** is the base case, i.e. there is 0 or 1 item

Merge Sort: Example

```
mergeSort(a[low…mid])
mergeSort(a[mid+1…high])
merge(a[low..mid],
      a[mid+1..high])
```



Divide Phase

Recursive call to **mergeSort()**

Conquer Phase

Merge steps

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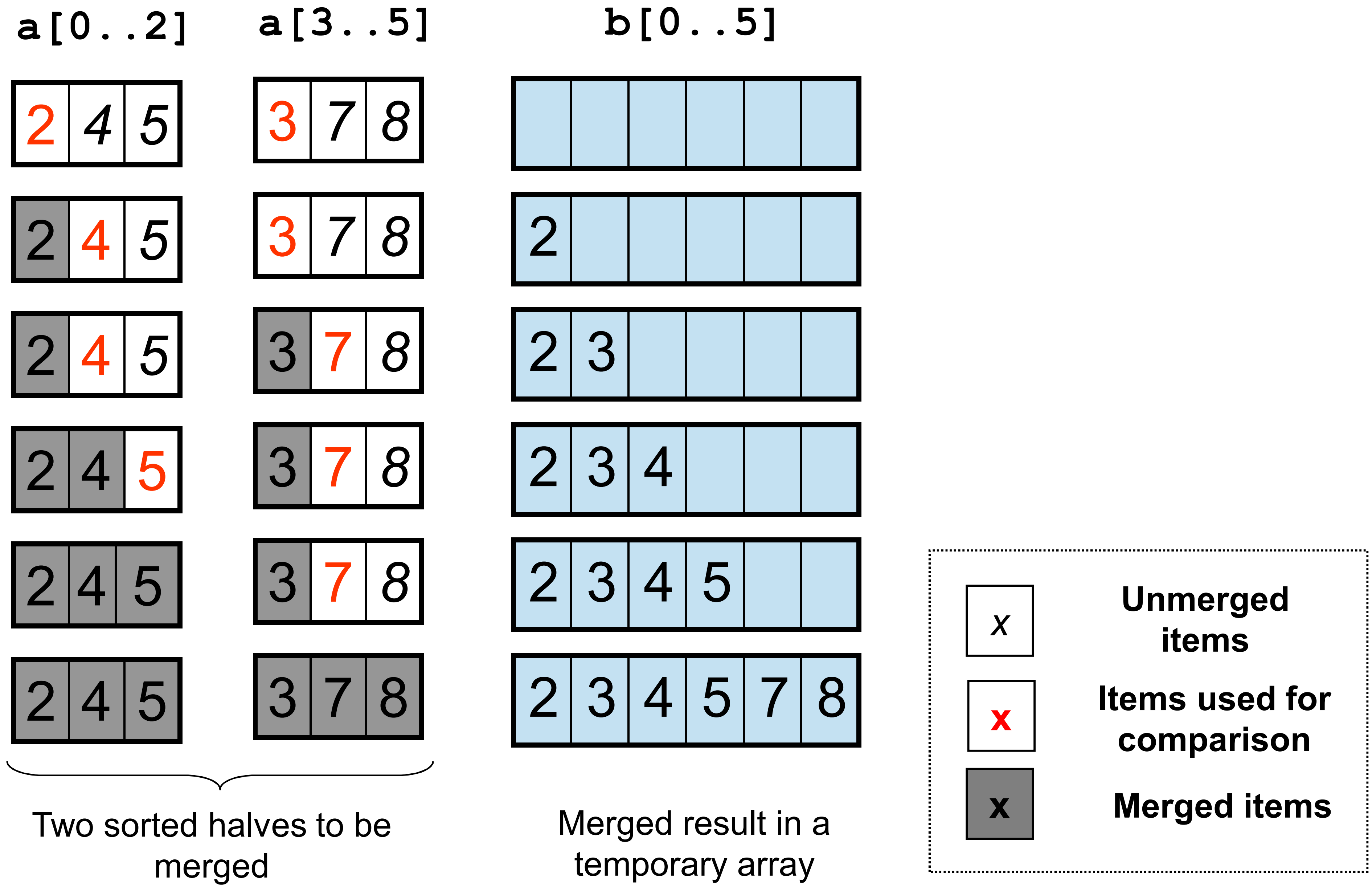
Merge Sort: Merge



Merge Sort: Merge Implementation

PS: C++ STL <algorithm> has merge subroutine too

```
void merge(int a[], int low, int mid, int high) {
  int n = high-low+1;
  int* b = new int[n];
  int left=low, right=mid+1, bIdx=0;

  while (left <= mid && right <= high) {
    if (a[left] <= a[right])
      b[bIdx++] = a[left++];
    else
      b[bIdx++] = a[right++];
  }

  // continue on next slide
```

b is a temporary array to store result

Normal Merging
Where both halves have unmerged items

Merge Sort: Merge Implementation

```
  // continued from previous slide

  while (left <= mid) b[bIdx++] = a[left++];
  while (right <= high) b[bIdx++] = a[right++];

  for (int k = 0; k < n; k++)
    a[low+k] = b[k];

  delete [] b;
}
```

Remaining items are copied into **b[]**

Merged result are copied back into **a[]**

Remember to free allocated memory

- Question
  - Why do we need a temporary array **b[]**?

Merge Sort: Analysis

- In **`mergeSort()`**, the bulk of work is done in the **merge** step
- For **`merge(a, low, mid, high)`**
  - Let total items = k = (high – low + 1)
  - Number of comparisons ≤ k – 1
  - Number of moves from original array to temporary array = k
  - Number of moves from temporary array back to original array = k
- In total, number of operations ≤ $3k - 1 = O(k)$
- The important question is
  - How many times is **`merge()`** called?

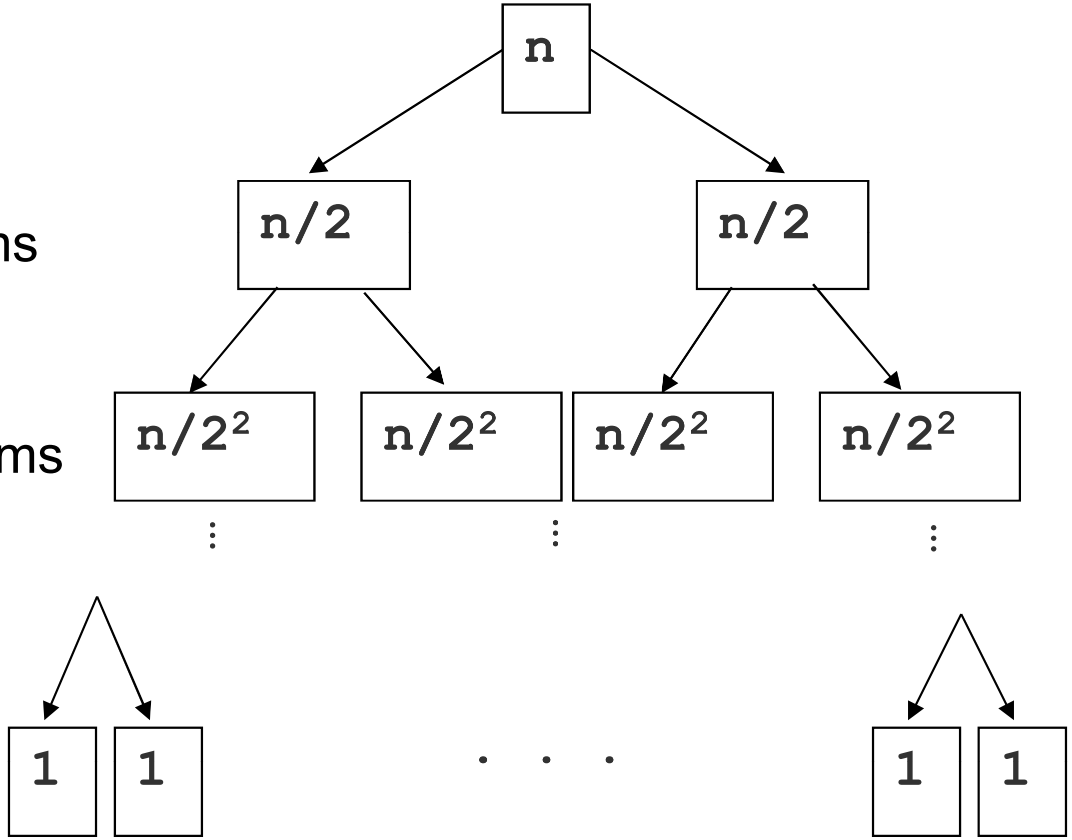
Merge Sort: Analysis

Level 0:
mergeSort `n` items

Level 1:
mergeSort `n/2` items

Level 2:
mergeSort $n/2^2$ items

Level (**lg *n***):
mergeSort **1** item



Level 0:
1 call to mergeSort

Level 1:
2 calls to mergeSort

Level 2:
$\mathbf{2^2}$ calls to mergeSort

Level (**lg *n***):
$\mathbf{2^{\lg n}(= n)}$ calls to mergeSort

$$n/(2^k) = 1 \rightarrow n = 2^k \rightarrow k = \lg n$$

Merge Sort: Analysis

- Level 0: **0** call to `merge()`
- Level 1: **1** calls to `merge()` with ***n*/2** items in each half,
  $O(1 \times 2 \times n/2) = O(n)$ time
- Level 2: **2** calls to `merge()` with ***n*/2²** items in each half,
  $O(2 \times 2 \times n/2^2) = O(n)$ time
- Level 3: **2²** calls to `merge()` with ***n*/2³** items in each half,
  $O(2^2 \times 2 \times n/2^3) = O(n)$ time
- …
- Level (lg *n*): $\mathbf{2^{\lg(n) - 1}}$ **(=** ***n*/2)** calls to `merge()` with $\mathbf{n/2^{\lg(n)}}$ **(= 1)** item in each half, $O(n)$ time
- Total time complexity = $\mathbf{O(n \lg(n))}$
- **Optimal** comparison-based sorting method

Merge Sort: Pros and Cons

- Pros
  - The performance is guaranteed, i.e. unaffected by original ordering of the input
  - Suitable for extremely large number of inputs
    - Can operate on the input portion by portion

- Cons
  - Not easy to implement
  - Requires additional storage during merging operation
    - $O(n)$ extra memory storage needed

Quick Sort

Quick Sort: Idea

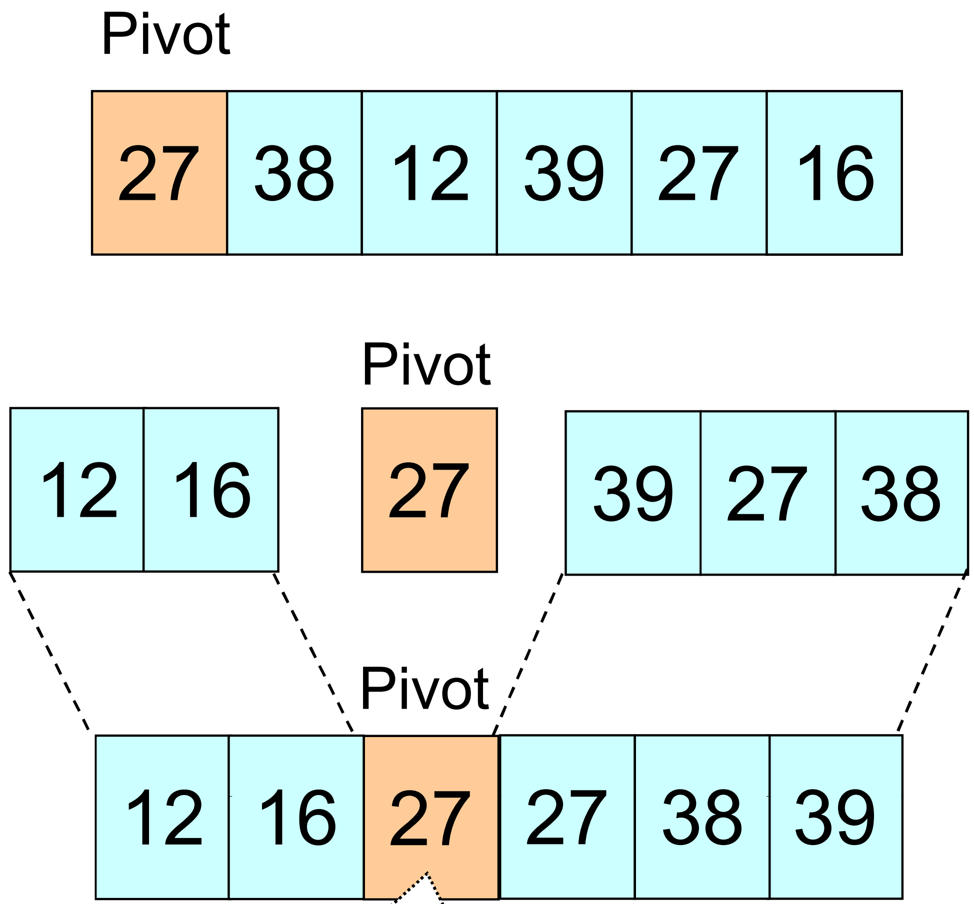
- **Quick Sort** is a divide-and-conquer algorithm
  - **Divide** step
    - Choose an item ***p*** (known as **pivot**) and partition the items of a[i...j] into two parts
      - Items that are smaller than ***p***
      - Items that are greater than or equal to ***p***
    - Recursively sort the two parts
  - **Conquer** step
    - Do nothing!
- In comparison, **Merge Sort** spends most of the time in conquer step but very little time in divide step

Quick Sort: Divide Step Example

Choose first element as pivot

Partition `a[]` about the pivot `27`

Recursively sort the two parts



Notice anything special about the position of pivot in the final sorted items?

Quick Sort: Implementation

```
void quickSort(int a[], int low, int high) {
  if (low < high) {
    int pivotIdx = partition(a, low, high);

    quickSort(a, low, pivotIdx-1);
    quickSort(a, pivotIdx+1, high);
  }
}
```

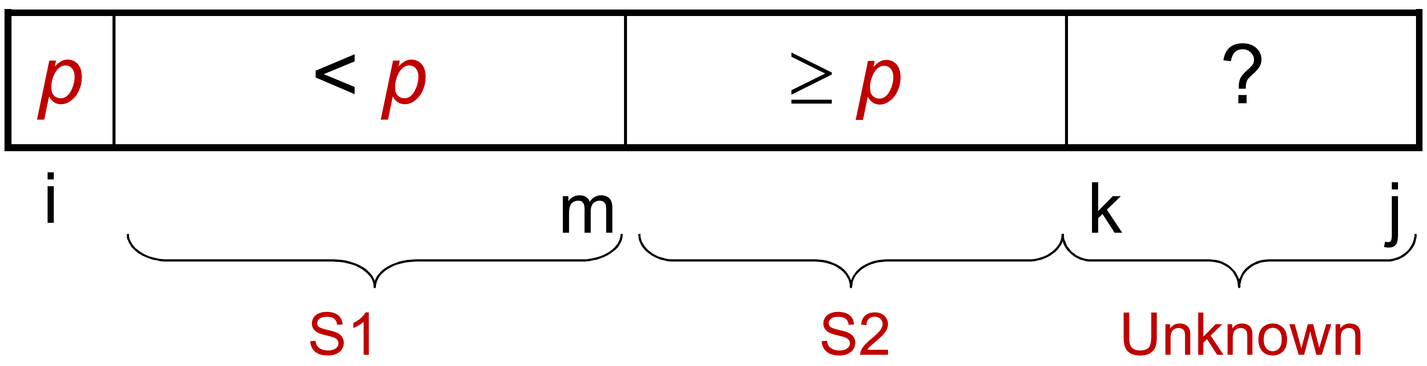
Partition **a[low...high]** and return the index of the pivot item

Recursively sort the two portions

- **partition()** splits **a[low...high]** into two portions
  - **a[low ... pivot–1]** and **a[pivot+1 ... high]**
- Pivot item does not participate in any further sorting

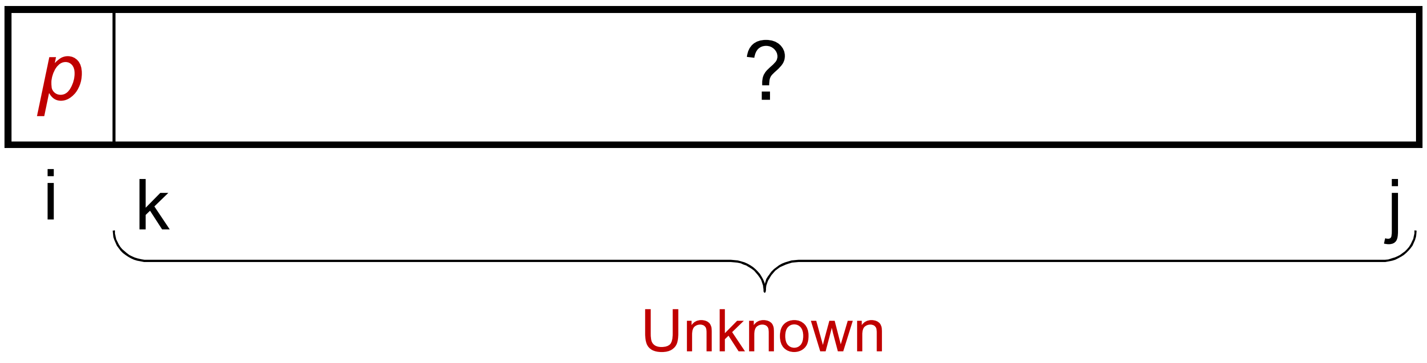
Quick Sort: Partition Algorithm

- To partition a[i...j], we choose a[i] as the pivot ***p***
  - Why choose a[i]? Are there other choices?
- The remaining items (i.e. a[i+1...j]) are divided into 3 regions
  - **S1** = a[i+1...m] where items < *p*
  - **S2** = a[m+1...k-1] where item ≥ *p*
  - **Unknown** (unprocessed) = a[k...j], where items are yet to be assigned to S1 or S2



Quick Sort: Partition Algorithm

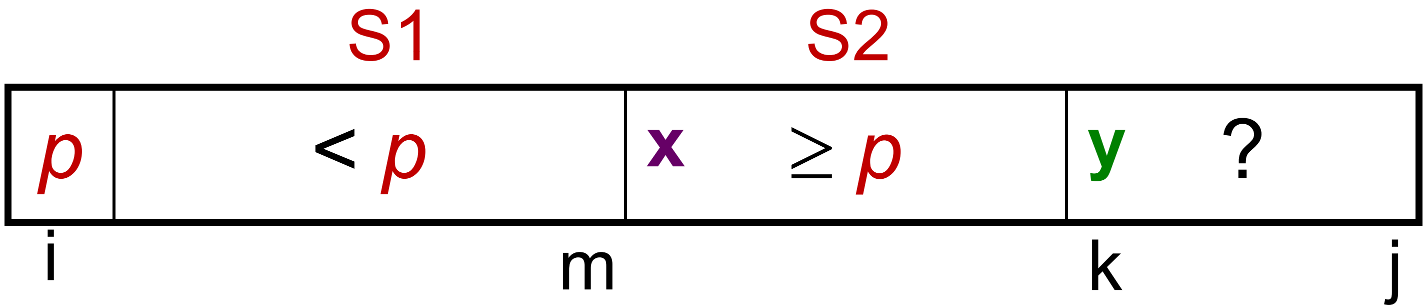
- Initially, regions S1 and S2 are empty
  - All items excluding *p* are in the unknown region
- For each item a[k] in the unknown region
  - Compare a[k] with *p*
    - If a[k] >= *p*, put it into S2
    - Otherwise, put a[k] into S1



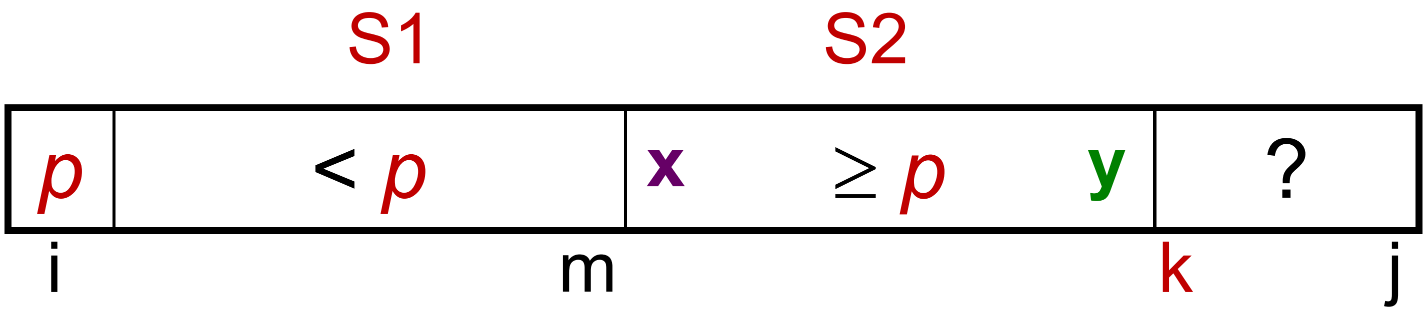
Quick Sort: Partition Algorithm

- Case 1: if a[k] >= *p*

If a[k]=y ≥ *p*,



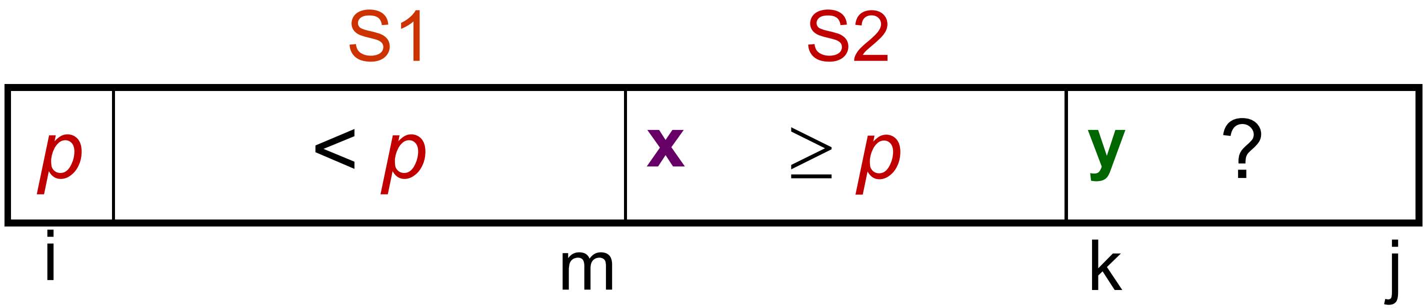
Increment k



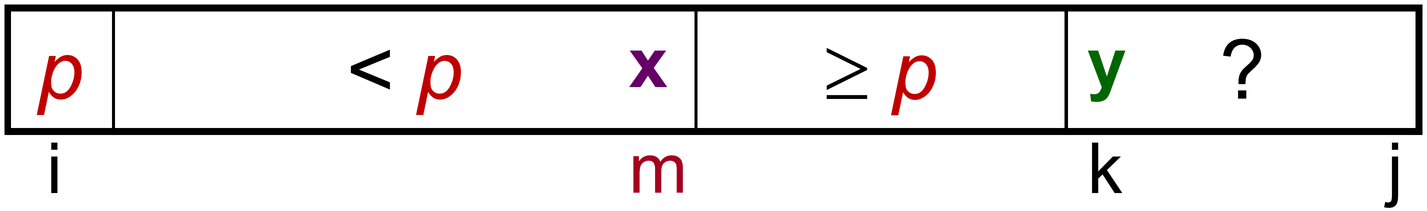
Quick Sort: Partition Algorithm

- Case 2: if a[k] < p

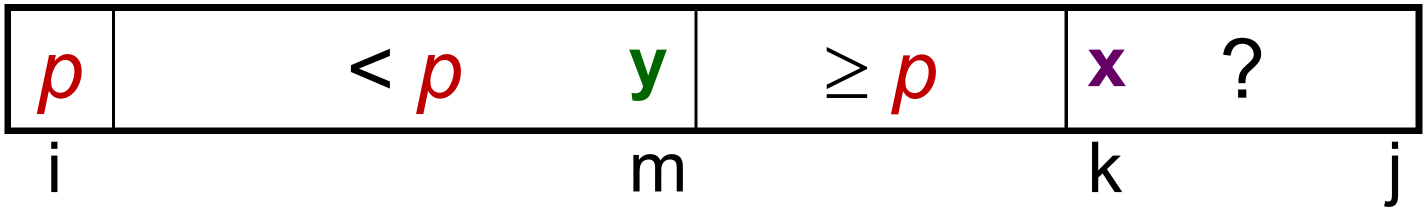
If a[k]=y < p



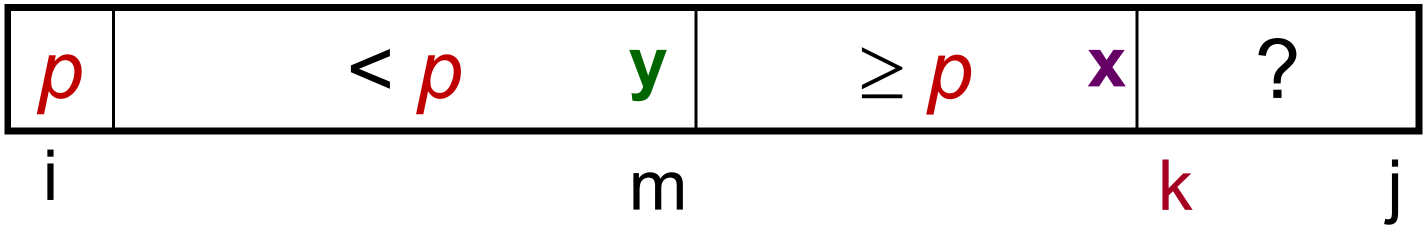
Increment m



Swap x and y



Increment k



Quick Sort: Partition Implementation

PS: C++ STL <algorithm> has partition subroutine too

```
int partition(int a[], int i, int j) {
  int p = a[i];
  int m = i;

  for (int k = i+1; k <= j; k++) {
    if (a[k] < p) {
      m++;
      swap(a[k], a[m]);
    }
    else {
    }
  }
  swap(a[i], a[m]);
  return m;
}
```

p is the pivot

S1 and **S2** empty initially

Go through each element in unknown region

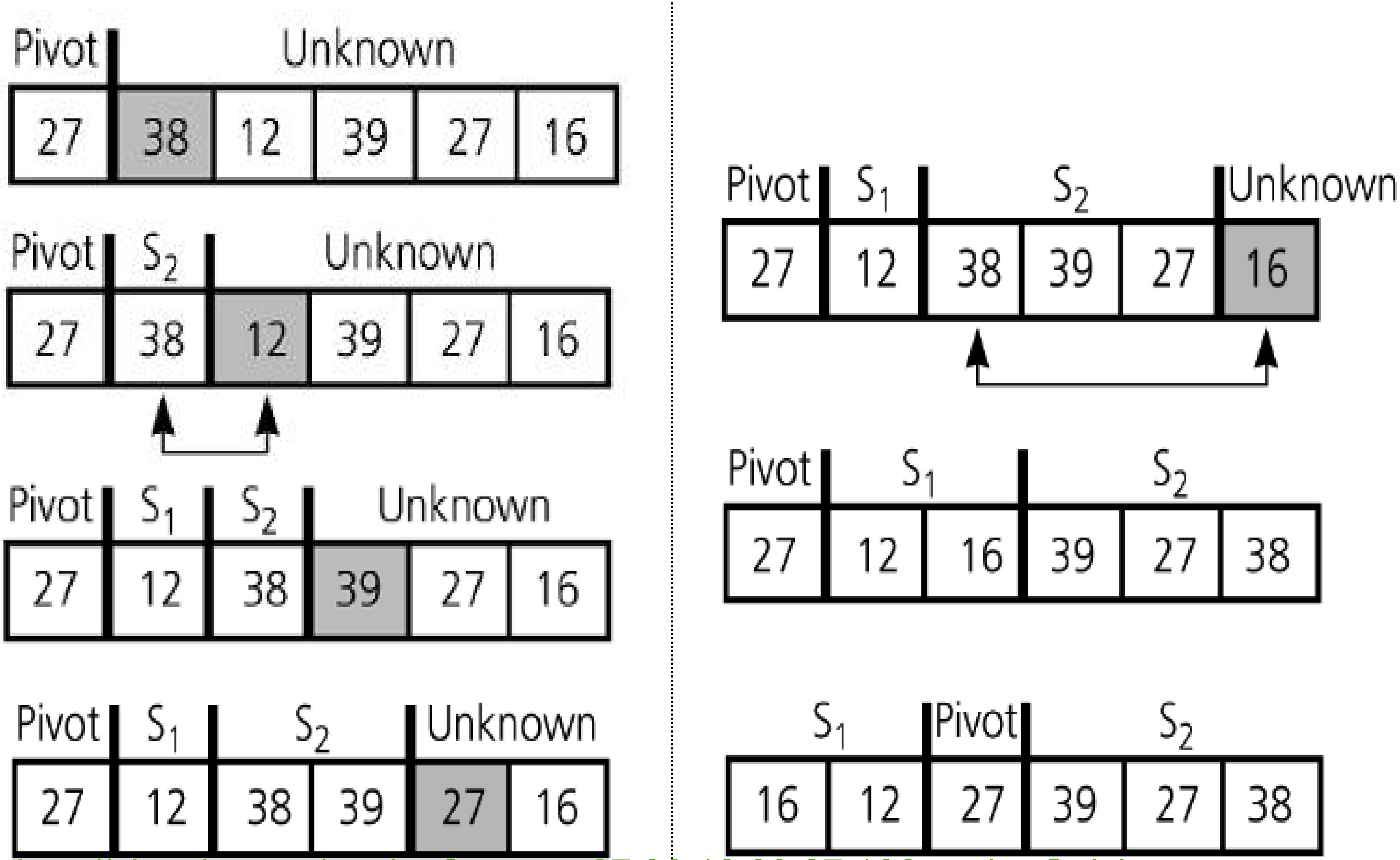
Case 2

Case 1: Do nothing!

Swap pivot with **a[m]**

m is the index of pivot

Quick Sort: Partition Example



http://visualgo.net/sorting?create=27,38,12,39,27,16&mode=Quick

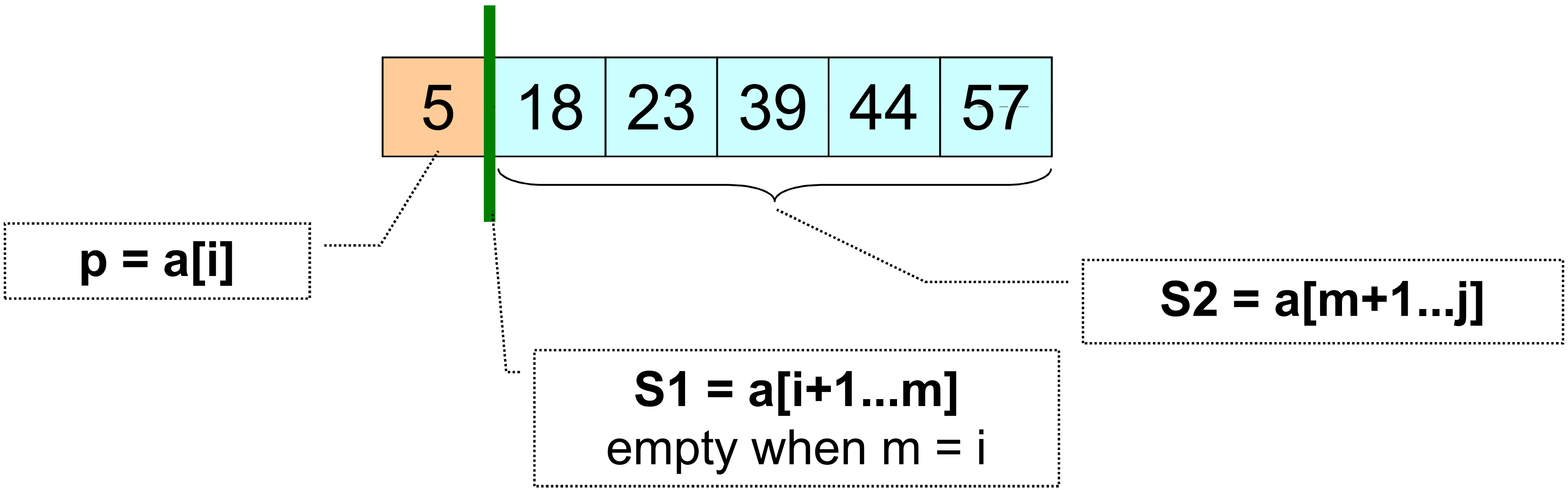
Quick Sort: Partition Analysis

- There is only a single for-loop
  - Number of iterations = number of items, *n*, in the unknown region
    - *n* = high – low
  - Complexity is O(*n*)

- Similar to **Merge Sort**, the complexity is then dependent on the number of times **partition()** is called

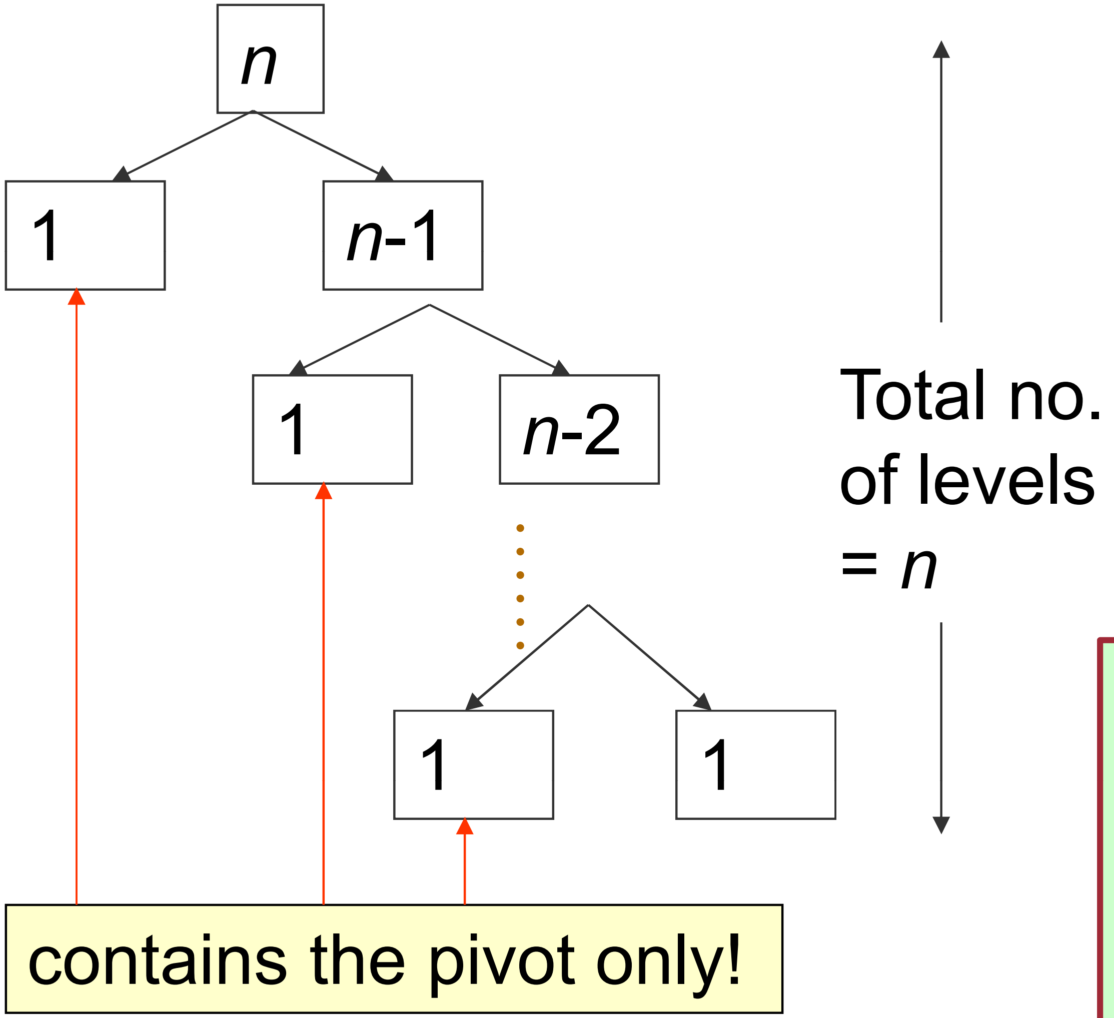
Quick Sort: Worst Case Analysis

- When the array is already in ascending order



- What is the pivot index returned by **partition()**?
  - What is the effect of swap(a, i, m)?
- S1 is empty, while S2 contains every item except the pivot

Quick Sort: Worst Case Analysis



As each partition takes linear time, the algorithm in its worst case has n levels and hence it takes time $n+(n-1)+...+1 = O(n^2)$

Quick Sort: Best/Average Case Analysis

- Best case occurs when partition always splits the array into **two equal halves**
  - Depth of recursion is log *n*
  - Each level takes *n* or fewer comparisons, so the time complexity is O(*n* log *n*)
- In practice, worst case is rare, and on the average we get some good splits and some bad ones (details in CS3230 :O)
  - Average time is also O(*n* log *n*)

Lower Bound: Comparison-Based Sort

- It is known that
  - All **comparison-based** sorting algorithms have a complexity **lower bound** of $n \log n$

- Therefore, any comparison-based sorting algorithm with **worst-case complexity** $O(n \log n)$ is **optimal**