Unit-V

Vector Integration

* **Line integral:** Any integral which is to be evaluated along a curve is called line integral. The line integral of $\bar{F}$ along the curve C is denoted by
$\int_C \bar{F}\cdot d\bar{r}$ or $\int_C \bar{F}\cdot \frac{d\bar{r}}{dt}\,dt$

* If the vector function $\bar{F}$ is replaced by a scalar function ϕ, then the corresponding line integral is denoted by $\int_C \phi\, d\bar{r}$

* If the parametric equations of the curve C are $x = x(t),\ y = y(t),\ z = z(t)$ and limits $t=a$ to $t=b$ is $\int_{t=a}^{b} \bar{F}\cdot d\bar{r}$ or $\int_{t=a}^{b} \left(\bar{F}\cdot \frac{d\bar{r}}{dt}\right)dt$

* If C is closed curve, then $\int_C$ is replaced by $\oint_C$

* The other type of line integral is $\int_C \bar{F}\times d\bar{r}$

* **Circulation:** If $\bar{V}$ represents the velocity of a fluid particle and 'c' is a closed curve then the integral $\oint_C \bar{V}\cdot d\bar{r}$ is called the circulation of $\bar{V}$ round the curve 'c'

* If $\int_C \bar{V}\cdot d\bar{r} = 0$, then the field $\bar{V}$ is called conservative. i,e no work is done and the energy is conserved.

* If the circulation of $\bar{V}$ along any closed curve in a region R vanishes, then $\bar{V}$ is said to be irrotational in region R

* **Work done by a force:** If we apply some force $\bar{F}$ on a particle, then it is moved from one place to another place is called work done on a particle. Hence the total work done by force $\bar{F}$ during the displacement from A to B is given by the line integral $\int_A^B \bar{F}\cdot d\bar{r}$

If the force $\bar{F}$ is conservative (i,e $\bar{F} = \nabla\phi$), then the work done is independent of the path and conversely. In this case $\text{curl}\,\bar{F} = \text{curl}(\overset{\nabla\phi}{\text{grad}\,\phi}) = \bar{0}$ and ϕ is called scalar potential.

Note: ① $\bar{F}$ is conservative force field, if $\nabla \times \bar{F} = \bar{0} \Leftrightarrow \exists$ a scalar point function ϕ such that $\bar{F} = \nabla\phi$

② A conservative force field is also irrotational

* If $\bar{F} = 3xy\,\bar{i} - y^2\bar{j}$ evaluate $\int_C \bar{F}\cdot d\bar{r}$, where C is the curve $y = 2x^2$ in the xy-plane

from $\underset{x_1\ y_1}{(0,0)}$ to $\underset{x_2\ y_2}{(1,2)}$

G.T $\bar{F} = 3xy\,\bar{i} - y^2\bar{j}$

$\because \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

In the xy-plane $z = 0$ so that $dz = 0$ (i.e the integration is performed in the xy-plane)

$\therefore d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + 0\,\bar{k} = dx\,\bar{i} + dy\,\bar{j}$

$\therefore \bar{F}\cdot d\bar{r} = (3xy\,\bar{i} - y^2\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j}) = 3xy\,dx - y^2 dy$ $[\because \bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = 1]$

$y = 2x^2$

$\frac{d}{dx} y = \frac{d}{dx} 2x^2$

$\frac{dy}{dx} = 2(2x) = 4x$

$dy = 4x\,dx$

$\int_C \bar{F}\cdot d\bar{r} = \int_C 3xy\,dx - y^2 dy$, where $C: y = 2x^2$

$\Rightarrow dy = 4x\,dx$ and $x: 0 \to 1$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2 4x\,dx$$

$$= \int_{x=0}^{1} 6x^3\,dx - 16x^5\,dx = 6\left(\frac{x^4}{4}\right)_{x=0}^{1} - 16\left(\frac{x^6}{6}\right)_{x=0}^{1} = \frac{6}{4}(1^4 - 0^4) - \frac{16}{6}(1^6 - 0^6)$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$

* Evaluate $\int_C \bar{f}\cdot d\bar{r}$, where $\bar{f} = (x^2+y^2)\,\bar{i} + (x^2-y^2)\,\bar{j}$ and c is the curve $y = x^2$, joining $(0,0)$ & $(1,1)$

G.T $\bar{f} = (x^2+y^2)\,\bar{i} + (x^2-y^2)\,\bar{j}$

$\because\ \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\because$ The integration is performed in xy-plane, $z=0 \Rightarrow dz = 0$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + 0\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

NOW $\bar{f}\cdot d\bar{r} = \left((x^2+y^2)\,\bar{i} + (x^2-y^2)\,\bar{j}\right)\cdot(dx\,\bar{i} + dy\,\bar{j})$

$\bar{f}\cdot d\bar{r} = (x^2+y^2)\,dx + (x^2-y^2)\,dy$

$\therefore \int_C \bar{f}\cdot d\bar{r} = \int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy$, where $C:\ y = x^2$, $dy = 2x\,dx$ and $x: 0 \to 1$

$= \int_{x=0}^{1} (x^2+(x^2)^2)\,dx + (x^2-(x^2)^2)\,2x\,dx$

$= \int_{x=0}^{1} (x^2 + x^4 + 2x^3 - 2x^5)\,dx \quad = \left(\frac{x^3}{3} + \frac{x^5}{5} + 2\frac{x^4}{4} - 2\frac{x^6}{6}\right)_{x=0}^{1}$

$= \frac{1}{3}(1^3-0^3) + \frac{1}{5}(1^5-0^5) + \frac{2}{4}(1^4-0^4) - \frac{2}{6}(1^6-0^6)$

$= 1/3 + 1/5 + 1/2 - 1/3 \quad = \frac{2+5}{10} = 7/10$

* If $\bar{F} = 3xy\,\bar{i} - y^2\bar{j}$ evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve $x = t,\ y = 2t^2$ from $t = 0$ to $t = 1$

G.T $\bar{F} = 3xy\,\bar{i} - y^2\bar{j}$

$\because \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

In the xy-plane $z = 0 \Rightarrow dz = 0$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$$

Now $\bar{F}\cdot d\bar{r} = (3xy\,\bar{i} - y^2\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j})$

$$\bar{F}\cdot d\bar{r} = 3xy\,dx - y^2 dy$$

$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C 3xy\,dx - y^2 dy$, where C is the curve $x = t,\ y = 2t^2$ and $t: 0 \to 1$

$dx = dt,\ dy = 4t\,dt$

$$= \int_{t=0}^{1} 3t(2t^2)\,dt - 4t^4(4t\,dt)$$

$$= \int_{t=0}^{1} 6t^3\,dt - 16t^5\,dt = 6\left(\frac{t^4}{4}\right)_{t=0}^{1} - 16\left(\frac{t^6}{6}\right)_{t=0}^{1}$$

$$= \frac{6}{4}(1^4 - 0^4) - \frac{16}{6}(1^6 - 0^6)$$

$$= \frac{3}{2} - \frac{8}{3} = \frac{9 - 16}{6} = -7/6$$

* Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along (i) the straight line from $(0,0,0)$ to $(2,1,3)$

(ii) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$

G.T $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

$\because \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\because$ The integration is performed in xyz-plane

Now $\bar{F} \cdot d\bar{r} = (3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}) \cdot (dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$

$\bar{F} \cdot d\bar{r} = 3x^2 dx + (2xz - y)dy + z\,dz$

$x = 2t$, $y = t$, $z = 3t$

$\overset{x}{(0},\overset{y}{0},\overset{z}{0)} \Rightarrow 0 = 2t \Rightarrow t = 0$; $0 = t \Rightarrow t = 0$; $0 = 3t \Rightarrow t = 0$ $\Rightarrow t = 0$

$\overset{x}{(2},\overset{y}{1},\overset{z}{3)} \Rightarrow 2 = 2t \Rightarrow t = 1$; $1 = t \Rightarrow t = 1$; $3 = 3t \Rightarrow t = 1$ $\Rightarrow t = 1$

(i) work done $= \int_C \bar{F} \cdot d\bar{r}$

$= \int_{C:OA} 3x^2 dx + (2xz - y)dy + z\,dz$, where C is the st line from $O\overset{x_1\ y_1\ z_1}{(0,0,0)}$ to $A\overset{x_2\ y_2\ z_2}{(2,1,3)}$

Eqn of OA is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} = t$ say

$\frac{x}{2} = t$	$\frac{y}{1} = t$	$\frac{z}{3} = t$
$x = 2t$	$y = t$	$z = 3t$
$dx = 2dt$	$dy = dt$	$dz = 3dt$

The point $(0,0,0)$ and $(2,1,3)$ correspond to $t=0$ & $t=1$ respectively

$\therefore x=2t,\ y=t,\ z=3t$

$\overset{x\ y\ z}{(0,0,0)} \Rightarrow \begin{matrix} 0=2t \\ t=0 \end{matrix} \Big| \ 0=t \ \Big| \begin{matrix} 0=3t \\ t=0 \end{matrix} \ . \Rightarrow t=0$

$\overset{x\ y\ z}{(2,1,3)} \Rightarrow \begin{matrix} 2=2t \\ t=1 \end{matrix},\ \ 1=t,\ \ \begin{matrix} 3=3t \\ t=1 \end{matrix} \Rightarrow t=1$

work done $= \int_{t=0}^{1} 3(2t)^2 2dt + (2(2t)(3t)-t)dt + 3t(3dt)$

$= \int_{t=0}^{1} 24t^2 dt + (12t^2-t)dt + 9tdt$

$= \int_{t=0}^{1} (24t^2+12t^2-t+9t)dt = \int_{t=0}^{1} (36t^2+8t)dt = \left(36\left(\frac{t^3}{3}\right) + 8\left(\frac{t^2}{2}\right)\right)_{t=0}^{1}$

$= 12(1^3-0^3) + 4(1^2-0^2)$

$= 12+4 = 16$

(ii) work done $= \int_C \bar{F}\cdot d\bar{r}$

$= \int_C 3x^2 dx + (2xz-y)dy + zdz$, where C is the curve $x^2=4y$ & $3x^3=8z$ & $x: 0\to 2$

$y = x^2/4 \qquad z = \frac{3x^3}{8}$

$dy = \frac{2x}{4}dx$ & $dz = \frac{3}{8}3x^2dx$

$dy = \frac{x}{2}dx$ & $dz = \frac{9}{8}x^2 dx$ & $x: 0 \to 2$ $\left[\because y = \frac{x^2}{4},\ z = \frac{3}{8}x^3\right]$

work done $= \int_{x=0}^{2} 3x^2 dx + \left(2x\left(\frac{3}{8}x^3\right) - \frac{x^2}{4}\right)\frac{x\,dx}{2} + \frac{3}{8}x^3 \frac{9}{8}x^2 dx$

$= \int_{x=0}^{2}\left(3x^2 + \left(\frac{3}{8}x^5 - \frac{x^3}{8}\right) + \frac{27}{64}x^5\right)dx$

$= 3\left(\frac{x^3}{3}\right)_{x=0}^{2} + \frac{3}{8}\left(\frac{x^6}{6}\right)_{x=0}^{2} - \frac{1}{8}\left(\frac{x^4}{4}\right)_{x=0}^{2} + \frac{27}{64}\left(\frac{x^6}{6}\right)_{x=0}^{2}$

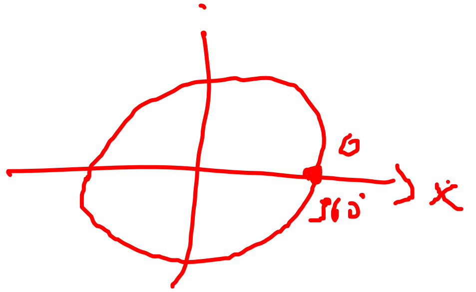
$= (8-0) + \frac{1}{16}(64-0) - \frac{1}{32}(16-0) + \frac{9}{128}(64-0)$

$= 8 + 4 - \frac{1}{2} + \frac{9}{2}$

$= 12 + 4$

work done $= 16$

Find the work done by $\bar{F} = (2x - y - z)\bar{i} + (x+y-z)\bar{j} + (3x - 2y - 5z)\bar{k}$ along a curve C in the xy-plane given by $x^2+y^2=9$, $z=0$



G.T $\bar{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$

$\because \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\because$ The integration is performed in xy-plane i.e $z=0 \Rightarrow dz = 0$

Now $\bar{F}\cdot d\bar{r} = \left((2x-y-0)\bar{i} + (x+y-0)\bar{j} + (3x-2y-5(0))\bar{k}\right)\cdot(dx\,\bar{i} + dy\,\bar{j} + 0\,\bar{k})$

$= (2x-y)\,dx + (x+y)\,dy$

$(x-h)^2 + (y-k)^2 = r^2$ which is eqn of the circle with centre (h,k) and radius r

work done $= \int_C \bar{F}\cdot d\bar{r}$

$= \int_C (2x-y)\,dx + (x+y)\,dy$, where C is the circle $x^2+y^2 = 9 = 3^2$ (centre $(0,0)$, $r=3$)

Take $x = r\cos\theta = 3\cos\theta$, $y = r\sin\theta = 3\sin\theta$

$dx = 3(-\sin\theta)\,d\theta$ & $dy = 3\cos\theta\,d\theta$ and θ; $0 \to 2\pi$

$= \int_{\theta=0}^{2\pi} (2(3\cos\theta) - 3\sin\theta)(-3\sin\theta\,d\theta) + (3\cos\theta + 3\sin\theta)\,3\cos\theta\,d\theta$

$= \int_{\theta=0}^{2\pi} (-18\cos\theta\sin\theta + 9\sin^2\theta)\,d\theta + (9\cos^2\theta + 9\sin\theta\cos\theta)\,d\theta$

$$\text{work done} = \int_{\theta=0}^{2\pi} \left(-18\cos\theta\sin\theta + \underline{9\sin^2\theta + 9\cos^2\theta} + 9\sin\theta\cos\theta\right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(9(\cos^2\theta + \sin^2\theta) - 9\cos\theta\sin\theta\right) d\theta \qquad \left[\because \sin^2\theta + \cos^2\theta = 1\right]$$

$$= \int_{\theta=0}^{2\pi} \left(9(1) - \frac{9}{2}(2\sin\theta\cos\theta)\right) d\theta \qquad \left[\because \sin 2\theta = 2\sin\theta\cos\theta\right]$$

$$= 9(\theta)_0^{2\pi} - \frac{9}{2}\int_{\theta=0}^{2\pi} \sin 2\theta \, d\theta$$

$$= 9(2\pi - 0) - \frac{9}{2}\left(-\frac{\cos 2\theta}{2}\right)_{\theta=0}^{2\pi} = 18\pi + \frac{9}{4}\left(\cos 4\pi - \cos 2(0)\right) \left[\because \cos n\pi = (-1)^n, n \in Z\right]$$

$$= 18\pi + \frac{9}{4}(1 - 1)$$

$$= 18\pi + \frac{0}{4} = 0$$

$$= 18\pi$$

H.W

* If $\vec{F} = xy\,\vec{i} - z\,\vec{j} + x^2\vec{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$.

Evaluate $\int_C \vec{F} \cdot d\vec{r}$

Ans $\int_C \vec{F} \cdot d\vec{r} = \frac{51}{70}$

If $\varphi = x^2 y z^3$, evaluate $\int_C \varphi \, d\bar{r}$ along the curve $x = t$, $y = 2t$, $z = 3t$ from $t = 0$ to $t = 1$

G.T $\varphi = x^2 y z^3$

$\because \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\int_C \varphi \, d\bar{r} = \int_C (x^2 y z^3)(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$, where C is the curve $x = t$, $y = 2t$, $z = 3t$

$dx = dt$, $dy = 2dt$ & $dz = 3dt$ & $t: 0 \to 1$

$= \int_{t=0}^{1} (t^2 (2t)\, 27t^3)(dt\,\bar{i} + 2dt\,\bar{j} + 3dt\,\bar{k})$

$= \int_{t=0}^{1} (54\, t^6\, \bar{i} + 108 t^6\, \bar{j} + 162 t^6\, \bar{k})\, dt$

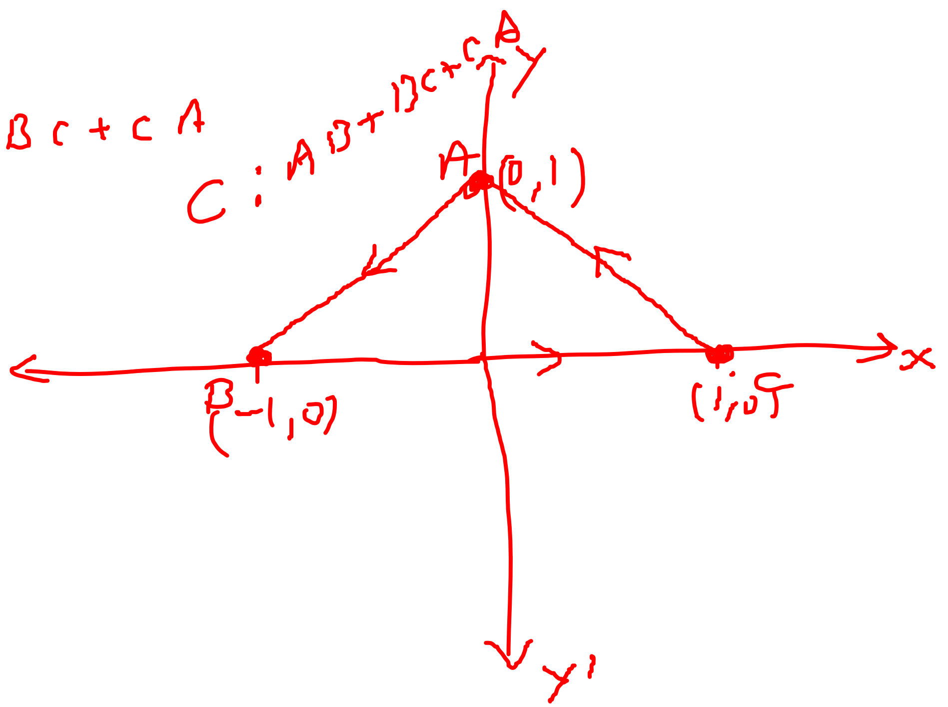
$= 54 \left(\frac{t^7}{7}\right)_{t=0}^{1} \bar{i} + 108\,\bar{j} \left(\frac{t^7}{7}\right)_{t=0}^{1} + 162\,\bar{k} \left(\frac{t^7}{7}\right)_{t=0}^{1}$

$= \frac{54}{7}\bar{i}(1^7 - 0^7) + \frac{108}{7}\bar{j}(1^7 - 0^7) + \frac{162}{7}\bar{k}(1^7 - 0^7)$

$\int_C \varphi \, d\bar{r} = \frac{54}{7}[\bar{i} + 2\bar{j} + 3\bar{k})$

* compute the line integral $\int y^2dx - x^2dy$ round the triangle whose vertices are (1,0) (0,1) (-1,0) in the xy-plane.

$C : AB + BC + CA$



let $A = (0,1)$, $B(-1,0)$, $C = (1,0)$

$$\int_C y^2dx - x^2dy = \int_{AB} y^2dx - x^2dy + \int_{BC} y^2dx - x^2dy + \int_{CA} y^2dx - x^2dy \quad (1)$$

Along AB: equation of AB is $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$(0,1)$ → $x_1\ y_1$, $(-1,0)$ → $x_2\ y_2$

$$y - 1 = \frac{0-1}{-1-0}(x - 0)$$

$$y - 1 = x$$

$$y = x + 1$$

$$\frac{dy}{dx} = 1 + 0 \Rightarrow dy = 1\,dx \quad \& \quad x : 0 \text{ to } -1$$

$$y = x + 1$$
$$\frac{dy}{dx} = \frac{d}{dx}(x+1) = 1 + 0 = 1$$

$$\int_{AB} y^2dx - x^2dy = \int_{x=0}^{-1} (x+1)^2\,dx - x^2\,dx$$

$$= \left.\frac{(x+1)^3}{3(1)}\right)_{x=0}^{-1} - \left(\frac{x^3}{3}\right)_{x=0}^{-1}$$

$$= \frac{1}{3}(0 - 1) - \frac{1}{3}\left((-1)^3 - 0^3\right) = -1/3 + 1/3 = 0$$

$$\left[\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(a)(n+1)} + c\right]$$

Along the curve BC is $y=0$ so that $dy=0$ and $x: -1$ to 1

$(-1,0)$ $(1,0)$

$$\int_{BC} y^2dx - x^2dy = \int_{x=-1}^{1} 0 - x^2(0) = \int_{-1}^{1} 0 = 0$$

$(-1,1)$ $(1,1)$ $(1,2)$ $(1,3)$

$y=1$ $x=1$

Along the curve CA is

$(1,0)$ $(0,1)$

$x_1\ y_1$ $x_2\ y_2$

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0 = \frac{1-0}{0-1}(x-1)$$

$$y = -(x-1) = 1-x$$

$$y = 1-x$$

$$\frac{dy}{dx} = 0-1 \Rightarrow dy = -dx \quad \& \quad x: 1 \to 0$$

$$\int_0^1 0\,dx = 0$$

$$\int 0\,dx = \text{Constant}$$

$$\int_{CA} y^2dx - x^2dy = \int_{x=1}^{0} (1-x)^2dx - x^2(-dx)$$

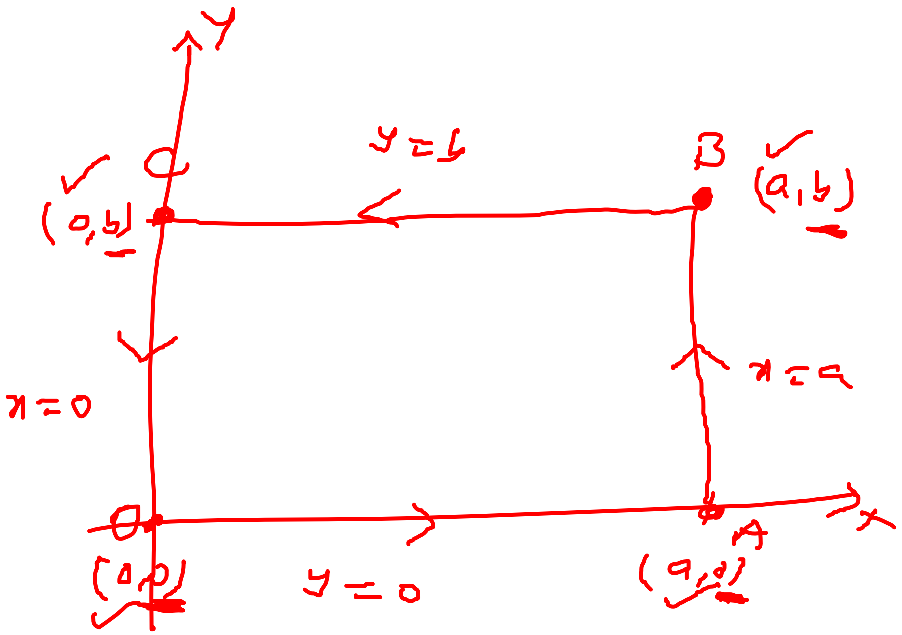
$$= \frac{(1-x)^3}{3(-1)}\Big)_1^0 + \left(\frac{x^3}{3}\right)_1^0$$

$$= \frac{-1}{3}(1-0) + \frac{1}{3}(0-1) = -1/3 - 1/3 = -2/3$$

$$\int_C y^2 dx - x^2 dy = 0 + 0 - 2/3 = -2/3$$

* If $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ evaluate $\oint \bar{F}\cdot d\bar{r}$ where curve c is the rectangle in xy-plane bounded by $y=0$, $y=b$, $x=0$, $x=a$

$C : OA + AB + BC + CO$



$\because$ The integration is performed in xy-plane ie $z=0 \Rightarrow dz=0$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$ $\quad [\because dz=0]$

$d\bar{r} = dx\,\bar{i} + dy\bar{j}$

Now $\bar{F}\cdot d\bar{r} = \left((x^2+y^2)\bar{i} - 2xy\bar{j}\right)\cdot\left(dx\bar{i} + dy\bar{j}\right) = (x^2+y^2)\,dx - 2xy\,dy$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r} \quad \text{--- (1)}$$

Along OA : Equ of OA is $y=0 \Rightarrow dy = 0$ and $x : 0 \to a$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{OA} (x^2+y^2)dx - 2xy\,dy$$

$$= \int_{x=0}^{a} (x^2+0)\,dx - 2x(0)(0) = \int_{x=0}^{a} x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{1}{3}(a^3-0) = a^3/3$$

Along AB : Eqn of AB is $x = a \Rightarrow dx = 0$ & $y : 0 \to b$

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_{AB} (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{y=0}^{b} (a^2+y^2)(0) - 2ay\,dy = \int_{y=0}^{b} -2ay\,dy = -2a\left(\frac{y^2}{2}\right)_{y=0}^{b} = -a\,(b^2-0) = -ab^2$$

Along BC : eqn of BC is $y = b \Rightarrow dy = 0$ & $x : a$ to 0

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{BC} (x^2+y^2)\,dx - 2xy\,dy = \int_{x=a}^{0} (x^2+b^2)\,dx - 0 = \left(\frac{x^3}{3} + b^2 x\right)_{x=a}^{0}$$

$$= \frac{1}{3}(0-a^3) + b^2(0-a)$$

$$= -\frac{a^3}{3} - ab^2$$

Along CO: Eqn of CO is $x = 0 \Rightarrow dx = 0$ & $y : b$ to 0

$$\int_{CO} \bar{F} \cdot d\bar{r} = \int_{CO} (x^2+y^2)\,dx - 2xy\,dy = \int_{y=b}^{0} 0 - 0 = \int_{y=b}^{0} 0 = 0$$

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \frac{a^3}{3} + (-ab^2) - \frac{a^3}{3} - ab^2 + 0 = -2ab^2$$

Evaluate $\int \bar{F}\cdot d\bar{r}$, $\bar{F} = (x-3y)\bar{i} + (y-2x)\bar{j}$ and C is the closed curve in xy-plane $x = 2\cos t$ and $y = 3\sin t$ and $t=0$ to $t = 2\pi$

G.T $\bar{F} = (x-3y)\bar{i} + (y-2x)\bar{j}$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\because$ The integration is performed in xy-plane (i.e $z=0$, $dz=0$)

Now $d\bar{r} = dx\bar{i} + dy\bar{j}$

$\therefore \bar{F}\cdot d\bar{r} = \left((x-3y)\bar{i} + (y-2x)\bar{j}\right)\cdot\left(dx\bar{i} + dy\bar{j}\right)$

$\bar{F}\cdot d\bar{r} = (x-3y)\,dx + (y-2x)\,dy$

$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C (x-3y)\,dx + (y-2x)\,dy$, where C is the closed curve $x = 2\cos t$, $y = 3\sin t$

$dx = 2(-\sin t\,dt)$, $dy = 3\cos t\,dt$ and

$t : 0 \rightarrow 2\pi$

$$= \int_{t=0}^{2\pi} (2\cos t - 3(3\sin t))(-2\sin t\,dt) + (3\sin t - 2(2\cos t))\,3\cos t\,dt$$

$$= \int_{t=0}^{2\pi} (-4\sin t\cos t + 18\sin^2 t + 9\sin t\cos t - 12\cos^2 t)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{2\pi} (18\sin^2 t - 12\cos^2 t + 5\sin t\cos t)\,dt$$

$\because$ $\cos 2t = 2\cos^2 t - 1 = 1 - 2\sin^2 t = \cos^2 t - \sin^2 \theta$

$2\cos^2 t = 1 + \cos 2t$ | $\cos 2t = 1 - 2\sin^2 t$

$\cos^2 t = \dfrac{1 + \cos 2t}{2}$ | $\sin^2 t = \dfrac{1 - \cos 2t}{2}$

& $\sin 2t = 2\sin t\cos t$

$$= \int_{t=0}^{2\pi} \left(18\left(\frac{1-\cos 2t}{2}\right) - 12\left(\frac{1+\cos 2t}{2}\right) + \frac{5}{2}\,2\sin t\cos t\right)dt$$

$$= \int_{t=0}^{2\pi} \left(9(1-\cos 2t) - 6(1+\cos 2t) + \frac{5}{2}\sin 2t\right)dt$$

$$= \int_{t=0}^{2\pi} \left(3 - 9\cos 2t - 6\cos 2t + \frac{5}{2}\sin 2t\right)dt$$

$$= \int_{t=0}^{2\pi} \left(3 - 15\cos 2t + \frac{5}{2}\sin 2t\right)dt$$

$$= 3\left(t\right)_{t=0}^{2\pi} - 15\left(\frac{\sin 2t}{2}\right)_{t=0}^{2\pi} + \frac{5}{2}\left(\frac{-\cos 2t}{2}\right)_{t=0}^{2\pi}$$

$$= 3(2\pi - 0) - \frac{15}{2}(\sin 4\pi - \sin 2(0)) - \frac{5}{4}(\cos(4\pi) - \cos 2(0))$$

$\because$ $\sin n\pi = 0,\ \forall n \in \mathbb{Z}$
i.e. $n = 0, \pm 1, \pm 2 \ldots$

$\cos n\pi = (-1)^n,\ n \in \mathbb{Z}$

$$= 6\pi - \frac{15}{2}(0 - 0) - \frac{5}{4}(1 - 1)$$

$$= 6\pi - 0 - 0$$

$$= 6\pi$$

* Find the work done in moving a particle in the force field (a) $\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$

How (b) $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

(b) ~~Ans~~ 16

(a) G.T $\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$

$\because \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\because$ The integration is performed in xyz-plane

Now $\bar{F} \cdot d\bar{r} = (3x^2\bar{i} + \bar{j} + z\bar{k}) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k}) = 3x^2 dx + dy + z\,dz$

work done $= \int_C \bar{F} \cdot d\bar{r}$, where C is the St line from $\overset{x_1, y_1, z_1}{(0,0,0)}$ to $\overset{x_2, y_2, z_2}{(2,1,3)}$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} \Rightarrow \frac{x}{2} = y = \frac{z}{3} = t \text{ say}$$

$x = 2t$, $y = t$ & $z = 3t$

$dx = 2dt$, $dy = dt$ & $dz = 3dt$

and t : 0 to 1

$\because \overset{x\ y\ z}{(0,0,0)} \Rightarrow \begin{matrix} 0 = 2t \\ t = 0 \end{matrix} \Big| \ 0 = t \ \Big|, \ \begin{matrix} 0 = 3t \\ t = 0 \end{matrix} \Rightarrow t = 0$

$(2,1,3) \Rightarrow \begin{matrix} 2 = 2t \\ t = 1 \end{matrix} \Big| \ 1 = t \ \Big| \ \begin{matrix} 3 = 3t \\ t = 1 \end{matrix} \Rightarrow t = 1$

$$= \int_{t=0}^{1} 3(2t)^2 2dt + dt + 3t(3dt)$$

$$= \int_{t=0}^{1} (24t^2 + 1 + 9t)\,dt$$

$$= 24\left(\frac{t^3}{3}\right)_{t=0}^{1} + (t)_{t=0}^{1} + 9\left(\frac{t^2}{2}\right)_{t=0}^{1}$$

$$= 8(1-0) + (1-0) + \frac{9}{2}(1-0)$$

work done $= 8 + 1 + \frac{9}{2} = 9 + \frac{9}{2} = \frac{27}{2}$

4 If $\bar{F} = 2y\bar{i} - z\bar{j} + x\bar{k}$, evaluate $\int_C \bar{F} \times d\bar{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$

G.T $\bar{F} = 2y\bar{i} - z\bar{j} + x\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$$\bar{F} \times d\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} \to R_1$$

$$\bar{F} \times d\bar{r} = \bar{i}(-z\,dz - x\,dy) - \bar{j}(2y\,dz - x\,dx) + \bar{k}(2y\,dy + z\,dx)$$

$$\therefore \int_C \bar{F} \times d\bar{r} = \int_C (-z\,dz - x\,dy)\,\bar{i} - \bar{j}(2y\,dz - x\,dx) + \bar{k}(2y\,dy + z\,dx)$$

where C is the curve

$x = \cos t$	$y = \sin t$	$z = 2\cos t$
$dx = -\sin t\,dt$	$dy = \cos t\,dt$	$dz = -2\sin t\,dt$

and $t : 0 \to \pi/2$

$$\int_C \bar{F} \times d\bar{r} = \int_{t=0}^{\pi/2} \left(-2\cos t(-2\sin t)dt - \cos t(\cos t)dt\right)\bar{i} - \bar{j}\left(2\sin t(-2\sin t\,dt) - \cos t(-\sin t\,dt)\right) + \bar{k}\left(2\sin t(\cos t\,dt) + 2\cos t(-\sin t\,dt)\right)$$

$$= \int_{t=0}^{\pi/2} (4\sin t\cos t\,dt - \cos^2 t\,dt)\,\bar{i} - \bar{j}\,(-4\sin^2 t\,dt + \sin t\cos t\,dt) + \bar{k}\,(0)$$

$\because \cos 2t = 2\cos^2 t - 1$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$\sin 2t = 2\sin t\cos t$

$$= \bar{i}\int_{t=0}^{\pi/2} \left(2\sin 2t\,dt\right) - \left(\left(\frac{1+\cos 2t}{2}\right)\right) dt - \bar{j}\left(-4\left(\frac{1-\cos 2t}{2}\right) + \frac{1}{2}\sin 2t\right) dt$$

$$= \bar{i}\left(2\left(\frac{-\cos 2t}{2}\right) - \frac{t}{2} - \frac{1}{2}\left(\frac{\sin 2t}{2}\right)\right)_{t=0}^{\pi/2} - \bar{j}\left(-2\left(t - \frac{\sin 2t}{2}\right) + \frac{1}{2}\left(\frac{-\cos 2t}{2}\right)\right)_{t=0}^{\pi/2}$$

$$= \bar{i}\left(-\left(\left(\cos 2\left(\tfrac{\pi}{2}\right) - \cos 0\right) - \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{1}{4}\left(\sin 2(\pi/2) - \sin 0\right)\right) - \bar{j}\left(-2\left(\left(\pi/2 - 0\right) - \frac{1}{2}\left(\sin 2\left(\tfrac{\pi}{2}\right) - \sin 0\right) - \frac{1}{4}\left(\cos 2\left(\tfrac{\pi}{2}\right) - \cos 0\right)\right)\right)$$

$$= \bar{i}\left(-(-1-1) - \frac{\pi}{4} - \frac{1}{4}(0-0)\right) - \bar{j}\left(-\pi + (0-0) - \frac{1}{4}(-1-1)\right)$$

$\because \cos n\pi = (-1)^n, n \in \mathbb{Z}$

$\sin n\pi = 0, \forall n \in \mathbb{Z}$

$$= \bar{i}\,(2 - \pi/4) - \bar{j}\left(-\pi + \frac{2}{4}\right)$$

$$= \bar{i}\,(2 - \pi/4) - \bar{j}\left(-\pi + \frac{1}{2}\right)$$

$$\int_C \bar{F} \times d\bar{r} = \bar{i}\,(2 - \pi/4) + \bar{j}\,(\pi - 1/2)$$

* P.T force field given by $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$ is conservative. Find the work done by moving particle from $(1,-1,2)$ to $(3,2,-1)$ in this force field.

G.T $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$

If $\bar{F}$ is conservative force field vector, then curl $\bar{F} = \nabla\times\bar{F} = \bar{0} \iff \exists$ a scalar potential function ϕ such that $\bar{F} = \nabla\phi$

curl $\bar{F} = \nabla\times\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \left(2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}\right)$

$$= \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix} \end{array} \rightarrow R_1$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(3x^2yz^2) - \frac{\partial}{\partial z}(x^2z^3)\right) - \bar{j}\left(\frac{\partial}{\partial x}(3x^2yz^2) - \frac{\partial}{\partial z}(2xyz^3)\right) + \bar{k}\left(\frac{\partial}{\partial x}(x^2z^3) - \frac{\partial}{\partial y}(2xyz^3)\right)$$

$$= \bar{i}(3x^2z^2 - 3x^2z^2) - \bar{j}(6xyz^2 - 6xyz^2) + \bar{k}(2xz^3 - 2xz^3)$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k}(0)$$

curl $\bar{F} = \nabla\times\bar{F} = \bar{0} \Rightarrow \bar{F}$ is irrotational vector

Hence $\bar{F}$ conservative force field $\Rightarrow \exists$ a scalar potential function $\phi \ni \bar{F} = \nabla\phi$

$$\bar{F} = \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$$

$$2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$$

$$\frac{\partial\varphi}{\partial x} = 2xyz^3, \quad \frac{\partial\varphi}{\partial y} = x^2z^3, \quad \frac{\partial\varphi}{\partial z} = 3x^2yz^2$$

w.k.T $\quad d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$

$$= \underline{2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz} \quad [\because d(uvw) = uv\,dw + uw\,dv + vw\,du]$$

$$d\varphi = d(x^2yz^3)$$

Integrating on both sides, we get

$$\varphi = x^2yz^3 + C$$

work done $= \int_C \bar{F}\cdot d\bar{r} = \int_C (2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$, where C is the curve joining $(1,-1,2)$ to $(3,2,-1)$

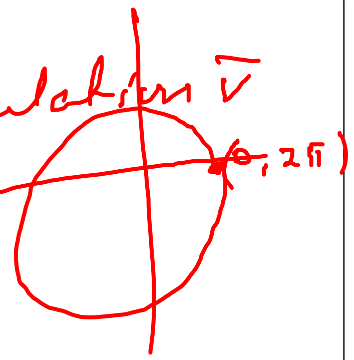
$$= \int_C \underline{2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz} = \int_{(1,-1,2)}^{(3,2,-1)} d(x^2yz^3) = (x^2yz^3)_{(1,-1,2)}^{(3,2,-1)}$$

$$= 3^2(2)(-1)^3 - 1^2(-1)(2^3)$$

$$= -18 + 8 = -10$$

* Find the circulation of $\bar F = (2x - y + 2z)\bar i + (x + y - z)\bar j + (3x - 2y - 5z)\bar k$ along the circle $x^2 + y^2 = 4$ in the xy-plane.

$\int_C \bar V \cdot d\bar r$ is called circulation $\bar V$ around the curve C



G.T $\bar F = (2x - y + 2z)\bar i + (x + y - z)\bar j + (3x - 2y - 5z)\bar k$

let $\bar r = x\bar i + y\bar j + z\bar k \Rightarrow d\bar r = dx\bar i + dy\bar j + dz\bar k$

∴ The integration is performed in xy-plane (i.e. $z = 0$, $dz = 0$)

Now $\bar F \cdot d\bar r = \left((2x - y + 2(0))\bar i + (x + y - 0)\bar j + (3x - 2y - 5(0))\bar k\right) \cdot (dx\,\bar i + dy\,\bar j + 0\,\bar k)$

$\bar F \cdot d\bar r = (2x - y)\,dx + (x + y)\,dy$

circulation $= \int_C \bar F \cdot d\bar r$, where $C: x^2 + y^2 = 2^2$ which is the eqn of circle with centre $(0,0)$ & radius $r = 2$

$x = r\cos\theta = 2\cos\theta$, $y = r\sin\theta = 2\sin\theta$

$dx = 2(-\sin\theta)\,d\theta$, $dy = 2\cos\theta\,d\theta$ & $\theta : 0 \to 2\pi$

$$= \int_{\theta=0}^{2\pi} \left(2(2\cos\theta) - 2\sin\theta\right)(-2\sin\theta\,d\theta) + (2\cos\theta + 2\sin\theta)\,2\cos\theta\,d\theta.$$

$$= \int_{\theta=0}^{2\pi} \left(-8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta\right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(4(\sin^2\theta + \cos^2\theta) - 4\sin\theta\cos\theta\right) d\theta$$

[∵ $\sin^2\theta + \cos^2\theta = 1$
$2\sin\theta\cos\theta = \sin 2\theta$]

$= \int_0^{2\pi} (4(1) - 2\sin 2\theta)\, d\theta$

circulation $= 4(\theta)_0^{2\pi} - \cancel{2}\left(\frac{-\cos 2\theta}{\cancel{2}}\right)_0^{2\pi}$

$= 4(2\pi - 0) + (\cos 2(2\pi) - \cos 2(0))$

$= 8\pi + (\cos 4\pi - \cos 0)$ $\quad [\because \cos n\pi = (-1)^n, n \in Z = 0, \pm 1, \pm 2 \ldots]$

$= 8\pi + (1 - 1)$

$= 8\pi$

* If $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$, P.T $\int_C \bar{F}\cdot d\bar{r}$ (i.e work done) is independent of the curve joining two points

G.T $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$

W.K.T F is conservative force field, if $\text{curl}\,\bar{F} = \nabla\times\bar{F} = \bar{0} \Rightarrow \exists$ a scalar point function ϕ such that $\bar{F} = \nabla\phi$ is independent of the path joining two points. and conversely

i.e we have to P.T $F = \nabla\phi$ is independent of the path joining two points $\Rightarrow$

$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \bar{0}$

$$\operatorname{curl}\bar{F} = \nabla\times\bar{F} = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy-3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix} \end{array} \to R_1$$

$$\left[\because \bar{F} = \nabla\varphi\right.$$

$$(4xy-3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k} = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2)\right) - \bar{j}\left(\frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy-3x^2z^2)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy-3x^2z^2)\right)$$

$$= \bar{i}(0-0) - \bar{j}\left(-6x^2z - (-6x^2z)\right) + \bar{k}\left(4x - (4x)\right)$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k}(0) = \bar{0}$$

$$\operatorname{curl}\bar{F} = \nabla\times\bar{F} = \bar{0} \to \int_C \bar{F}\cdot d\bar{r}$$

$\therefore$ The workdone is independent of the path joining two points.

* **Surface integral**: Any integral which is to be evaluated along the surface is called surface integral.

* let $\bar{F}$ be a continuous vector point function and S is any surface and let $\bar{n}$ be the unit normal vector to the surface S, then the surface integral is defined as $\int_S \bar{F}\cdot d\bar{s}$ (or) $\int_S \bar{F}\cdot\bar{n}\, ds$, where $d\bar{s} = \bar{n}\, ds$ &

$\downarrow$ $\qquad$ $\downarrow$

$\iint_S \bar{F}\cdot d\bar{s}$ (or) $\iint_S \bar{F}\cdot\bar{n}\, ds$

$\bar{n}$ is a unit outward drawn normal at P to the surface

* The other types of surface integrals are

$$\int_S \bar{F}\times d\bar{s} \quad \text{and} \quad \int_S \phi\, d\bar{s} = \int_S \phi\, \bar{n}\, ds$$

__Note:__ To evaluate any surface integral it is convenient to evaluate the double integral its projection in the xy-plane (or) yz-plane (or) zx-plane

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_S \bar{F}\cdot d\bar{s} = \iint_R \bar{F}\cdot\bar{n}\, \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

, where R is the projection of S in the xy-plane

Here $ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

· Similarly, R be the projection of surface in yz-plane, $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

& R be the projection of surface in zx-plane, $ds = \frac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$

✱ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = (x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

G.T $\bar{F} = (x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$ and S be the surface of the plane

i.e $S = 2x+y+2z-6 = 0$

Normal to the surface S is ∇S

i.e $\nabla S = \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z}$

$= \bar{i}\frac{\partial}{\partial x}(2x+y+2z-6) + \bar{j}\frac{\partial}{\partial y}(2x+y+2z-6) + \bar{k}\frac{\partial}{\partial z}(2x+y+2z-6)$

$\nabla S = \bar{i}2 + \bar{j}(1) + \bar{k}(2)$

$|\nabla S| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3$

∴ $\int_S \bar{F}\cdot\bar{n}\,ds$ or $\int_S \bar{F}\cdot d\bar{s}$

$\iint_S \bar{F}\cdot\bar{n}\,ds$ or $\iint_S \bar{F}\cdot d\bar{s}$

where ds is the projection of surface S in the xy-plane or yz-plane or zx-plane let $\bar{n}$ = unit normal to the surface S

i.e $\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{\sqrt{2^2+1^2+2^2}} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{\sqrt{9}} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{3}$

let R be the projection of the surface S in xy-plane, so $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$ds = \frac{dx\,dy}{\left|\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}\cdot\bar{k}\right|} = \frac{dx\,dy}{\frac{2}{3}} = \frac{3}{2}\,dx\,dy$$

Now $\bar{F}\cdot\bar{n} = \left((\cancel{x}+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}\right)\cdot\left(\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}\right) = \frac{1}{3}\left(2(x+y^2) - 2x + 4yz\right) = \frac{1}{3}(2y^2+4yz) = \frac{2}{3}(y^2+2yz)$

$\because$ R be the projection of surface S in xy-plane (i,e $z=0$)

Given surface becomes $2x+y+2z=6$ $[\because z=0]$

$2x+y=6$

$y=6-2x$

equation of x-axis, $y=0$ $[\because z=0]$

$2x=6$

$x=3$

& $y: 0 \to 6-2x$

$x: 0 \to 3$

$[\because$ In the first octant means lower limits must be zeros$]$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{3}\int_{y=0}^{6-2x} \frac{2}{3}(y^2+2yz)\,\frac{3}{2}\,dx\,dy$$

$$\left[\because \begin{aligned} &\text{Given surface is} \\ &2x+y+2z=6 \\ &2z = 6-2x-y \\ &z = \frac{6-2x-y}{2} \end{aligned}\right]$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} y^2+2y\left(\frac{6-2x-y}{2}\right)dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} \left(y^2+\cancel{y}(6-2x-y)\right)dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} \left(y^2+6y-2xy-y^2\right)dx\,dy$$

$$= \int_{x=0}^{3}\left(\int_{y=0}^{6-2x} \underbrace{(6y-2xy)}_{\downarrow\, x \text{ as constant}}dy\right)dx$$

$$= \int_{x=0}^{3}\left(6\left(\frac{y^2}{2}\right)-2x\left(\frac{y^2}{2}\right)\right)_{y=0}^{6-2x} dx$$

$$= \int_{x=0}^{3}\left(3\left((6-2x)^2-0\right)-x\left((6-2x)^2-0\right)\right)dx$$

$$= 3\left(\frac{(6-2x)^3}{3(-2)}\right)^3_{x=0} - \left[(x)\left(\frac{(6-2x)^3}{3(-2)}\right) - (1)\left(\frac{-1}{6}\right)\left(\frac{(6-2x)^4}{4(-2)}\right)\right]^3_{x=0}$$

$$= \frac{-1}{2}(0-6^3) - \left[\frac{-1}{6}(3(0)-0)\right] + \frac{1}{6}\left(\frac{-1}{8}\right)(0-6^4)\Big]$$

$$= \frac{6^3}{2} - \left[0 + \frac{1}{48}(+6^4)\right]$$

$$= \frac{1}{2}(6^3) - \frac{6^4}{48}$$

$$= \frac{24(6^3) - 6^4}{48} = \frac{24(216) - 1296}{48} = \frac{3888}{48}$$

$$= 81$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\, dS = 81$$

let R be the projection of S in yz plane. So $ds = \dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$$ds = \frac{dy\,dz}{\left|\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}\cdot\bar{i}\right|} = \frac{dy\,dz}{\frac{2}{3}} = \frac{3}{2}\,dy\,dz$$

Now $\bar{F}\cdot\bar{n} = \left((x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}\right)\cdot\left(\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}\right) = \frac{1}{3}\left(2(x+y^2) - 2x + 4yz\right)$

$$= \frac{1}{3}\left(2x + 2y^2 - 2x + 4yz\right)$$

$$= \frac{1}{3}\left(2y^2 + 4yz\right) = \frac{2}{3}\left(y^2 + 2yz\right)$$

$\because$ R be the projection of S in yz-plane (i.e $x=0$)

Given surface becomes $2x+y+2z=6$ $[\because x=0]$

$y+2z=6$ ✓

$y = 6-2z$

$y+2z=6$

eq. of z axis is $y=0$ $[\because x=0]$

$2z=6 \Rightarrow z=3$

and $y: 0 \to 6-2z$

$z: 0 \to 3$

$[\because$ In the first octant means lower limits must be zeros$]$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\, ds = \int_{z=0}^{3}\int_{y=0}^{6-2z} \frac{2}{3}(y^2+2yz)\cdot\frac{3}{2}\,dy\,dz$$

$$= \int_{z=0}^{3}\left(\int_{y=0}^{6-2z} \underbrace{(y^2+2yz)}_{z \text{ as constant}}\, dy\right) dz$$

$$= \int_{z=0}^{3}\left(\frac{y^3}{3} + 2z\frac{y^2}{2}\right)_{y=0}^{6-2z} dz$$

$$= \int_{z=0}^{3}\left(\frac{1}{3}\left((6-2z)^3-0\right) + z\left((6-2z)^2-0\right)\right) dz$$

$$= \int_{z=0}^{3}\left(\frac{1}{3}(6-2z)^3 + z(6-2z)^2\right) dz$$

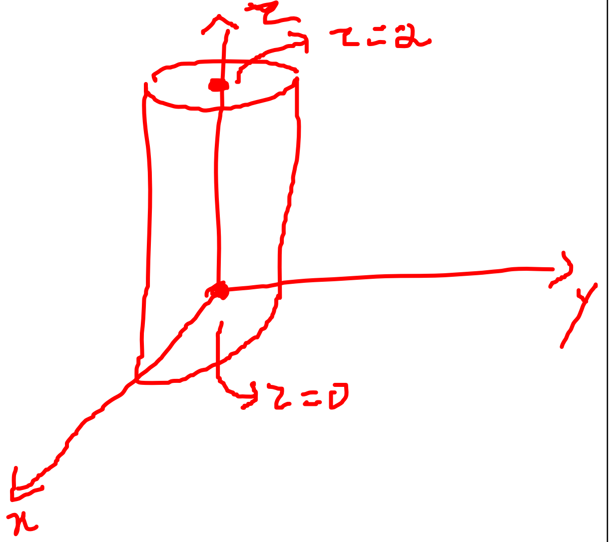
$$= \frac{1}{3}\left(\frac{(6-2z)^4}{4(-2)}\right)_{z=0}^{3} + \left[z\left(\frac{(6-2z)^3}{3(-2)}\right) - (1)\left(-\frac{1}{6}\left(\frac{(6-2z)^4}{4(-2)}\right)\right)\right]_{z=0}^{3}$$

$$= \frac{1}{3}\left(-\frac{1}{8}\right)\left(0-6^4\right) + \left(-\frac{1}{6}\left(3(0)-0\right)\right) + \frac{1}{6}\left(-\frac{1}{8}\right)\left(0-6^4\right)$$

$$= \frac{6^4}{24} + \frac{6^4}{48}$$

$$= \frac{2\,6^4 + 6^4}{48} = \frac{3(6^4)}{48} = \frac{3888}{48} = 81$$

* Evaluate $\iint_S \overline{F} \cdot \overline{ds}$ [$\overrightarrow{n}\,ds$], if $\overline{F} = yz\,\overline{i} + 2y^2\,\overline{j} + xz^2\,\overline{k}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ contained in the first octant b/w the planes $z = 0$ & $z = 2$



G.T $\overline{F} = yz\,\overline{i} + 2y^2\,\overline{j} + xz^2\,\overline{k}$

Given surface be $S : x^2 + y^2 - 9 = 0$

Normal to the surface S is $\nabla S = \overline{i}\frac{\partial S}{\partial x} + \overline{j}\frac{\partial S}{\partial y} + \overline{k}\frac{\partial S}{\partial z}$

$$= \overline{i}(2x) + \overline{j}(2y) + \overline{k}(0)$$

$$= 2x\,\overline{i} + 2y\,\overline{j} + 0\,\overline{k}$$

$$|\nabla S| = |2x\,\overline{i} + 2y\,\overline{j} + 0\,\overline{k}| = \sqrt{(2x)^2 + (2y)^2 + 0^2} = 2\sqrt{x^2 + y^2} \quad [\because x^2 + y^2 = 9]$$

$$= 2\sqrt{9}$$

$$= 2(3) = 6$$

let $\bar{n}$ be the unit normal vector to the surface

i.e $\bar{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{2x\bar{i} + 2y\bar{j} + 0\bar{k}}{6} = \dfrac{2(x\bar{i} + y\bar{j} + 0\bar{k})}{6}$

$F \cdot \bar{n} = (yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}) \cdot \frac{1}{3}(x\bar{i} + y\bar{j} + 0\bar{k})$

$F \cdot \bar{n} = \frac{1}{3}(xyz + 2y^3 + 0) = \frac{1}{3}(xyz + 2y^3)$

let R be the projection of surface S in yz-plane (or zx plane)

so $dS = \dfrac{dy\,dz}{|\bar{n} \cdot \bar{i}|} = \dfrac{dy\,dz}{\left|\frac{1}{3}(x\bar{i} + y\bar{j} + 0\bar{k}) \cdot \bar{i}\right|} = \dfrac{dy\,dz}{\frac{x}{3}} = \frac{3}{x}\,dy\,dz$

$\because$ R be the projection of surface S in yz-plane (i.e $x = 0$)

Given surface becomes $x^2 + y^2 - 9 = 0$ [$\because x = 0$]

$y^2 = 9 \Rightarrow y = 3$

& $y : 0 \to 3$

$z : 0 \to 2$ [$\because$ In the first octant lower limits must be zeros]

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_S \bar{F}\cdot\bar{n}\,dS = \int_{z=0}^{2}\int_{y=0}^{3} \frac{1}{\cancel{3}}(xyz+2y^3)\frac{\cancel{3}}{x}\,dy\,dz$$

$$\left[\because x^2+y^2=9 \quad x^2=9-y^2 \quad x=\sqrt{9-y^2}\right]$$

$$= \int_{z=0}^{2}\int_{y=0}^{3}\left(yz+\frac{2}{x}y^3\right)dy\,dz$$

$$= \int_{z=0}^{2}\left(\int_{y=0}^{3}\left(yz+\frac{2}{\sqrt{9-y^2}}y^3\right)dy\right)dz$$

z as constant

$$= \int_{z=0}^{2}\left(z\left(\frac{y^2}{2}\right)_0^3 + 2\int_{y=0}^{3}\frac{y^3}{\sqrt{9-y^2}}\,dy\right)dz$$

$$= \int_{z=0}^{2}\left(\frac{z}{2}(9-0) + 2\left(\int_{\theta=\pi/2}^{0}\frac{(3\cos\theta)^3}{\sqrt{9-(3\cos\theta)^2}}(-3\sin\theta\,d\theta)\right)\right)dz$$

$$= \int_{z=0}^{2}\left(\frac{9z}{2} + 2\int_{\theta=0}^{\pi/2}\frac{27\cos^3\theta\,(+\cancel{3}\sin\theta\,d\theta)}{\cancel{3}\sqrt{1-\cos^2\theta}}\right)dz$$

$$= \int_{z=0}^{2}\left[\left(\frac{9z}{2}\right) + 2\left(27\int_{\theta=0}^{\pi/2}\frac{\cos^3\theta}{\sin\theta}\sin\theta\,d\theta\right)\right]dz$$

$$\left[\because \sin^2\theta+\cos^2\theta=1\right]$$

consider

$$\int_{y=0}^{3}\frac{y^3}{\sqrt{9-y^2}}\,dy$$

Put $y = 3\cos\theta$ (or $3\sin\theta$)

$dy = -3\sin\theta\,d\theta$

and $y: 0 \to 3$

$\theta: \pi/2 \to 0$

$y = 3\cos\theta$

$y=0,\ 0 = 3\cos\theta \Rightarrow \cos\theta = 0$

$\cos\theta = \cos\pi/2$

$\theta = \pi/2$

$y=3,\ 3 = 3\cos\theta$

$\cos\theta = 1$

$\cos\theta = \cos 0$

$\theta = 0$

$$\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$-\int_{\theta=\pi/2}^{0} f(\theta)\,d\theta = \int_0^{\pi/2} f(\theta)\,d\theta$$

$$= \int_{z=0}^{2} \left(\frac{9z}{2} + 2\left[27\int_0^{\pi/2} \cos^3\theta \, d\theta\right]\right) dz$$

$$= \frac{9}{2}\left(\frac{z^2}{2}\right)_0^2 + 2\left(27\left(\frac{2}{3}\right)\right)(z)_0^2$$

$\rightarrow 18$

$$= \frac{9}{4}(4-0) + 2(18)(2-0)$$

$$= 9 + 72$$

$$= 81.$$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$\int_0^{\pi/2} \sin^2\theta \cos^3\theta \, d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, 2\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma 2}{\Gamma\frac{1}{2}+2}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}\,(1!)}{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\sqrt{\pi}} = \frac{1}{\cancel{3}} \quad \frac{2}{3}$$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots\ldots} \cdot \frac{\pi}{2}$$

if m & n are even

$$= \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots\ldots}, \text{ if}$$

m & n are odd

$$\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\,dy = \frac{1}{3}\int_0^3 \frac{y^3}{\sqrt{1-\frac{y^2}{9}}}\,dy$$

Put $\frac{y^2}{9} = t$

$y^2 = 9t$

$y = 3(t^{1/2})$

$dy = \frac{3}{2} t^{1/2-1}\,dt$

As $y: 0 \rightarrow 3$

$t: 0 \rightarrow 1$

$$= \frac{1}{3}\int_{t=0}^{1} \frac{3^3\, t^{3/2}}{\sqrt{1-t}} \frac{3}{2} t^{1/2-1}\,dt$$

$$= \frac{1}{3}\int_{t=0}^{1} \frac{27}{2}(3)\, t^{3/2+1/2-1}(1-t)^{-1/2}\,dt$$

$$= \frac{27}{2}\int_{t=0}^{1} t^{2-1}(1-t)^{-1/2+1-1}\,dt$$

$$= \frac{27}{2}\beta(2, 1/2) = \frac{27}{2}\frac{\Gamma 2\,\Gamma 1/2}{\Gamma(2+1/2)} = \frac{27}{2}\frac{(1!)\sqrt{\pi}}{\Gamma 5/2} = \frac{27}{2}\frac{\sqrt{\pi}}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}} = \frac{27}{2}\cdot\frac{4}{3}$$

$$= 9\times 2 = 18$$

* Evaluate $\int \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ to $z=5$

G.T $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$

let s be the given surface (ie $S: x^2+y^2-16=0$)

normal to the surface S is $\nabla S = \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z} = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(0)$

$|\nabla S| = \sqrt{(2x)^2+(2y)^2+0^2} = 2\sqrt{x^2+y^2} = 2\sqrt{16} = 2(4) = 8$

len $\bar{n}$ = unit normal vector to the surface

i.e $\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\bar{i}+2y\bar{j}+0\bar{k}}{8} = \frac{2}{8}(x\bar{i}+y\bar{j}+0\bar{k}) = \frac{1}{4}(x\bar{i}+y\bar{j})$

$\bar{F}\cdot\bar{n} = (z\bar{i}+x\bar{j}-3y^2z\bar{k})\cdot\left(\frac{2}{8}(x\bar{i}+y\bar{j}+0\bar{k})\right)$

$\bar{F}\cdot\bar{n} = \frac{2}{8}(zx+xy-0) = \frac{1}{4}(zx+xy)$

let R be the projection of surface S in yz-plane (or zx-plane)

so $dS = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{|\frac{1}{4}(x\bar{i}+y\bar{j})\cdot\bar{i}|} = \frac{dy\,dz}{x/4} = \frac{4}{x}dy\,dz$

$\therefore$ R be the projection of surface S in yz-plane (i.e $x=0$)

Given surface becomes (i.e $x^2+y^2=16$) [$\because x=0$

$$y^2=16 \Rightarrow y=4$$

and $y: 0 \to 4$

$z: 0 \to 5$ [$\because$ In the first octant means lower limits must be zeros]

$$\therefore \iint_S \bar{F} \cdot \bar{n}\, ds = \int_{z=0}^{5} \int_{y=0}^{4} \frac{1}{4}(zx+xy)\,\frac{4\,dy\,dz}{x}$$

$$= \int_{z=0}^{5} \int_{y=0}^{4} \cancel{x}(z+y)\,\frac{dy\,dz}{\cancel{x}}$$

$$= \int_{z=0}^{5} \Big(\underbrace{\int_{y=0}^{4} (z+y)\,dy}_{z \text{ as constant}} \Big)\, dz$$

$$= \int_{z=0}^{5} \left(\int_{y=0}^{4} (z+y)\, dy \right) dz$$

$$= \int_{z=0}^{5} \left(zy + \frac{y^2}{2} \right)_{y=0}^{4} dz$$

$$= \int_{z=0}^{5} \left(z(4-0) + \frac{1}{2}(4^2-0) \right) dz$$

$$= \int_{z=0}^{5} \left(4z + \frac{16}{2} \right) dz$$

$$= \left(4\left(\frac{z^2}{2}\right) + 8z \right)_{z=0}^{5}$$

$$= 2(25-0) + 8(5-0)$$

$$= 50 + 40$$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = 90$$

* Evaluate $\iint_S \bar{F}\cdot\bar{n}\, ds$, where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$, where S is the surface of the cylinder $x^2+y^2=1$ in the first octant between $z=0$ to $z=2$ [Hint: Projection in yz or zx-plane]

Ans $\iint_S \bar{F}\cdot\bar{n}\, ds = \underline{3}$

*

Hint: Projection in yz-plane (i.e $x=0$)

Given surface $S: x^2+y^2=1$ becomes ($\because x=0$)

$y^2=1$

$y=1$

$y: 0 \to 1$

$z: 0 \to 2$

$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{\bar{i}\,2x + \bar{j}\,2y + \bar{k}\,0}{\sqrt{(2x)^2+(2y)^2+0^2}}$

$= \frac{2[x\bar{i}+y\bar{j}+0\bar{k}]}{\cancel{2}\sqrt{x^2+y^2}}$

$= \frac{x\bar{i}+y\bar{j}+0\bar{k}}{\sqrt{1}}$

$\bar{n} = \frac{x\bar{i}+y\bar{j}+0\bar{k}}{1}$

$ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$= \frac{dy\,dz}{\left|\frac{x\bar{i}+y\bar{j}+z\bar{k}}{1}\cdot\bar{i}\right|}$

$ds = \frac{dy\,dz}{x}$

$\bar{F}\cdot\bar{n} = (z\bar{i}+x\bar{j}-3y^2z\bar{k})\cdot\left(\frac{x\bar{i}+y\bar{j}+0\bar{k}}{1}\right)$

$= zx + xy + 0$

$\bar{F}\cdot\bar{n} = zx + xy$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \int_{z=0}^{2}\int_{y=0}^{1} (zx+xy)\, \frac{dy\,dz}{x}$$

$$= \int_{z=0}^{2}\int_{y=0}^{1} (z+y)\, dy\, dz$$

z as constant

$$= \int_{z=0}^{2} \left(zy + \frac{y^2}{2}\right)_{y=0}^{1} dz$$

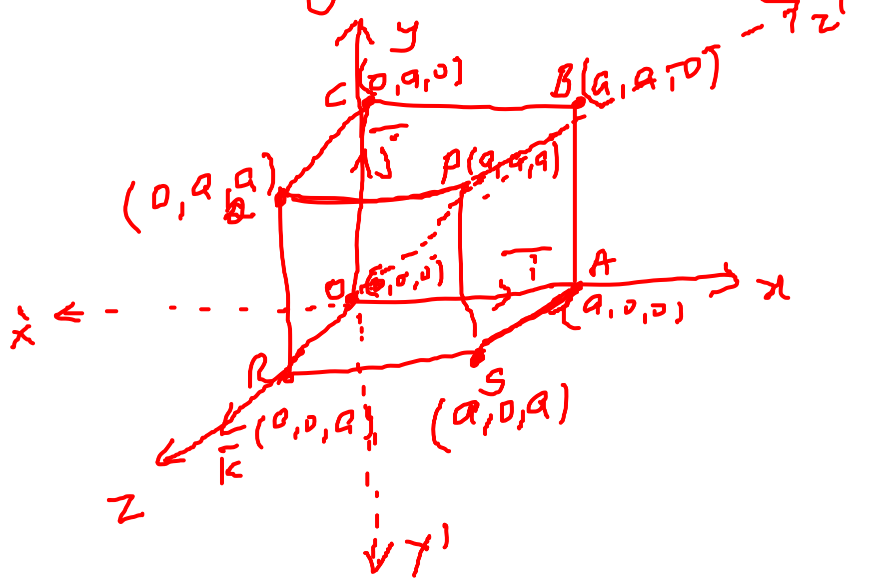
$$= \int_{z=0}^{2} \left(z+\frac{1}{2}\right) dz = \left(\frac{z^2}{2}+\frac{1}{2}z\right)_{z=0}^{2}$$

$\frac{1}{2}(4-0) + \frac{1}{2}(2-0)$

$2+1$

$=3$

* If $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$, evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, where S is the surface of the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$



The surface of the cube consists of 6 smooth surfaces is given by S_1: PBAS, S_2: OCQR, S_3: PBCQ, S_4: OASR

S_5: PQRS, S_6: OABC

G.T $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_{S_1} + \int_{S_2} + \int_{S_3} + \int_{S_4} + \int_{S_5} + \int_{S_6} \quad \text{——— (1)}$$

Along S_1: eqn of S_1: PBAS is $x=a$

let $\bar{n}$ be the unit outward drawn normal to the surface S_1 is along the +ve side of x-axis, then

$\bar{n} = \frac{\bar{i}}{|\bar{i}|} = \frac{\bar{i}}{\sqrt{1^2}} = \frac{\bar{i}}{1} = \bar{i}$ & R be the projection of surface S_1 in yz-plane, so $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{|\bar{i}\cdot\bar{i}|} = \frac{dy\,dz}{1}$

$\bar{F}\cdot\bar{n} = (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot\bar{i} = 4xz = 4az$ [$\because x=a$]

& $y: 0 \to a$

$z: 0 \to a$

$$\iint_{S_1} \bar{F}\cdot\bar{n}\, ds = \int_{z=0}^{a}\int_{y=0}^{a} 4az\, dy\, dz$$

$$= 4a \int_{z=0}^{a} z\, dz \int_{y=0}^{a} dy = 4a\left(\frac{z^2}{2}\right)_0^a (y)_0^a = 2a(a^2-0)(a-0) = 2a^4 \quad \text{——} (2)$$

Along S_2: Eqn of S_2 : OCQR is $x=0$

let $\bar{n} = \frac{-\bar{i}}{|-\bar{i}|} = \frac{-\bar{i}}{\sqrt{(-1)^2}} = -\bar{i}$, $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{|\bar{i}\cdot -\bar{i}|} = \frac{dy\,dz}{|-1|} = \frac{dy\,dz}{1}$

and $y: 0 \to a$

$z: 0 \to a$

$$\bar{F}\cdot\bar{n} = (4xz\bar{i} - y^2\bar{j} + yz\bar{k})\cdot -\bar{i} = -4xz = -4(0)z = 0$$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\, ds = \int_{z=0}^{a}\int_{y=0}^{a} 0\cdot dy\, dz = 0 \quad \text{——} (3)$$

Along S_3 ; Eqn of S_3 : PBCQ is $y=a$

$\bar{n} = \frac{\bar{j}}{|\bar{j}|} = \frac{\bar{j}}{\sqrt{1^2}} = \frac{\bar{j}}{1} = \bar{j}$, $ds = \frac{dz\,dx}{|\bar{n}\cdot\bar{j}|} = \frac{dz\,dx}{|\bar{j}\cdot\bar{j}|} = \frac{dz\,dx}{1} = dz\,dx$

And $z: 0 \to a$

$x: 0 \to a$

$\bar{F}\cdot\bar{n} = (4xz\,\bar{i} - y^2\bar{j} + yz\bar{k})\cdot(\bar{j}) = -y^2 = -a^2 \quad [\because y=a]$

$\iint_{S_3} \bar{F}\cdot\bar{n}\,dS = \int_{x=0}^{a}\int_{z=0}^{a} -a^2 \; dz\,dx = -a^2 (z)_0^a (x)_0^a = -a^2(a-0)(a-0) = -a^4$ —— (4)

Along S_4: Equ of S_4: OASR is $y=0$

let $\bar{n} = -\bar{j}$, $dS = \dfrac{dz\,dx}{|\bar{n}\cdot\bar{j}|} = \dfrac{dz\,dx}{|-\bar{j}\cdot\bar{j}|} = \dfrac{dz\,dx}{|-1|} = \dfrac{dz\,dx}{1}$

and $z: 0 \to a$

$x: 0 \to a$

$\bar{F}\cdot\bar{n} = (4xz\,\bar{i} - y^2\bar{j} + yz\bar{k})\cdot(-\bar{j}) = y^2 = 0 \quad [\because y=0]$

$\iint_{S_4} \bar{F}\cdot\bar{n}\,dS = \int_{x=0}^{a}\int_{z=0}^{a} 0 \; dz\,dx = 0$ —— (5)

Along S_5: Equ of S_5: PQRS is $z=a$

let $\bar{n} = +\bar{k}$, $dS = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \dfrac{dx\,dy}{|\bar{k}\cdot\bar{k}|} = \dfrac{dx\,dy}{1}$.

and $x: 0 \to a$

$y: 0 \to a$ & $\bar{F}\cdot\bar{n} = (4xz\,\bar{i} - y^2\bar{j} + yz\bar{k})\cdot\bar{k} = yz = ay \quad [\because z=a]$

$$\iint_{S_5} \bar F\cdot\bar n\, ds = \int_{x=0}^{a}\int_{y=0}^{a} ay\, dx\,dy = a\left(\frac{y^2}{2}\right)_0^a (x)_0^a = a\,\frac{1}{2}(a^2-0)\,(a-0)$$

$$= \frac{a^4}{2} \quad \text{------ } (6)$$

Along S_6: Eq of S_6: OABC is $z=0$

let $\bar n = -\bar k$, $ds = \dfrac{dx\,dy}{|\bar n\cdot\bar k|} = \dfrac{dx\,dy}{|-\bar k\cdot\bar k|} = \dfrac{dx\,dy}{|-1|} = \dfrac{dx\,dy}{1}$

and $x: 0\to a$

$y: 0\to a$

& $\bar F\cdot\bar n = (4xz\bar i - y^2\bar j + yz\bar k)\cdot(-\bar k) = -yz = -y(0) = 0$

$$\iint_{S_6} \bar F\cdot\bar n\, ds = \int_{x=0}^{a}\int_{y=0}^{a} 0\, dx\,dy = 0 \quad \text{------ } (7)$$

sub (2), (3), (4), (5), (6) & (7) in (1), we get

$$\iint_S \bar F\cdot\bar n\, ds = 2a^4 + 0 - a^4 + 0 + \frac{a^4}{2} + 0 = a^4 + \frac{a^4}{2} = \frac{3a^4}{2}$$

* If $\phi = \frac{3}{8}xyz$, find $\int_S \phi \bar{n}\, ds$, where S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ to $z=5$

G.T $\phi = \frac{3}{8}xyz$

$S:\ x^2+y^2-16=0$

Normal to the surface S is $\nabla S = \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z} = \bar{i}\,2x + \bar{j}\,2y + \bar{k}(0)$

$|\nabla S| = \sqrt{(2x)^2+(2y)^2} = 2(\sqrt{x^2+y^2}) = 2(\sqrt{16}) = 2(4) = 8$

$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\bar{i}+2y\bar{j}+0\bar{k}}{8} = \frac{2}{8}(x\bar{i}+y\bar{j}+0\bar{k}) = \frac{1}{4}(x\bar{i}+y\bar{j}+0\bar{k})$

$\phi\bar{n} = \frac{3}{8}xyz\left(\frac{1}{4}\right)(x\bar{i}+y\bar{j}+0\bar{k}) = \frac{3}{32}xyz(x\bar{i}+y\bar{j}+0\bar{k}) = \frac{3}{32}(x^2yz\bar{i}+xy^2z\bar{j}+0)$

let R be the projection of S in yz (or zx)-plane

so, $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{|\frac{1}{4}(x\bar{i}+y\bar{j}+0\bar{k})\cdot\bar{i}|} = \frac{dy\,dz}{\frac{x}{4}} = \frac{4}{x}dy\,dz$

$\because$ R be the projection of S in yz-plane (i.e $x=0$)

Given surface becomes $(x^2+y^2=16)$ $[\because x=0]$

$y^2=16 \Rightarrow y=4$

and $y : 0 \to 4$

$z : 0 \to 5$

$$\therefore \iint_S \phi \bar{n}\, ds = \int_{z=0}^{5}\int_{y=0}^{4} \frac{3}{32}(x^2yz\,\bar{i} + xy^2z\,\bar{j})\,\frac{4}{x}\,dy\,dz$$

$$= \frac{12}{32}\int_{z=0}^{5}\left(\int_{y=0}^{4}(xyz\,\bar{i} + y^2z\,\bar{j})\,dy\right)dz$$

$\left[\because \text{Given surface } x^2+y^2=16,\ x^2 = 16-y^2,\ x = \sqrt{16-y^2}\right]$

$$= \frac{12}{32}\left[\int_{z=0}^{5}\left(\int_{y=0}^{4}\left(\sqrt{16-y^2}\cdot yz\,\bar{i} + y^2z\,\bar{j}\right)dy\right]dz\right]$$

$$= \frac{3}{8}\left[\int_{z=0}^{5}\left(\frac{z\,\bar{i}}{-2}\int_{y=0}^{4}\sqrt{16-y^2}\,(-2)y + z\,\bar{j}\int_{y=0}^{4}y^2\,dy\right)dz\right]$$

$\left[\because \int \sqrt{f(x)}\,f'(x)\,dx = \frac{(f(x))^{3/2}}{3/2} + c\right]$

$$= \frac{3}{8}\left[\int_{z=0}^{5}\left(\frac{-z\,\bar{i}}{2}\left(\frac{(16-y^2)^{3/2}}{3/2}\right) + z\,\bar{j}\left(\frac{y^3}{3}\right)\right)_{y=0}^{4} dz\right]$$

$$= \frac{3}{8}\left[\int_{z=0}^{5}\left(\frac{-z}{2}\left(\frac{2}{3}\right)\bar{i}\,(0-(16)^{3/2}) + \frac{z\,\bar{j}}{3}(4^3-0)\right)dz\right]$$

$$= \frac{3}{8}\left[\int_{z=0}^{5}\left(-\frac{z}{3}\,\bar{i}\,(-4^3) + \frac{z\,\bar{j}}{3}(4^3)\right)dz\right]$$

$$= \frac{3}{8}\left[\int_{z=0}^{5}\left(\frac{64}{3}(\bar{i}\,z) + \frac{64}{3} z\bar{j}\right) dz\right]$$

$$= \frac{\cancel{8}}{8}\left[\frac{64}{\cancel{3}}\right]\left(\bar{i}\left(\frac{z^2}{2}\right) + \bar{j}\left(\frac{z^2}{2}\right)\right)_{z=0}^{5}$$

$$= 8\left[\frac{\bar{i}}{2}(25-0) + \frac{\bar{j}}{2}(25-0)\right]$$

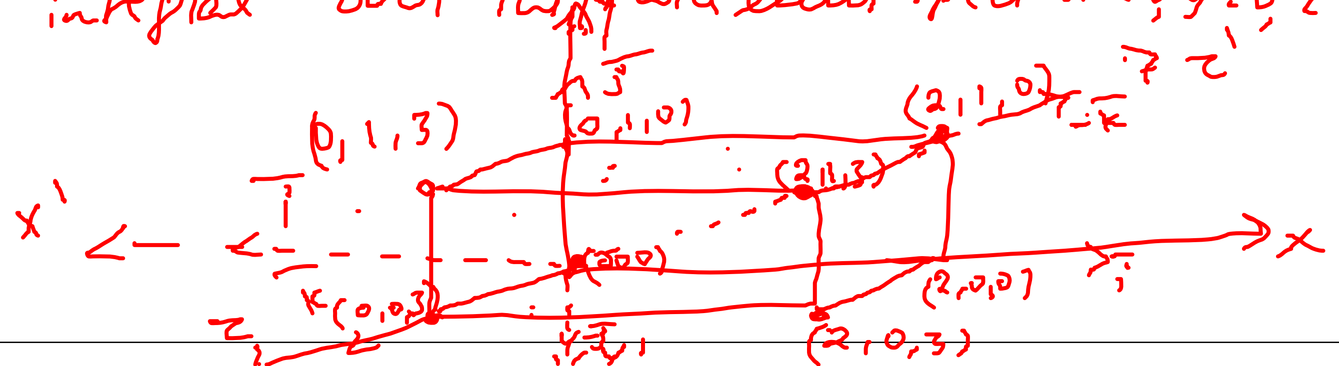
$$= \frac{8}{2}\left[25\bar{i} + 25\bar{j}\right] = \frac{25\times 8}{2}(\bar{i}+\bar{j})$$

$$= 25\times 4(\bar{i}+\bar{j})$$

$$= 100(\bar{i}+\bar{j})$$

$$\iint_S \phi\,\bar{n}\,ds = 100(\bar{i}+\bar{j})$$

* If $\bar{F} = 2xy\bar{i} + yz^2\bar{j} + xz\bar{k}$, find the surface integral over the parallelopiped $x=0, y=0, z=0$, $x=2, y=1, z=3$



Ans $\iint_S \bar{F}\cdot\bar{n}\,ds = 30$

* **Volume integrals** : Any integral which is to be evaluated over a volume is called volume integral. If V is a volume bounded by a surface S, then

(i) $$\iiint_V \bar{F}\, dV = \iiint_V (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\, dx\,dy\,dz$$

$$= \bar{i}\iiint F_1\, dx\,dy\,dz + \bar{j}\iiint F_2\, dx\,dy\,dz + \bar{k}\iiint F_3\, dx\,dy\,dz$$

(ii) $$\iiint_V \phi\, dV = \iiint_V \phi\, dx\,dy\,dz$$

* If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ evaluate $\int_V \bar{F}\,dV$, where V is the region bounded by the surfaces $x=0,\ x=2,\ y=0,\ y=6,\ z=x^2,\ z=4$

G.T $\bar{F} = 2xz\,\bar{i} - x\bar{j} + y^2\bar{k} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ say

Here $F_1 = 2xz$, $F_2 = -x$, $F_3 = y^2$

Given limits are $x: 0 \to 2$

$y: 0 \to 6$

$z: x^2 \to 4$

$$\int_V \bar{F}\,dv = \bar{i}\iiint F_1\,dx\,dy\,dz + \bar{j}\iiint F_2\,dx\,dy\,dz + \bar{k}\iiint F_3\,dx\,dy\,dz$$

$$\int_V \bar{F}\,dv = \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,dx\,dy\,dz + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} -x\,dx\,dy\,dz + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$$

$$= 2\bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\left(\frac{z^2}{2}\right)_{z=x^2}^{4} x\,dy\,dx - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}(z)_{z=x^2}^{4}\,x\,dy\,dx + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}(z)_{x^2}^{4}\,y^2\,dx\,dy$$

$$= \cancel{2}\bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\frac{1}{\cancel{2}}(16-x^4)\,x\,dy\,dx - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}(4-x^2)\,x\,dy\,dx + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}(4-x^2)\,y^2\,dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}(16x-x^5)\,dy\,dx - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}(4x-x^3)\,dy\,dx + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}(4-x^2)\,y^2\,dx\,dy$$

$$= \bar{i}\left(16\left(\frac{x^2}{2}\right)-\frac{x^6}{6}\right)_{x=0}^{2}(y)_{y=0}^{6} - \bar{j}\left[\left(4\left(\frac{x^2}{2}\right)-\frac{x^4}{4}\right)_{x=0}^{2}(y)_{y=0}^{6}\right] + \bar{k}\left(\left(4x-\frac{x^3}{3}\right)_{x=0}^{2}\left(\frac{y^3}{3}\right)_{y=0}^{6}\right)$$

$$= \bar{i}\left(8(4-0)-\frac{1}{6}(64-0)\right)(6-0) - \bar{j}\left(2(4-0)-\frac{1}{4}(16-0)\right)(6-0) + \bar{k}\left(4(2-0)-\frac{1}{3}(8-0)\right)\left(\frac{1}{3}(216-0)\right)$$

$$= \bar{i}\left(32-\frac{64}{6}\right)(6) - \bar{j}(8-4)(6-0) + \bar{k}\left(8-\frac{8}{3}\right)(72)$$

$$= \bar{i}\left(\frac{192-64}{6}\right)(6) - \bar{j}(4)(6) + \bar{k}\left(\frac{24-8}{3}\right)(72) = 2\bar{i}(128) - 24\bar{j} + \bar{k}\left(\frac{16}{\cancel{3}}\right)\overset{24}{(\cancel{72})}$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k}$$

* If $\bar{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$, then evaluate (i) $\int_V \nabla\cdot\bar{F}\,dv$ and (ii) $\int_V \nabla\times\bar{F}\,dv$, where V is the closed region bounded by $x=0$, $y=0$, $z=0$, $\underline{2x+2y+z=4}$

G.T $\bar{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ say

$$\nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left((2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(2x^2-3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$\nabla\cdot\bar{F} = 4x - 2x - 0 = 2x$$

Given limits are $z : 0 \to 4-2x-2y$

$y : 0 \to \frac{4-2x}{2} = 2-x$

$x : 0 \to 2$

$2x+2y = 4$

$2y = 4-2x$

$y = \frac{4-2x}{2}$

$y = 2-x$

$2x = 4$

$x = 2$

$2x + 2y + z = 4$

$z = 4-2x-2y$

(i) $$\int_V \nabla\cdot\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dx\,dy\,\underline{dz}$$

x & y are constants

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (z)_{z=0}^{4-2x-2y}\,\underline{2x}\,dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (4-2x-2y-0)\,2x\,dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left(8x - 4x^2 - 4xy\right) dy\, dx$$

x as constant

$$= \int_{x=0}^{2} \left(8xy - 4x^2y - 4x\left(\frac{y^2}{2}\right)\right)\Bigg|_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2} \left(8x(2-x-0) - 4x^2(2-x-0) - 2x\left((2-x)^2 - 0\right)\right) dx$$

$$= \int_{x=0}^{2} \left(16x - 8x^2 - 8x^2 + 4x^3 - 2x(4 + x^2 - 4x)\right) dx$$

$$= \int_{x=0}^{2} \left(16x - 8x^2 - \cancel{8x^2} + 4x^3 - 8x - 2x^3 + \cancel{8x^2}\right) dx$$

$$= \int_{x=0}^{2} \left(2x^3 - 8x^2 + 8x\right) dx$$

$$= \left(2\left(\frac{x^4}{4}\right) - 8\left(\frac{x^3}{3}\right) + 8\left(\frac{x^2}{2}\right)\right)\Bigg|_{x=0}^{2}$$

$$= \frac{1}{2}(16-0) - \frac{8}{3}(8-0) + 4(4-0)$$

$$= 8 - \frac{64}{3} + 16 = 24 - \frac{64}{3} = \frac{72-64}{3} = \frac{8}{3}$$

(ii) G.T $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \overset{+}{\bar{i}} & \overset{-}{\bar{j}} & \overset{+}{\bar{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix} \to R_1 \qquad \left(\because \nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$$

$$= \bar{i}(0-0) - \bar{j}(-4-(-3)) + \bar{k}(-2y - 0)$$

$$= \bar{j} - 2y\bar{k}$$

$$\int_V \nabla \times \bar{F}\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{z=4-2x-2y} \underbrace{(\bar{j} - 2y\bar{k})}_{\text{constant}}\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (\bar{j} - 2y\bar{k})\,(z)_{z=0}^{4-2x-2y}\, dx\, dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (\bar{j} - 2y\bar{k})\,(\underbrace{4-2x-2y}_{U.L} - \underbrace{0}_{L.L})\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \left(\bar{j}\,\underbrace{(4-2x-2y)}_{x \text{ as constant}} - 2y\bar{k}\,\underbrace{(4-2x-2y)}_{x \text{ as constant}}\right) dy\, dx$$

$$= \bar{j}\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} dx\, dy\, dz - 2\bar{k}\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} y\, dx\, dy\, dz$$

$$= \bar{j}\int_{x=0}^{2}\int_{y=0}^{2-x} (z)_{z=0}^{4-2x-2y}\, dy\, dx - 2\bar{k}\int_{x=0}^{2}\int_{y=0}^{2-x} (z)_{z=0}^{4-2x-2y}\, y\, dy\, dx$$

$$= \bar{j}\int_{x=0}^{2}\int_{y=0}^{2-x} \underbrace{(4-2x-2y-0)}_{x \text{ as constant}}\, dy\, dx - 2\bar{k}\int_{x=0}^{2}\int_{y=0}^{2-x} \underbrace{(4-2x-2y-0)y}_{x \text{ as constant}}\, dy\, dx$$

$$= \bar{j}\int_{x=0}^{2}\left(4y - 2xy - 2\frac{y^2}{2}\right)_{y=0}^{2-x} dx - 2\bar{k}\int_{x=0}^{2}\left(4\left(\frac{y^2}{2}\right) - 2x\left(\frac{y^2}{2}\right) - 2\left(\frac{y^3}{3}\right)\right)_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left(\bar{j}\left((4-2x) - 2y\right) - \bar{k}\left((4-2x)\,2y - 4y^2\right)\right) dy\, dx$$

$$= \int_{x=0}^{2} \left[\bar{j}\left((4-2x)y - \cancel{2}\left(\frac{y^2}{2}\right)\right) - \bar{k}\left((4-2x)\,\cancel{2}\left(\frac{y^2}{\cancel{2}}\right) - 4\left(\frac{y^3}{3}\right)\right)\right]_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2} \left[\bar{j}\left((4-2x)(2-x-0) - \left((2-x)^2 - 0\right)\right) - \bar{k}\left((4-2x)\left((2-x)^2 - 0\right) - \frac{4}{3}\left((2-x)^3 - 0\right)\right]\right] dx$$

$$= \int_{x=0}^{2} \left(\bar{j}\left(\cancel{2}(2-x)^2 - (2-x)^2\right) - \bar{k}\left(\left((2)(2-x)^3 - \frac{4}{3}(2-x)^3\right)\right)\right) dx$$

$$= \bar{j}\left(2\left(\frac{(2-x)^3}{3(-1)}\right) - \frac{(2-x)^3}{3(-1)}\right)_{x=0}^{2} - \bar{k}\left(2\left(\frac{(2-x)^4}{4(-1)}\right) - \frac{4}{3}\left(\frac{(2-x)^4}{4(-1)}\right)\right)_{x=0}^{2}$$

$$= \bar{j}\left(-\frac{2}{3}(0-8) + \frac{1}{3}(0-8)\right) - \bar{k}\left(-\frac{1}{2}(0-16) + \frac{1}{3}(0-16)\right)$$

$$= \bar{j}\left(\frac{16}{3} - \frac{8}{3}\right) - \bar{k}\left(\frac{16}{2} - \frac{16}{3}\right) = \frac{8}{3}\left[\bar{j}\right] - \bar{k}\left(\frac{48-32}{6}\right) = \frac{8}{3}\bar{j} - \frac{16\bar{k}}{6} = \frac{8}{3}\bar{j} - \frac{8}{3}\bar{k}$$

$$= \frac{8}{3}\left(\bar{j} - \bar{k}\right)$$

* Evaluate $\iiint_V \bar{F}\, dv$, where $\bar{F} = 2x\bar{i} + y\bar{j}$ and V is the volume bounded by the cylinder $z = 4 - x^2$ and the plane $x = 0$, $y = 0$, $y = 2$ and $z = 0$

Hint: The limits are
$z : 0 \to 4 - x^2$
$y : 0 \to 2$
$x : 0 \to 2$

$$\iiint_V \bar{F}\, dv = \iiint (2x\bar{i} + y\bar{j})\, dx\, dy\, dz$$

$$= 2\bar{i} \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} x\, dx\, dy\, dz + \bar{j}\int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} y\, dx\, dy\, dz$$

$z = 0$, $z = 4 - x^2$
$0 = 4 - x^2$
$x^2 = 4 = 2^2$
$x = 2$

Ans $\iiint_V \bar{F}\, dv = \frac{80}{3}$

* If $\phi = 45x^2y$, evaluate $\iiint_V \phi\, dv$, where V is the closed region bounded by the planes $4x + 2y + z = 8$, $y = 0$, $z = 0$, $x = 0$

eqn of xy-plane, $z = 0$, $4x + 2y = 8$
$2y = 8 - 4x \Rightarrow y = 4 - 2x$
eqn of x-axis $y = 0$, $z = 0$, $4x = 8$
$x = 2$

Ans $\iiint_V \phi\, dv = 128$

The limits are $z : 0 \to 8 - 4x - 2y$
$y : 0 \to \frac{8 - 4x}{2} = 4 - 2x$
$x : 0 \to 2$

* Evaluate $\int_V \bar{F}\, dv$, where $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and V is the region bounded by $x = 0$, $y = 0$, $y = 6$
$z = 4$, $z = x^2$

The limits are $z : x^2 \to 4$
$y : 0 \to 6$
$x : 0 \to 2$

G.T $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\int_V \bar{F}\, dV = \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x \, dx\, dy\, dz + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y \, dx\, dy\, dz + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} z \, dx\, dy\, dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} (z)_{x^2}^{4}\, x\, dx\, dy + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} (z)_{x^2}^{4}\, dx\, y\, dy + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6} \left(\frac{z^2}{2}\right)_{z=x^2}^{4} dx\, dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} (4-x^2)\, x\, dx\, dy + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} (x^2-4)\, dx\, y\, dy + \frac{\bar{k}}{2}\int_{x=0}^{2}\int_{y=0}^{6} (16-x^4)\, dx\, dy$$

$$= \bar{i}\left(4\left(\frac{x^2}{2}\right) - \left(\frac{x^4}{4}\right)\right)_{x=0}^{2} (y)_{y=0}^{6} + \bar{j}\left(\frac{x^3}{3} - 4x\right)_{x=0}^{2} \left(\frac{y^2}{2}\right)_{y=0}^{6} + \frac{\bar{k}}{2}\left(6x - \frac{x^5}{5}\right)_{x=0}^{2} (y)_{y=0}^{6}$$

$$= \bar{i}\left(2(4-0) - \frac{1}{4}(16-0)\right)(6-0) + \bar{j}\left(\frac{1}{3}(8-0) - 4(2-0)\right)\frac{1}{2}(36-0) + \bar{k}\frac{1}{2}\left(6(2-0) - \frac{1}{5}(32-0)\right)(6-0)$$

$$= \bar{i}(8-4)6 + \bar{j}\left(\frac{8}{3} - 8\right)(18) + \bar{k}\frac{1}{2}\left(12 - \frac{32}{5}\right)(6)$$

$$= \bar{i}\,24 + \bar{j}\left(\frac{8-24}{3}\right)(18) + \bar{k}\,3\left(\frac{60-32}{5}\right)$$

$$= 24\bar{i} - \frac{16}{3}\bar{j}(18) + \bar{k}\,3\left(\frac{28}{5}\right) = 24\bar{i} - 96\bar{j} + \frac{84}{5}\bar{k}$$

* If $\bar{F} = 2xz\,\bar{i} - x\bar{j} + y^2\bar{k}$, evaluate $\int_V \bar{F}\,dV$, where V is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$, $z=4$

Ans $\int_V \bar{F}\,dV = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz\,\bar{i} - x\bar{j} + y^2\bar{k})\,dx\,dy\,dz = 128\bar{i} - 24\bar{j} + 384\bar{k}$

$$= 2\bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} xz\,dx\,dy\,dz - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$$

$$= 2\bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\left(\frac{z^2}{2}\right)_{z=x^2}^{4} x\,dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}(z)_{x^2}^{4}\, x\,dx\,dy + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}(z)_{x^2}^{4}\, y^2\,dx\,dy$$

Vector integral Theorems :

Green's Theorem in a plane (Transformation b/w line integral and double integral)

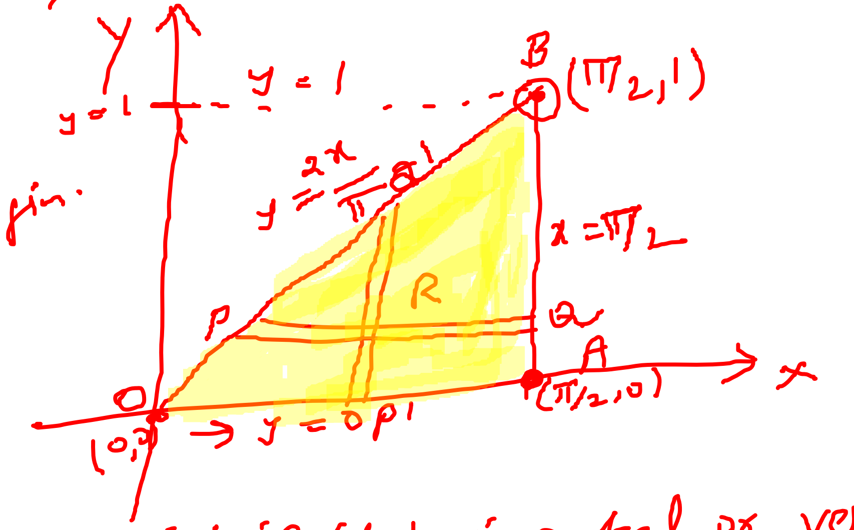
Statement: If R is a closed region in xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x & y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

, where C is traversed in the +ve direction (i.e Anticlockwise direction.

Problems:

* Apply green's theorem to evaluate $\int_C (y - \sin x)\,dx + \cos x\,dy$ where C is the plane triangle enclosed by the lines $y = 0$, $x = \pi/2$ and $\pi y = 2x$
i.e $y = \frac{2x}{\pi}$
$y = mx$ line passes through origin.



(strip is horizontal or vertical)

By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = y - \sin x$ & $N = \cos x$

$\frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial x} = -\sin x$

$$\oint_C (y-\sin x)\,dx + \cos x\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (-\sin x - 1)\,dx\,dy$$

The limits of integration are $x : \frac{y\pi}{2} \to \pi/2$

$y : 0 \to 1$

[∵ strip 'PQ' is horizontal ⇒ x_1, x_2 in terms of y; y_1, y_2 are constants]

$$\oint_C (y-\sin x)\,dx + \cos x\,dy = \int_{y=0}^{1}\left(\underbrace{\int_{x=\frac{y\pi}{2}}^{\pi/2}(-\sin x - 1)\,dx}_{y \text{ as constant}}\right)dy$$

$$= \int_{y=0}^{1}\left(-(-\cos x) - x\right)_{x=y\pi/2}^{\pi/2} dy = \int_{y=0}^{1}(\cos x - x)_{x=y\pi/2}^{\pi/2}\,dy$$

$$= \int_{y=0}^{1}\left(\left(\cos\pi/2 - \cos(y\pi/2)\right) - \left(\pi/2 - y\pi/2\right)\right)dy$$

$$= \int_{y=0}^{1}\left(\left(0 - \cos y\frac{\pi}{2}\right) - \left(\pi/2 - y\pi/2\right)\right)dy$$

$$= \left(-\frac{\sin y\frac{\pi}{2}}{\pi/2} - \pi/2(y) + \left(\frac{y^2}{2}\right)\frac{\pi}{2}\right)_{y=0}^{1} = -\frac{2}{\pi}\left(\sin\pi/2 - \sin 0\right) - \pi/2\,(1-0) + \frac{\pi}{4}\left(1^2 - 0^2\right)$$

$$= -\frac{2}{\pi}(1-0) - \pi/2 + \frac{\pi}{4} = -\frac{2}{\pi} - \pi/2 + \pi/4$$

$$= \frac{-8 - 2\pi^2 + \pi^2}{4\pi} = \frac{-8 - \pi^2}{4\pi}$$

$$\oint_C (y-\sin x)\,dx + \cos x\,dy = -\left(\frac{2}{\pi}+\frac{\pi}{4}\right)$$

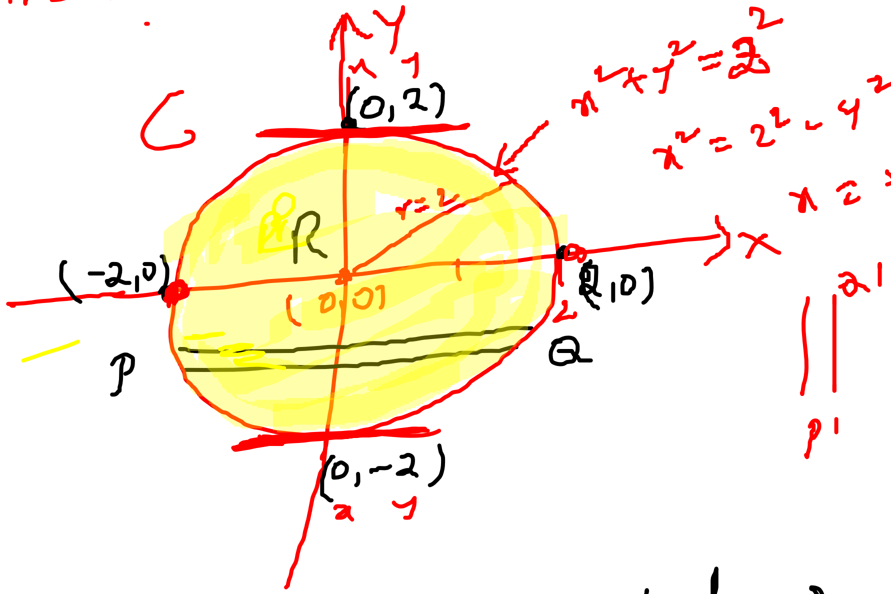
Hint: suppose strip P'Q' is vertical ⇒ y_1, y_2 are in terms of x; x_1, x_2 are constants

$y: 0 \to \frac{2x}{\pi}$

$x: 0 \to \pi/2$

$$\oint_C (y-\sin x)\,dx + \cos x\,dy = \int_{x=0}^{\pi/2}\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dx\,dy = -\left(\frac{2}{\pi}+\frac{\pi}{4}\right)$$

* Evaluate $\oint_C \underbrace{(3x+4y)}_{M}dx + \underbrace{(2x-3y)}_{N}dy$ where C is the circle $x^2+y^2=4$.



$x^2+y^2=2^2$

$x^2 = 2^2 - y^2$

$x = \pm\sqrt{2^2-y^2}$

By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

Here $M = 3x+4y$ & $N = 2x-3y$

$\frac{\partial M}{\partial y} = 4$ & $\frac{\partial N}{\partial x} = 2$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2-4 = -2$

$$\oint_C (3x+4y)\,dx + (2x-3y)\,dy = \iint_R -2\,dx\,dy$$

The limits of integration are $x: -\sqrt{4-y^2}$ to $\sqrt{4-y^2}$ (Horizontal strip PQ)

$y: -2$ to 2

Strip PQ is horizontal ⇒ x_1, x_2 are in terms of y; y_1, y_2 are constants

vertical strip: $y: -\sqrt{2^2-x^2}$ to $\sqrt{2^2-x^2}$

$x: -2$ to 2

$$\oint (3x+4y)\,dx + (3x-2y)\,dy = \int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} -2\,dx\,dy = -2\int_{y=-2}^{2}\left(\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 1\,dx\right)dy$$

$$= -2\int_{y=-2}^{2} 2\left(\int_{x=0}^{\sqrt{4-y^2}} 1\,dx\right)dy$$

$\left[\because f(x)=1\right.$

$f(-x)=1 = f(x) \Rightarrow f(x)$ is even

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is even

$= 0$, if $f(x)$ is odd $\left.\right]$

$f(-x) = -f(x) \Rightarrow f(x)$ is odd

$$= -4\int_{=-2}^{2} (x)_0^{\sqrt{4-y^2}}\,dy$$

$$= -4\int_{y=-2}^{2} \left(\sqrt{4-y^2}-0\right)dy$$

$$= -4\left(2\int_{y=0}^{2}\sqrt{4-y^2}\,dy\right)$$

$\left[\because f(y)=\sqrt{4-y^2}\right.$

$f(-y)=\sqrt{4-(-y)^2} = \sqrt{4-y^2} = f(y) \Rightarrow f(y)$ is an even fun $\left.\right]$

$$= -8\left[\int_{y=0}^{2}\sqrt{2^2-y^2}\,dy\right]$$

$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + C\right]$

$$= -8\left[\frac{y}{2}\sqrt{2^2-y^2} + \frac{2^2}{2}\sin^{-1}(y/2)\right]_{y=0}^{2}$$

$$= -8\left[\frac{1}{2}(0-0) + \frac{4}{2}\left(\sin^{-1}(1) - \sin^{-1}(0)\right)\right]$$

$$= -8\left(\frac{4}{2}\left(\sin^{-1}(\sin\pi/2) - \sin^{-1}(\sin 0)\right)\right) = -8\left[\frac{4}{2}\left(\pi/2 - 0\right)\right] = \frac{-32}{2}\frac{\pi}{2} = -8\pi$$

Aliter: In polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$ so that $dx\,dy = |J|\,dr\,d\theta$, where $J = J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = r$

i.e. $dx\,dy = r\,dr\,d\theta$ and $r : 0 \to 2$

$\theta : 0 \to 2\pi$

$$\oint_C (2x+4y)\,dx + (2x-3y)\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} -2\; r\,dr\,d\theta$$

$$= -2\left(\frac{r^2}{2}\right)_{r=0}^{2}(\theta)_0^{2\pi} = -(2^2-0^2)(2\pi-0)$$

$$= -4(2\pi) = -8\pi$$

* Apply Green's theorem to evaluate $\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$, where C is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$

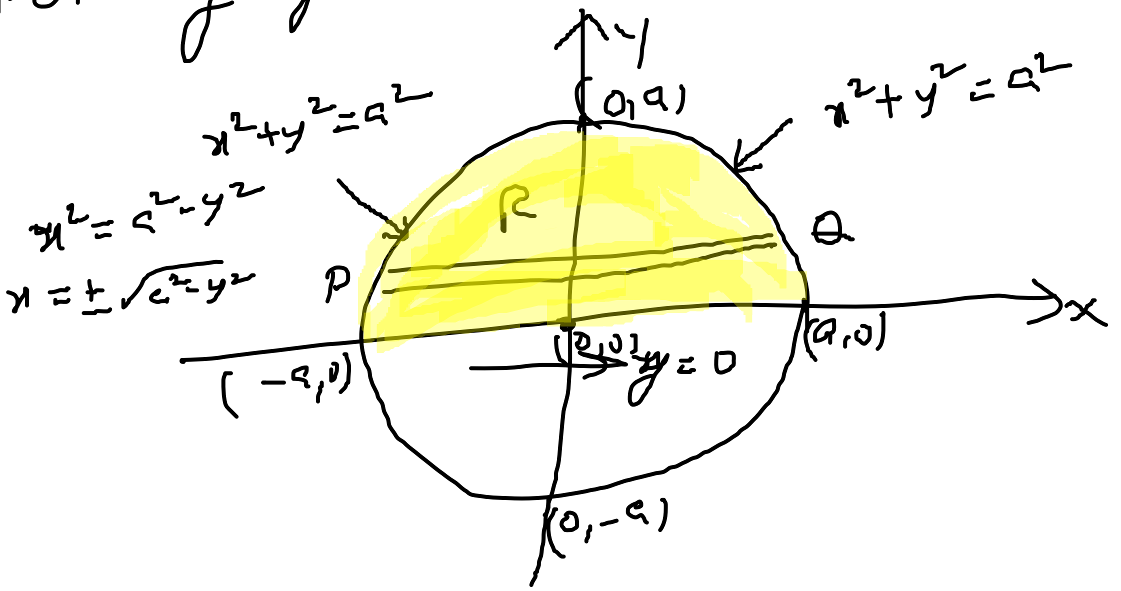
By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \oint_C M\,dx + N\,dy \quad \text{say}$$

Here $M = 2x^2-y^2$ & $N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y$ & $\frac{\partial N}{\partial x} = 2x$



$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - (-2y) = 2(x+y)$$

The limits of integration are $x : -\sqrt{a^2-y^2}$ to $\sqrt{a^2-y^2}$

$y : 0$ to a

$$\oint_C (2x^2+y^2)\,dx + (x^2+y^2)\,dy = \int_{y=0}^{a}\left(\int_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 2(x+\underbrace{y}_{y \text{ as constant}})\,dx\right)dy$$

$$= 2\int_{y=0}^{a}\left(\frac{x^2}{2} + yx\right)_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy$$

$$= 2\int_{y=0}^{a}\left[\frac{1}{2}\left(\underbrace{a^2-y^2}_{U.L} - \underbrace{(a^2-y^2)}_{L.L}\right) + y\left(\underbrace{\sqrt{a^2-y^2}}_{U.L} - \underbrace{\left(-\sqrt{a^2-y^2}\right)}_{L.L}\right)\right]dy$$

$$= 2\int_{y=0}^{a}\left(\frac{1}{2}(0) + y\,2\sqrt{a^2-y^2}\right)dy$$

$$= 2\int_{y=0}^{a} 2y\sqrt{a^2-y^2}\,dy$$

$$= 2\int_{t=a^2}^{0}\sqrt{t}\,(-dt)$$

Put $a^2-y^2 = t$

$-2y\,dy = dt$

$2y\,dy = -dt$

As $y : 0 \to a$

$t : a^2 \to 0$

$$= 2\left(-\int_{t=a^2}^{0}\sqrt{t}\,dt\right)$$

$$= 2\left[\int_{t=0}^{a^2} t^{1/2}\,dt\right]$$

$$= 2\left(\frac{t^{3/2}}{3/2}\right)_{t=0}^{a^2}$$

$$= 2\left(\frac{2}{3}\right)\left((a^2)^{3/2} - 0\right)$$

$$= \frac{4}{3}(a^3)$$

$$= \frac{4}{3}a^3$$

~~Suppose~~ PQ strip is vertical

$y : 0 \to \sqrt{a^2-x^2}$

$x : -a \to a$

Aliter:

$$\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \iint_R 2(x+y)\,dx\,dy$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ so that $dx\,dy = r\,dr\,d\theta$ and $r: 0 \to a$
$\theta: 0 \to \pi$

$$\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \int_{\theta=0}^{\pi}\int_{r=0}^{a} 2(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a} 2r(\cos\theta+\sin\theta)\,r\,dr\,d\theta.$$

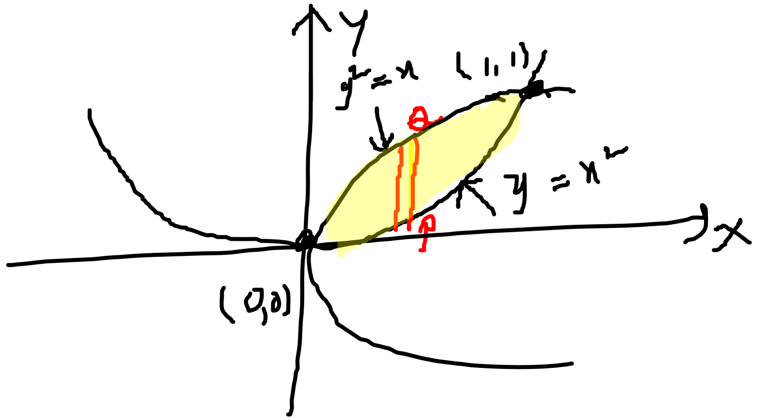
$$= 2\left(\frac{r^3}{3}\right)_{r=0}^{a}\left(+\sin\theta - \cos\theta\right)_{\theta=0}^{\pi}$$

$$= \frac{2}{3}(a^3-0)\left(\sin\pi - \sin 0 - (\cos\pi - \cos 0)\right)$$

$$= \frac{2}{3}a^3\left(0-0-(-1-1)\right)$$

$$= \frac{2}{3}a^3\left(-(-2)\right) = \frac{4}{3}a^3$$

* Using Green's theorem, evaluate $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$, where C is the closed curve of the region bounded by $y = x^2$ & $y^2 = x$



Given curves are $y = x^2$ —(1)

$y^2 = x$ —(2)

solve equations (1) & (2), we get point of intersection

sub (2) in (1), we get

$$y = (y^2)^2 \Rightarrow y^4 - y = 0$$
$$y(y^3-1) = 0$$
$$y = 0 \text{ or } y^3 - 1 = 0$$
$$y^3 = 1 = 1^3$$
$$y = 1$$

Put $y=0$ in (1) (or (2)) $\Rightarrow x^2 = 0 \Rightarrow x = 0 \Rightarrow (0,0)$

Put $y=1$ in (1) (or (2)) $\Rightarrow 1 = x^2 \Rightarrow x = 1 \Rightarrow (1,1)$

(0,0) (1,1) are intersecting points of curves (1) & (2)

strip is vertical $\Rightarrow$

y_1, y_2 are in terms of x

x_1, x_2 are constants

The limits of integration are $y : x^2 \to \sqrt{x}$

$x : 0 \to 1$

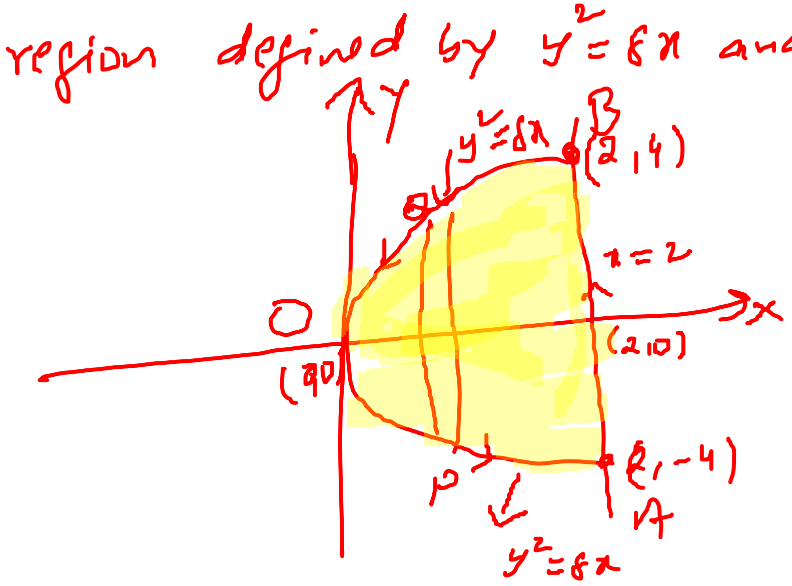
Here $M = 2xy - x^2$ & $N = x^2 + y^2$

$\frac{\partial M}{\partial y} = 2x$ & $\frac{\partial N}{\partial x} = 2x$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$$

$$\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 0 \; dx\,dy = 0$$

* Evaluate $\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$ around the boundary of the region defined by $y^2=8x$ and $x=2$



(i) directly (ii) By using Green's theorem

Given curves are $y^2=8x$ —— (1)

$x=2$ —— (2)

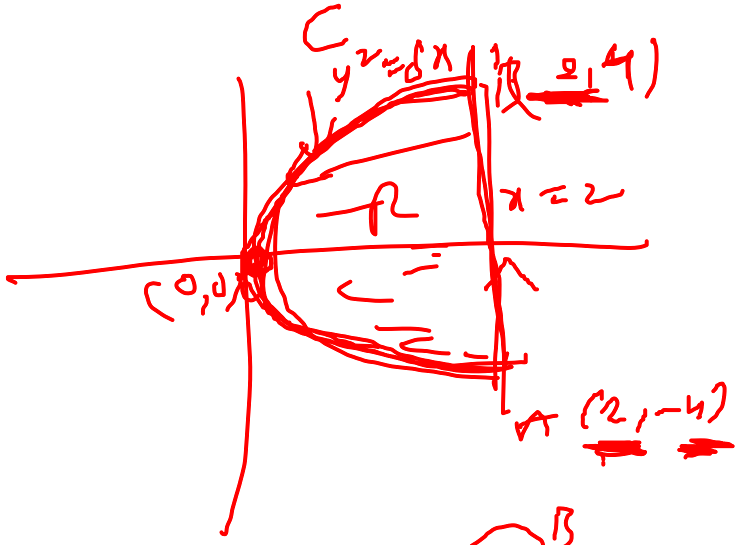
Solve (1) & (2), we get point of intersection

Sub (2) in (1), we get

$y^2 = 16 \Rightarrow y^2=4^2$

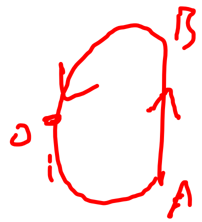
$y=\pm 4$

(2,4) (2,-4) are points of intersection of the curves (1) & (2)



$$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

(i) directly



$$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = \int_{AB} + \int_{BOA} \quad \text{—— (3)}$$

Along AB: Eqn of AB is $x = 2$, $dx = 0$ and $y: -4$ to 4

$$\int_{AB} (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy = \int_{y=-4}^{4} \underbrace{(4 - 4y)(0)} + (4y+3)\,dy$$

$$= \int_{y=-4}^{4} (4y+3)\,dy$$

$$= \left.\frac{(4y+3)^2}{2(4)}\right)_{y=-4}^{4} = \frac{1}{8}\left((19)^2 - (-13)^2\right)$$

$$= \frac{1}{8}(361 - 169) = \frac{192}{8} = 24 \quad \text{—— } 4$$

Along the curve BOA: Eqn of BOA is $y^2 = 8x$

$$x = \frac{y^2}{8}$$

$$dx = \frac{2y\,dy}{8} = \frac{y\,dy}{4} \quad \text{and} \quad y: 4 \to -4$$

$$\int_{BOA} (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy = \int_{y=4}^{-4} \left(\frac{y^4}{64} - 2\frac{y^2}{8}\cdot y\right)\left(\frac{y}{4}dy\right) + \left(\frac{y^4}{64}\cdot y + 3\right)dy$$

$$= \int_{y=4}^{-4} \left(\underset{\text{odd}}{\frac{y^5}{256}} - \underset{\text{even}}{\frac{y^4}{16}} + \underset{\text{odd}}{\frac{y^5}{64}} + \underset{\text{even}}{3}\right) dy = -\int_{y=-4}^{4} \left(\underset{\text{odd}}{\frac{y^5}{256}} - \underset{\text{even}}{\frac{y^4}{16}} + \underset{\text{odd}}{\frac{y^5}{64}} + \underset{\text{even}}{3}\right) dy$$

$$= -\int_{y=-4}^{4}\left(\frac{-y^4}{16}+3\right)dy$$

$$= -2\int_{y=0}^{4}\left(\frac{-y^4}{16}+3\right)dy = -2\left[\frac{-1}{16}\left(\frac{y^5}{5}\right)+3(y)\right]_{y=0}^{4} = -2\left[\frac{-1}{16}\left(\frac{1}{5}\right)(4^5-0)+3(4-0)\right]$$

$$= -2\left[\frac{-1}{80}4^5+12\right]$$

$$= -2\left[\frac{-1024}{80}+12\right]$$

$$= \frac{1024}{40}-24 = 1.6 = \frac{8}{5} \quad \text{—— (5)}$$

Sub (4) & (5) in (3), we get

$$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = 24+\frac{8}{5} = \frac{120+8}{5} = \frac{128}{5}$$

(ii) <u>By using Green's theorem :</u>

By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = \oint_C M\,dx + N\,dy$ say

Here $M = x^2-2xy$ & $N = x^2y+3$

$\frac{\partial M}{\partial y} = -2x$ & $\frac{\partial N}{\partial x} = 2xy$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2xy - (-2x) = 2x(y+1)$

The limits of integration are y: $-\sqrt{8x}$ to $\sqrt{8x}$

x: 0 to 2

$\therefore \oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = \int_{x=0}^{2}\left(\int_{y=-\sqrt{8x}}^{\sqrt{8x}} \underbrace{2x(y+1)}_{x \text{ as constant}}\,dy\right)dx$

$= \int_{x=0}^{2} \cancel{2}x \left(\frac{(y+1)^2}{\cancel{2}(1)}\right)_{y=-\sqrt{8x}}^{\sqrt{8x}} dx$

$= \int_{x=0}^{2} x\left((\sqrt{8x}+1)^2 - (-\sqrt{8x}+1)^2\right) dx$

$= \int_{x=0}^{2} x\left(\cancel{8x}+\cancel{1}+2\sqrt{8x} - (\cancel{8x}+\cancel{1}-2\sqrt{8x})\right) dx$

$= \int_{x=0}^{2} x\,(+4\sqrt{8x})\,dx$

$= +4\sqrt{8}\int_{x=0}^{2} x\,(x)^{1/2}\,dx$

$= +4\sqrt{8}\int_{x=0}^{2} x^{3/2}\,dx$

$= +4\sqrt{8}\left(\frac{x^{5/2}}{5/2}\right)_{x=0}^{2}$

$= +4\sqrt{8}\left(\frac{2}{5}\right)(2^{5/2}-0)$

$= \frac{+8\sqrt{8}}{5}(\sqrt{2})^5$

$= \frac{+8\sqrt{8}\;4\sqrt{2}}{5} = \frac{+32}{5}\sqrt{16}$

$= \frac{+32}{5}(4)$

$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = \frac{+128}{5}$

Verification problems:

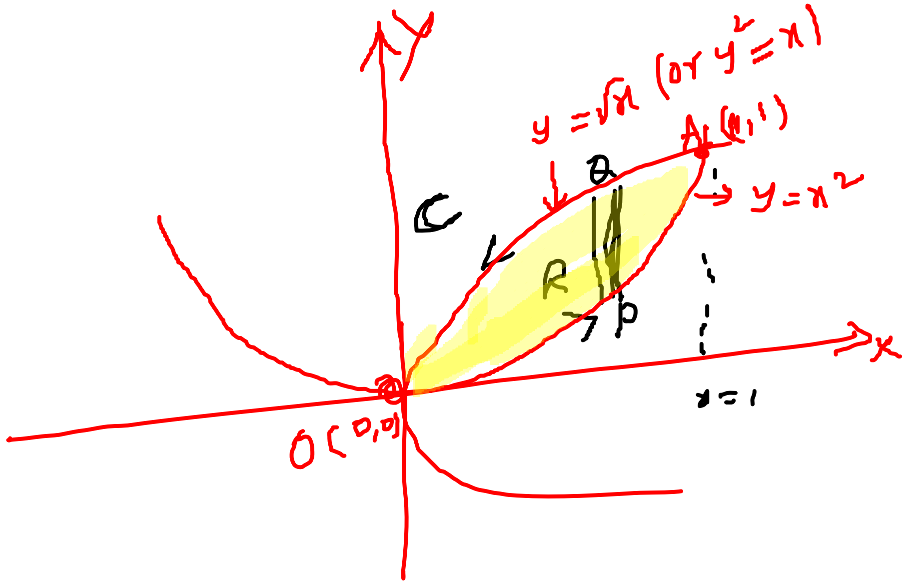
Verify Green's theorem in plane for $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the region bounded by $y=\sqrt{x}$ and $y=x^2$

$y^2=x \rightarrow y^2=4ax$ curve symmetric about x-axis

$x^2=4ay \rightarrow$ symmetric about y-axis

By Green's theorem, we have

$$\oint_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy$$



$$L.H.S = \oint_C M\,dx+N\,dy = \oint_C (3x^2-8y^2)\,dx+(4y-6xy)\,dy$$

where C is the region bounded by $y=\sqrt{x}$ —— (1)

$y=x^2$ —— (2)

Solve (1) & (2), we get point of intersection

sub (2) in (1), we get

$x^2=\sqrt{x}$

$(x^2)^2=(\sqrt{x})^2$

$x^4=x \Rightarrow x^4-x=0$

$x(x^3-1)=0$

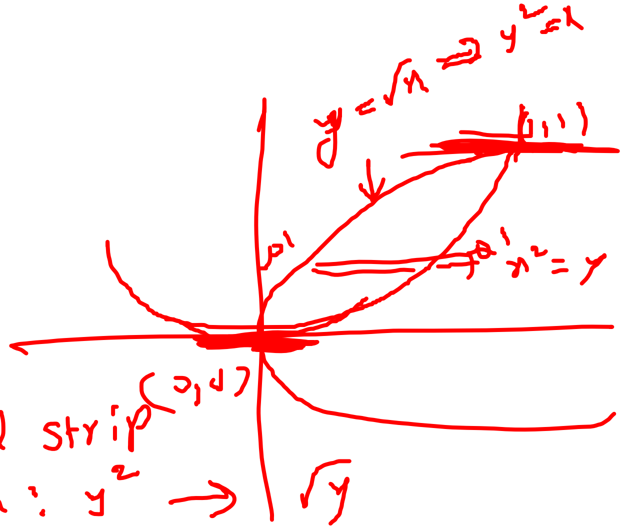
$x=0$ or $x^3-1=0$

$x^3=1=1^3$

$x=1$

put $x=0$ sub in (1) (or (2)), we get $y=0 \Rightarrow (0,0)$

$x=1$ sub in (1) (or (2)), we get $y=\sqrt{1}=1 \Rightarrow (1,1)$



P'Q' is Horizontal strip

$x: y^2 \rightarrow \sqrt{y}$

$y: 0 \rightarrow 1$

$$L.H.S = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{OA} + \int_{AO} \quad \text{—— (1)}$$

Along OA: Equ of OA is $y = x^2$ & $x : 0 \to 1$

$dy = 2x\,dx$

$$\int_{OA} (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{x=0}^{1} (3x^2 - 8(x^4))\,dx + \int (4x^2 - 6x(x^2))\,2x\,dx$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \int_{x=0}^{1} (-20x^4 + 8x^3 + 3x^2)\,dx = -20\left(\frac{x^5}{5}\right)_0^1 + 8\left(\frac{x^4}{4}\right)_0^1 + 3\left(\frac{x^3}{3}\right)_0^1$$

$$= -4(1-0) + 2(1-0) + (1-0)$$

$$= -4+2+1 = -1 \quad \text{—— (2)}$$

Along AO: Equ of AO is $y = \sqrt{x}$ (or $y^2 = x$) and $y : 1 \to 0$

$dx = 2y\,dy$

$$\int_{AO} (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{y=1}^{0} (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^2 y)\,dy$$

$$= \int_{y=1}^{0} (6y^5 - 16y^3 + 4y - 6y^3)\,dy = \int_{y=1}^{0} (6y^5 - 22y^3 + 4y)\,dy$$

$$= 6\left(\frac{y^6}{6}\right)_1^0 - 22\left(\frac{y^4}{4}\right)_1^0 + 4\left(\frac{y^2}{2}\right)_1^0$$

$$= \frac{6}{6}(0-1) - \frac{22}{4}(0-1) + 2(0-1)$$

$$= -1 + \frac{22}{4} - 2$$

$$= \frac{-4+22-8}{4} = \frac{-12+22}{4} = \frac{10}{4} = \frac{5}{2} \quad \text{------} \ (3)$$

Sub (2) & (3) in (1), we get

L.H.S = $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = -1 + \frac{5}{2} = 3/2$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

we have $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y - (-16y) = -6y + 16y = 10y$$

The limits of integration are (i.e strip PQ is vertical $\Rightarrow y_1, y_2$ are in terms of x & x_1, x_2 are constants)

$y : x^2 \to \sqrt{x}$

$x : 0 \to 1$

$$\therefore R.H.S = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10\, y\, dx\, dy$$

$$= \int_{x=0}^{1} 10\left(\int_{y=x^2}^{\sqrt{x}} y\, dy\right) dx = \int_{x=0}^{1} 10 \left(\frac{y^2}{2}\right)_{y=x^2}^{\sqrt{x}} dx$$

x as constant

$$= \frac{10}{2}\int_{x=0}^{1} \left((\sqrt{x})^2 - (x^2)^2\right) dx$$

$$= 5\int_{x=0}^{1} (x - x^4)\, dx = 5\left(\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^5}{5}\right)_0^1\right)$$

$$= 5\left(\frac{1}{2}(1-0) - \frac{1}{5}(1-0)\right)$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = 5\left(\frac{5-2}{10}\right) = \frac{5(3)}{10}$$

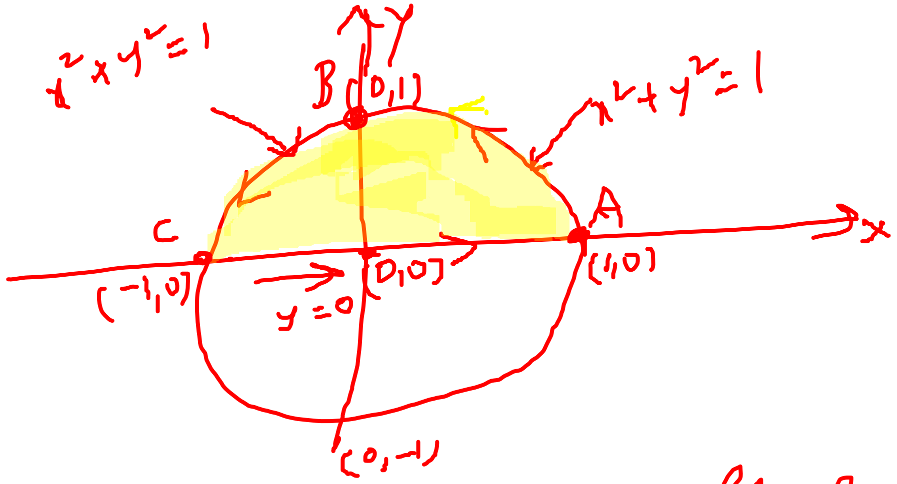
$$= \frac{3}{2}$$

$$\therefore R.H.S = \frac{3}{2}$$

$$\therefore L.H.S = R.H.S = 3/2$$

Hence Green's theorem is verified.

Verify Green's theorem in a plane with respect to $\oint_C (2x^2-y^2)dx + (x^2+y^2)dy$, where c is the boundary of the region in the xoy plane enclosed by the x-axis and the upper half of the circle $x^2+y^2=1$



By Green's theorem, we have

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

L.H.S $= \oint_C M dx + N dy = \oint_C (2x^2-y^2)dx + (x^2+y^2)dy$, where c is the upper half of the circle $x^2+y^2=1$

$$= \int_{ABC} + \int_{CA} \quad \text{——— (1)}$$

Along ABC : Equ of ABC is $x^2+y^2=1$ [∵ $r=1$, $C(0,0)$]

The parametric equations are $x = r\cos\theta = 1\cos\theta$, $y = r\sin\theta = 1\sin\theta$ and $\theta : 0 \to \pi$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\therefore \int_{ABC} (2x^2-y^2)dx + (x^2+y^2)dy = \int_{\theta=0}^{\pi} \left(2(\cos^2\theta) - \sin^2\theta\right)(-\sin\theta)d\theta + (\cos^2\theta + \sin^2\theta)\cos\theta\, d\theta \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \int_{\theta=0}^{\pi} \left(-2\cos^2\theta \sin\theta + \sin^3\theta + 1\cos\theta\right) d\theta$$

$$= \int_{\theta=0}^{\pi} \left(-2(1-\sin^2\theta)\sin\theta + \sin 3\theta + \cos\theta\right) d\theta$$

$$= \int_{\theta=0}^{\pi} \left(-2\sin\theta + \underbrace{2\sin^3\theta + \sin^3\theta}_{3\sin^3\theta} + \cos\theta\right) d\theta$$

$$\left[\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta \quad 4\sin^3\theta = 3\sin\theta - \sin 3\theta\right]$$

$$= \int_{\theta=0}^{\pi} \left(-2\sin\theta + 3\left(\frac{3\sin\theta - \sin 3\theta}{4}\right) + \cos\theta\right) d\theta$$

$$= -2\left(\frac{-\cos\theta}{1}\right)_0^{\pi} + \frac{3}{4}\left[3\left(\frac{-\cos\theta}{1}\right) - \left(\frac{-\cos 3\theta}{3}\right)\right]_{\theta=0}^{\pi} + \left(\frac{\sin\theta}{1}\right)_{\theta=0}^{\pi}$$

$$\left(\because \cos n\pi = (-1)^n, n \in \mathbb{Z} \quad \sin n\pi = 0, \forall n \in \mathbb{Z}\right)$$

$$= 2(\cos\pi - \cos 0) + \frac{3}{4}\left(-3(\cos\pi - \cos 0) + \frac{1}{3}(\cos 3\pi - \cos 3(0))\right) + (\sin\pi - \sin 0)$$

$$= 2(-1-1) + \frac{3}{4}\left(-3(-1-1) + \frac{1}{3}(-1-1)\right) + (0-0)$$

$$= -4 + \frac{3}{4}\left(6 - \frac{2}{3}\right).$$

$$= -4 + \frac{\cancel{3}}{4}\left(\frac{18-2}{\cancel{3}}\right) = -4 + \frac{16}{4} = -4+4 = 0$$

$$\int_{ABC} (2x^2 - y^2)\,dx + (x^2+y^2)\,dy = 0 \quad \text{——} \quad (2)$$

Along CA : Equ of CA is $y = 0$, $dy = 0$ and $x : -1 \to 1$

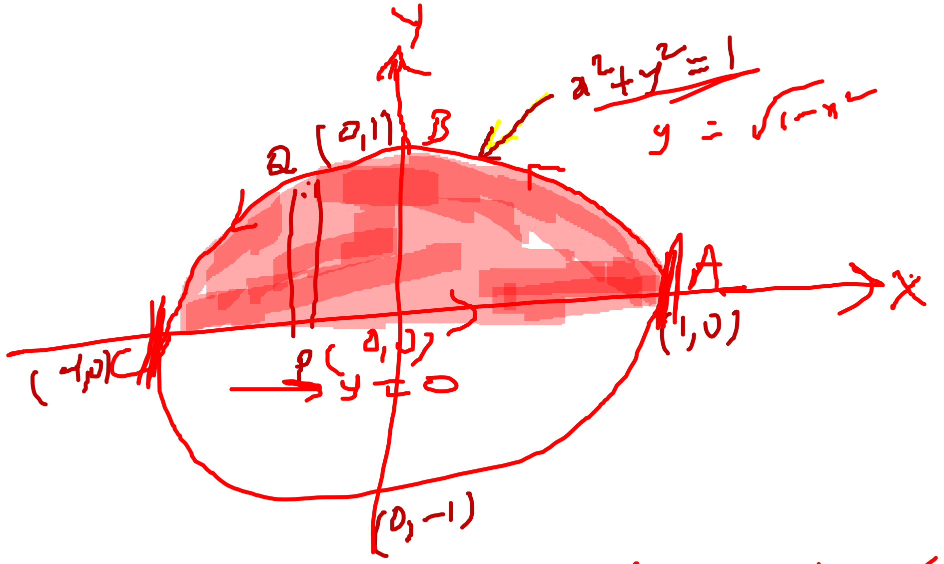
$$\int_{CA} (2x^2-y^2)\,dx + \cancel{(x^2+y^2)\,dy} = \int_{x=-1}^{1} (2x^2-0)\,dx + (x^2+0)\,0 = \int_{x=-1}^{1} 2x^2\,dx = 2\left(\frac{x^3}{3}\right)_{-1}^{1}$$

$$= \frac{2}{3}\left((1)^3-(-1)^3\right) = \frac{2}{3}\left(1-(-1)\right) = \frac{4}{3}$$

(3)

Sub (2) & (3) in (1), we get

$$L.H.S = \oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = 0 + \frac{4}{3} = 4/3$$



$$R.H.S = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

We have $M = 2x^2-y^2$ & $N = x^2+y^2$

$$\frac{\partial M}{\partial y} = -2y \quad \& \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x-(-2y) = 2(x+y)$$

The limits of integration are (i.e strip PQ is vertical $\Rightarrow$ y_1, y_2 are in terms of x & x_1, x_2 are constants)

Aliter $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$r : 0 \to 1$

$\theta : 0 \to \pi$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{1} 2(r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{1} 2r^2\,dr\,(\cos\theta+\sin\theta)\,d\theta$$

$$2\left(\frac{r^3}{3}\right)_0^1 \left(\sin\theta - \cos\theta\right)_{\theta=0}^{\pi}$$

$$\frac{2}{3}(1-0)\left(0-(-1-1)\right) = \frac{4}{3}$$

$y: 0 \to \sqrt{1-x^2}$

$x: -1 \to 1$

$$R.H.S = \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} 2(x+y)\, dx\, dy$$

$$= 2\int_{x=-1}^{1} \left[\int_{y=0}^{\sqrt{1-x^2}} \underbrace{(x+y)\, dy}_{x \text{ as constant}} \right] dx$$

$$= 2\int_{x=-1}^{1} \left(xy + \frac{y^2}{2} \right)_{y=0}^{\sqrt{1-x^2}} dx$$

$$= 2\int_{x=-1}^{1} \left(x\left(\sqrt{1-x^2} - 0\right) + \frac{1}{2}\left(\left(\sqrt{1-x^2}\right)^2 - 0\right)\right) dx$$

$$= 2\int_{x=-1}^{1} \left(\underbrace{x\sqrt{1-x^2}}_{\text{odd}} + \underbrace{\frac{1}{2}(1-x^2)}_{\text{even}} \right) dx$$

$\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ if } f(x) \text{ is even (i.e } f(-x) = f(x)) \right.$

$\left. = 0, \text{ if } f(x) \text{ is odd (i.e } f(-x) = -f(x) \right]$

$$R.H.S = 2\left[2\int_{x=0}^{1} \frac{1}{2}(1-x^2)\, dx \right] = 2\left(x - \frac{x^3}{3} \right)_0^1 = 2\left((1-0) - \frac{1}{3}(1-0)\right) = 2\left(1 - \frac{1}{3}\right) = 2(2/3)$$

$R.H.S = 4/3$

$\therefore L.H.S = R.H.S = 4/3$

* verify Green's theorem in a plane to evaluate $\oint_C x^2(1+y)dx + (x^3+y^3)dy$, where C is the square formed by $x = \pm 1$ & $y = \pm 1$

By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

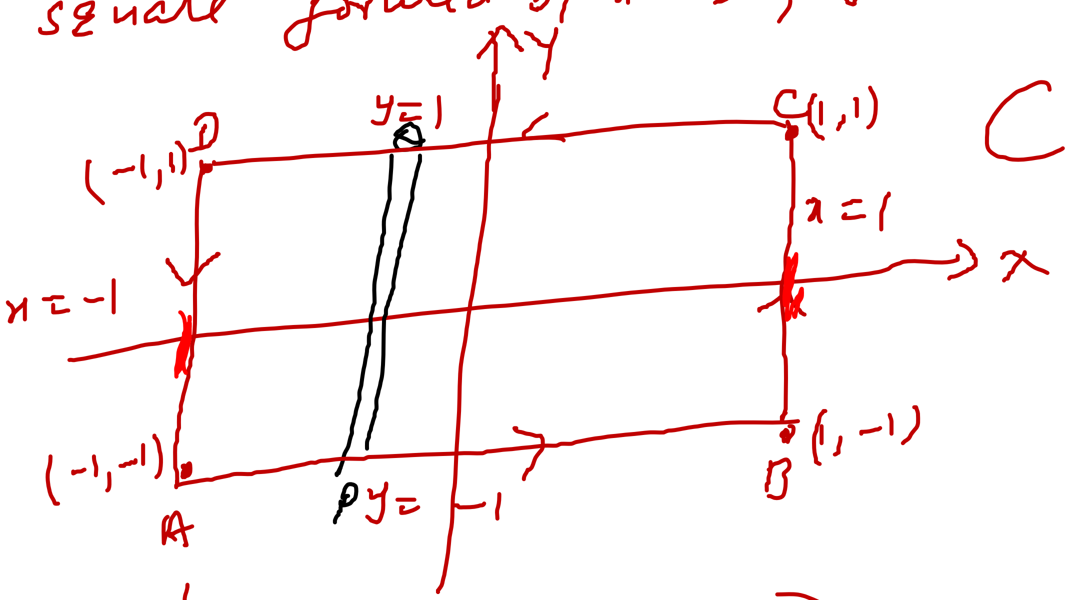
Here $M = x^2(1+y)$ & $N = x^3+y^3$

$$\frac{\partial M}{\partial y} = x^2(0+1) = x^2, \quad \frac{\partial N}{\partial x} = 3x^2$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x^2 - x^2 = 2x^2$$

$$L.H.S = \oint_C M\,dx + N\,dy = \oint_C \underbrace{x^2(1+y)dx + (x^3+y^3)dy}$$, where C is the square formed by $x = \pm 1$, $y = \pm 1$

$$= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \quad \text{——} \ (1)$$



Along AB: Equation of AB is $y = -1$, $dy = 0$ & $x : -1$ to 1

$$\int_{AB} \left(x^2(1+y)dx + (x^3+y^3)dy\right) = \int_{x=-1}^{1} x^2(1-1)dx + (x^3+(-1)^3)0 = \int_{x=-1}^{1} 0\,dx = 0 \quad \text{——} \ (2)$$

Along BC: Eqn of BC is $x = 1$, $dx = 0$ and $y : -1$ to 1

$$\int_{BC} x^2(1+y)dx + (x^3+y^3)dy = \int_{y=-1}^{1} 0 + (1+y^3)\,dy = \left(y + \frac{y^4}{4}\right)_{y=-1}^{1} = (1-(-1)) + \frac{1}{4}(1-(-1)^4)$$

$$= 2 + \frac{0}{4} = 2 \quad \text{——} \ (3)$$

Along CD : eqn of CD is $y=1$, $dy=0$ and $x: 1 \to -1$

$$\int_{CD} x^2(1+y)\,dx + (x^3+y^3)\,dy = \int_{x=1}^{-1} x^2(1+1)\,dx + 0 = 2\left(\frac{x^3}{3}\right)_{x=1}^{-1}$$

$$= \frac{2}{3}\left((-1)^3 - (1)^3\right)$$

$$= \frac{2}{3}(-1-1) = \frac{-4}{3} \quad \text{--- (4)}$$

Along DA : Eqn of DA is $x=-1$, $dx=0$ and $y: 1 \to -1$

$$\int_{DA} x^2(1+y)\,dx + (x^3+y^3)\,dy = \int_{y=1}^{-1} 0 + (-1+y^3)\,dy = \left(-y + \frac{y^4}{4}\right)_{y=1}^{-1}$$

$$= -(-1-1) + \frac{1}{4}\left((-1)^4 - (1)^4\right)$$

$$= 2 - \frac{1}{4}(1-1) = 2 \quad \text{--- (5)}$$

Sub (2), (3), (4) & (5) in (1), we get

$$L.H.S = \oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy = 0 + 2 - \frac{4}{3} + 2 = -4/3 + 4$$

$$= \frac{-4+12}{3} = \frac{8}{3}$$

$L.H.S = 8/3$

$R.H.S = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

The limits of integration are (i.e strip PQ is vertical $\Rightarrow$ y_1, y_2 interms of x
x_1, x_2 are constants)

$y : -1 \to 1$

$x : -1 \to 1$

$R.H.S = \int_{x=-1}^{1}\int_{y=-1}^{1} x^2\,dx\,dy = \int_{x=-1}^{1} 2x^2\,dx \cdot \int_{y=-1}^{1} 1 \cdot dy$

[$\because f(x) = x^2$
$f(-x) = (-x)^2 = x^2 = f(x) \Rightarrow f(x)$ is even

$f(y) = 1$
$f(-y) = 1 = f(y) \Rightarrow f(y)$ is even]

$= 2\left(2\int_{x=0}^{1} x^2\,dx\right)\left(2\int_{y=0}^{1} dy\right)$

$= 2\left(2\left(\frac{x^3}{3}\right)_0^1\right)\left(2\left(y\right)_0^1\right)$

$R.H.S = \frac{4}{3}(1-0)\ 2(1-0) = \frac{8}{3}$

$\therefore L.H.S = R.H.S = \frac{8}{3}$

* verify Green's theorem in the plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region defined by $x=0$, $y=0$, $x+y=1$

By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

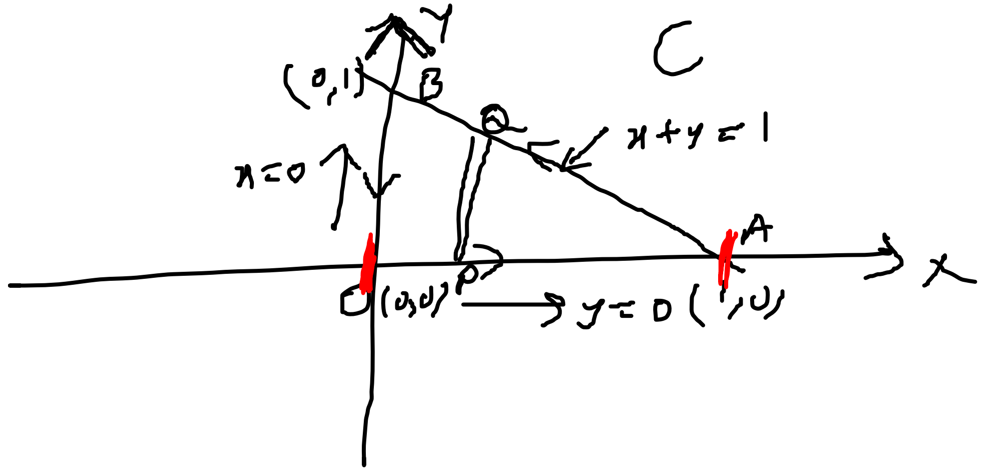
Here $M = 3x^2-8y^2$ $\quad N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y-(-16y) = 10y$

$$L.H.S = \oint_C M\,dx + N\,dy = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_{OA} + \int_{AB} + \int_{BO} \quad \text{--- (1)}$$



Along OA: Equ of OA is $y=0$, $dy=0$ and $x: 0 \to 1$

$$\int_{OA} (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{x=0}^{1} 3x^2\,dx = \cancel{3}\left(\frac{x^3}{\cancel{3}}\right)_{x=0}^{1} = 1-0 = 1 \quad \text{--- (2)}$$

Along AB: Equ of AB is $x+y=1$ & $x: 1 \to 0$

$y = 1-x$

$\frac{dy}{dx} = 0-1 \Rightarrow dy = -dx$

$$\int_{AB} (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{x=1}^{0} (3x^2 - 8(1-x)^2)\,dx + (4(1-x) - 6x(1-x))(-dx)$$

$$= \int_{x=1}^{0} \left(3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)\right) dx$$

$$= \int_{x=1}^{0} \left(3x^2 - 8(1+x^2-2x) - 4 + 4x + 6x - 6x^2\right) dx$$

$$= \int_{x=1}^{0} 3x^2 - 8 - 8x^2 + 16x - 4 + 10x - 6x^2)\,dx = \int_{x=1}^{0} (-11x^2 + 26x - 12)\,dx$$

$$= \left(-11\left(\frac{x^3}{3}\right) + 26\left(\frac{x^2}{2}\right) - 12x\right)_{x=1}^{0}$$

$$= -\frac{11}{3}(0-1) + 13(0-1) - 12(0-1) = +\frac{11}{3} - 13 + 12$$

$$= \frac{11}{3} - 1 = \frac{11-3}{3} = 8/3$$

Along BO: equ of BO is $x=0$, $dx=0$ and $y: 1 \to 0$

$$\int_{BO} (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{y=1}^{0} 0 + (4y-0)\,dy = 4\left(\frac{y^2}{2}\right)_{y=1}^{0} = 2(0-1^2) = -2 \quad \text{—— (4)}$$

Sub (2), (3) & (4) in (1), we get

$$L.H.S = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = 1 + 8/3 - 2 = -1 + 8/3 = 5/3$$

$$R.H.S = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R 10y\,dx\,dy$$

The limits of integration are (i.e. strip is vertical $\Rightarrow$ y_1, y_2 are in terms of x
x_1, x_2 are constants)

y : $0 \longrightarrow 1-x$

x : $0 \longrightarrow 1$

$$R.H.S = \int_{x=0}^{1} \int_{y=0}^{1-x} 10y\,dx\,dy$$

$$= 10\int_{x=0}^{1} \left(\int_{y=0}^{1-x} y\,dy \right) dx$$

$$= 10\int_{x=0}^{1} \left(\frac{y^2}{2}\right)_{y=0}^{1-x} dx$$

$$= \frac{10}{2}\int_{x=0}^{1} \left((1-x)^2 - 0\right) dx$$

$$R.H.S = 5\left(\frac{(1-x)^3}{3(-1)}\right)_{x=0}^{1} = -\frac{5}{3}(0-1) = 5/3$$

$\therefore L.H.S = R.H.S = 5/3$

H.W

* Verify Green's theorem for $\int_C (xy+y^2)\,dx + x^2\,dy$ where C is bounded by $y=x$ & $y=x^2$

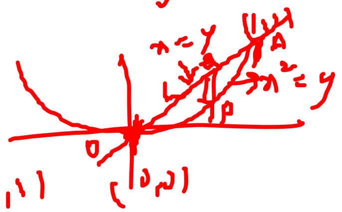
$x^2 = x$

$x^2 - x = 0$

$x(x-1)=0$

$x=0,\ x=1$

$(0,0)(1,1)$



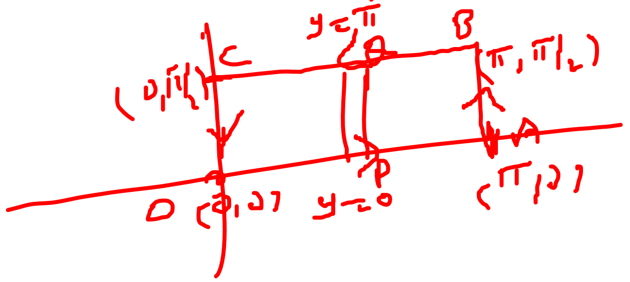
Ans $L.H.S = R.H.S = \frac{-1}{20}$ $N = e^{-x}\cos y$

H.W

* Evaluate $\oint_C e^{-x}(\sin y\,dx + \cos y\,dy)$, where C is the rectangle with vertices $(0,0)\ (\pi,0)\ (\pi,\pi/2)$ and $(0,\pi/2)$

$y: x^2 \to x$

$x: 0 \to 1$



$y: 0 \to \pi/2$

$x: 0 \to \pi$

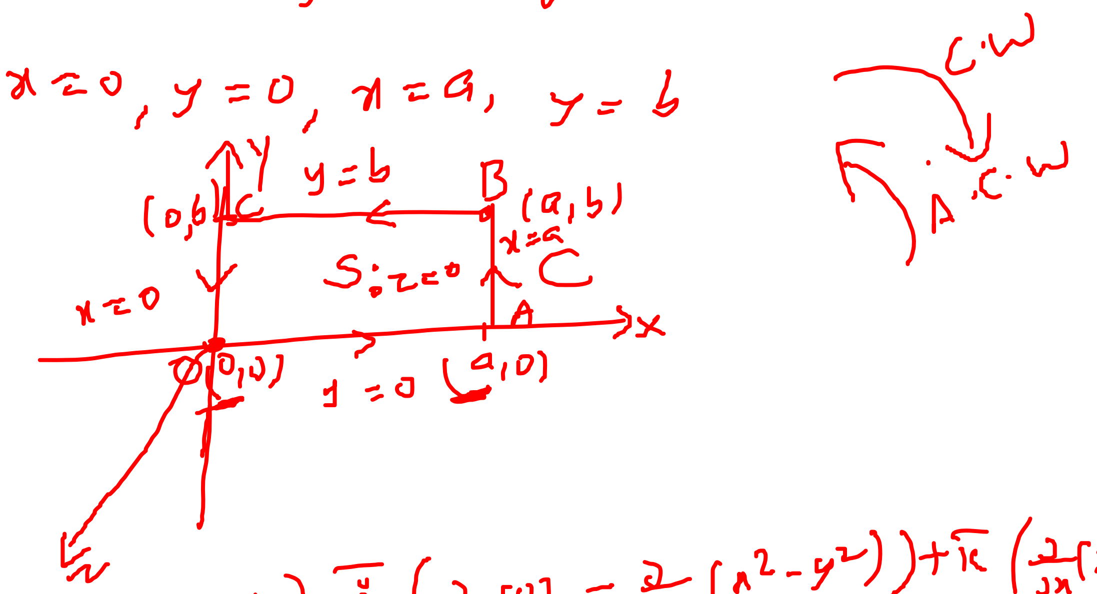
Ans $2(e^{-\pi} - 1)$

* Stokes Theorem (Transformation b/w line and surface integrals):

If S be an open surface bounded by a closed curve C and $\bar{F}$ be any vector point function having continuous first order partial derivatives, then

$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$, where C is traversed in its positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface

* verify stokes theorem for the function $\overline{F} = (x^2-y^2)\overline{i} + 2xy\overline{j}$ integrate around the rectangle plane $z=0$ and bounded by the lines $x=0,\ y=0,\ x=a,\ y=b$



By Stokes theorem, we have

$$\oint_C \overline{F}\cdot d\overline{r} = \iint_S \text{curl}\,\overline{F}\cdot\overline{n}\,ds$$

$R.H.S = \iint_S \text{curl}\,\overline{F}\cdot\overline{n}\,ds$

Given $\overline{F} = (x^2-y^2)\overline{i} + 2xy\overline{j}$

$$\text{curl}\,\overline{F} = \nabla\times\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix} \to R_1$$

$$= \overline{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right) - \overline{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2-y^2)\right) + \overline{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right)$$

$$= \overline{i}(0-0) - \overline{j}(0-0) + \overline{k}(2y-(-2y))$$

$$= \overline{i}(0) - \overline{j}(0) + \overline{k}(4y)$$

let S be the surface of the plane (i.e $S: z=0$)

normal to the surface S is $\nabla S = \overline{i}\frac{\partial S}{\partial x} + \overline{j}\frac{\partial S}{\partial y} + \overline{k}\frac{\partial S}{\partial z} = \overline{i}(0) + \overline{j}(0) + \overline{k}(1)$ $\left[\because \frac{\partial S}{\partial z} = \frac{\partial}{\partial z}z = 1\right]$

$$|\nabla S| = |\overline{i}(0)+\overline{j}(0)+\overline{k}(1)| = \sqrt{0^2+0^2+1^2} = 1$$

$$\overline{n} = \frac{\nabla S}{|\nabla S|} = \frac{\overline{k}}{1} = \overline{k}$$

let R be the projection of surface $S: z=0$ in xy-plane

$$ds = \frac{dx\,dy}{|\bar{n} \cdot \bar{k}|} = \frac{dx\,dy}{|\bar{k} \cdot \bar{k}|} = \frac{dx\,dy}{1}$$

In the projection, $x;\ 0 \to a$

$y;\ 0 \to b$

$$R.H.S = \iint_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds$$

$$= \int_{y=0}^{b} \int_{x=0}^{a} (4y\,\bar{k} \cdot \bar{k})\, dx\, dy = \int_{y=0}^{b} \int_{x=0}^{a} 4y\, dx\, dy = 4 \left[(x)_0^a \left(\frac{y^2}{2}\right)_0^b \right]$$

$$= 4(a-0)\ \frac{1}{2}(b^2-0)$$

$$\underline{R.H.S = 2ab^2}$$

$$L.H.S = \oint_C \bar{F} \cdot d\bar{r}$$

G.T $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$ $\quad \left[\because z=0,\ dz=0\right]$

Now $d\bar{r} = dx\bar{i} + dy\bar{j}$

$$\bar{F} \cdot d\bar{r} = ((x^2-y^2)\bar{i} + 2xy\bar{j}) \cdot (dx\bar{i} + dy\bar{j}) = (x^2-y^2)dx + 2xy\,dy$$

$$L\cdot H\cdot S = \oint_C \bar{F}\cdot d\bar{r} = \oint_C (x^2-y^2)\,dx + 2xy\,dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} \quad \text{—— (1)}$$

Along OA : eqn of OA is $y=0$, $dy=0$ and $x : 0 \to a$

$$\int_{OA} (x^2-y^2)\,dx + \cancel{2xy\,dy} = \int_{x=0}^{a} x^2\,dx = \left(\frac{x^3}{3}\right)_0^a = \frac{1}{3}(a^3-0) = \frac{a^3}{3} \quad \text{—— (2)}$$

Along AB : eqn of AB is $x=a$, $dx=0$ & $y : 0 \to b$

$$\int_{AB} (x^2-y^2)\,dx + 2xy\,dy = \int_{y=0}^{b} 0 + 2(a)\,y\,dy = 2a\left(\frac{y^2}{2}\right)_0^b = a(b^2-0) = ab^2 \quad \text{—— (3)}$$

Along BC : eqn of BC is $y=b$, $dy=0$ and $x : a \to 0$

$$\int_{BC} (x^2-y^2)\,dx + 2xy\,dy = \int_{x=a}^{0} (x^2-b^2)\,dx = \left(\frac{x^3}{3} - b^2x\right)_{x=a}^{0} = \frac{1}{3}(0-a^3) - b^2(0-a)$$

$$= -\frac{a^3}{3} + ab^2 \quad \text{—— (4)}$$

Along CO : eqn of CO is $x=0$, $dx=0$ and $y : b \to 0$

$$\int_{CO} (x^2-y^2)\,dx + 2xy\,dy = \int_{y=b}^{0} 0 = 0 \quad \text{—— (5)}$$

sub ②, ③, ④ & ⑤ in ①, we get

$$L.H.S = \oint_C \bar{F} \cdot d\bar{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2$$

$$L.H.S = R.H.S = 2ab^2$$

Hence Stokes theorem is verified.

* verify stokes theorem for $\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$, where S is the surface of the cube $x=0, y=0, z=0, x=2, y=2, z=2$ above the xy-plane

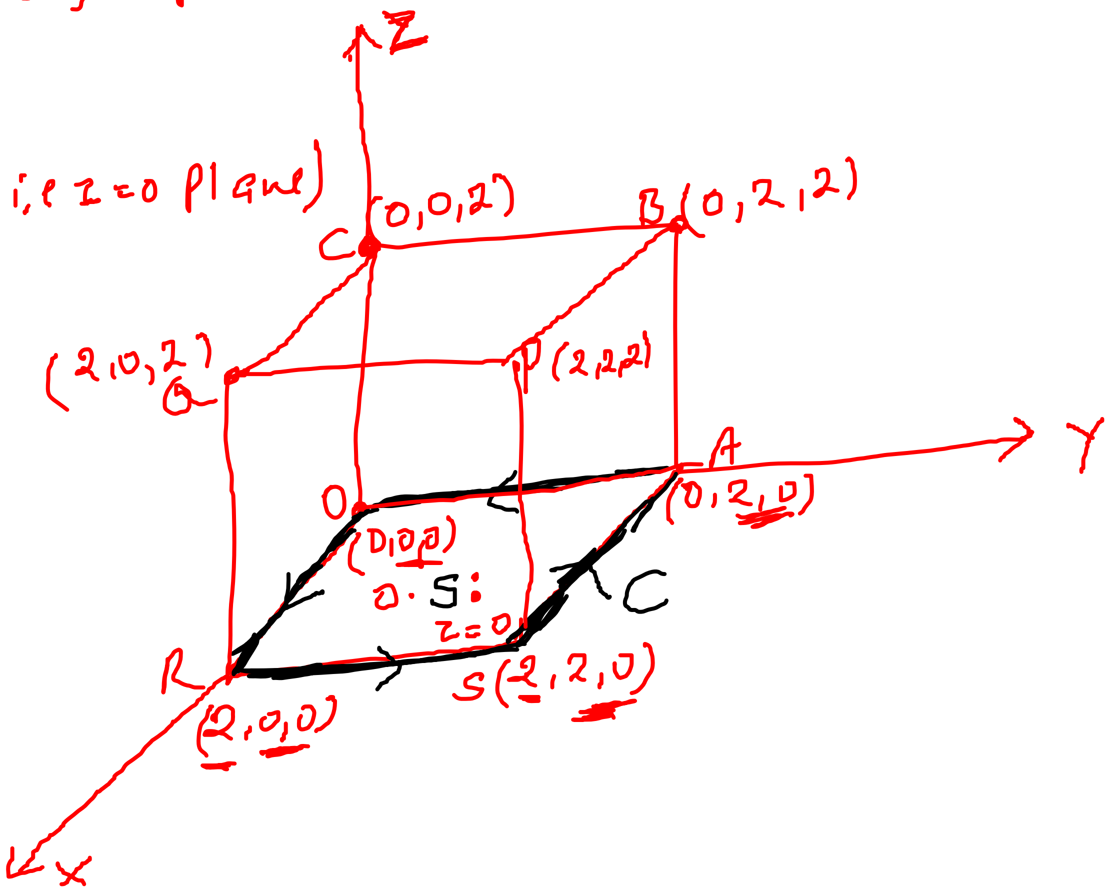
G.T $\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$

let S be the surface of cube above the xy-plane (i.e z=0 plane)

By stokes theorem, we have

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl}\bar{F} \cdot \bar{n}\, ds$$

$L.H.S = \oint_C \bar{F} \cdot d\bar{r}$, where the boundary C of the surface lies on the plane $z=0$ (or xy-plane)



let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ Then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

Now $\bar{F} \cdot d\bar{r} = \left((y - z + 2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}\right) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k})$

$\bar{F} \cdot d\bar{r} = (y - z + 2)\,dx + (yz+4)\,dy + (-xz)\,dz$, $\left[\because C \text{ lies in } xy\text{-plane (i.e } z=0,\ dz=0)\right]$

$\bar{F} \cdot d\bar{r} = (y+2)\,dx + 4\,dy$

L.H.S $= \oint_C \bar{F} \cdot d\bar{r} = \oint_C (y+2)\,dx + 4dy = \int_{OR} + \int_{RS} + \int_{SA} + \int_{AO}$ —— (1)

Along OR : Equ of OR is $y=0, z=0$ and $x : 0 \to 2$
$dy=0,\ dz=0$

$\int_{OR} (y+2)\,dx + 4dy = \int_{x=0}^{2} (0+2)\,dx + 4(0) = \int_{x=0}^{2} 2\,dx = 2(x)_0^2 = 2(2-0) = 4$ —— (2)

Along RS : Equ of RS is $x=2, z=0$ and $y : 0 \to 2$
$dx=0,\ dz=0$

$\int_{RS} (y+2)\,dx + 4dy = \int_{y=0}^{2} 0 + 4\,dy = 4\,(y)_{y=0}^{2} = 4(2-0) = 8$ —— (3)

Along SA : Equ of SA is $y=2, z=0$ and $x : 2 \to 0$
$dy=0,\ dz=0$

$$\int_{SA} (y+2)\,dx + 4\,dy = \int_{x=2}^{0} (2+2)\,dx + 0 = 4\,(x)_2^0 = 4\,(0-2) = -8 \quad \text{—— (4)}$$

Along AO : Eqn of AO is $x=0,\ z=0$ and $y : 2 \to 0$

$dx=0,\ dz=0$

$$\int_{AO} (y+2)\,dx + 4\,dy = \int_{y=2}^{0} 0 + 4\,dy = 4\,(y)_2^0 = 4\,(0-2) = -8 \quad \text{—— (5)}$$

Sub (2), (3), (4) & (5) in (1), we get

$$L.H.S = 4 + \cancel{8} - \cancel{8} - 8 = -4$$

$$R.H.S = \iint_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds.$$

G.T $\bar{F} = (y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}$

$$\operatorname{curl} \bar{F} = \nabla \times \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \left((y-z+2)\bar{i} + (yz+4)\bar{j} - xz\bar{k}\right)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix} \xrightarrow{R_1}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(yz+4)\right) - \bar{j}\left(\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(y-z+2)\right) + \bar{k}\left(\frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2)\right)$$

$$= \bar{i}\,(0-y) - \bar{j}\,(-z+1) + \bar{k}\,(0-1)$$

$$= -y\bar{i} - \bar{j}(-z+1) - \bar{k}$$

let S be the surface of plane $z=0$

i.e $S: z=0$ $\qquad s=z$

$\therefore \bar{n}=\bar{k}$ $\qquad \left[\because \bar{n}=\frac{\nabla S}{|\nabla S|}\right.$, $\because \nabla S=\bar{i}\frac{\partial S}{\partial x}+\bar{j}\frac{\partial S}{\partial y}+\bar{k}\frac{\partial S}{\partial z}=\bar{i}\frac{\partial}{\partial x}z+\bar{j}\frac{\partial}{\partial y}z+\bar{k}\frac{\partial}{\partial z}z=\bar{i}(0)+\bar{j}(0)+\bar{k}(1)\Big]$

$\nabla S=\bar{k}$, $|\nabla S|=|\bar{k}|=\sqrt{1^2}=1$

$\text{curl}\,\bar{F}\cdot\bar{n}=(-y\bar{i}-\bar{j}(-z+1)-\bar{k})\cdot\bar{k}=\underline{-1}$ & $dS=\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}=\frac{dx\,dy}{|\bar{k}\cdot\bar{k}|}=\frac{dx\,dy}{1}$

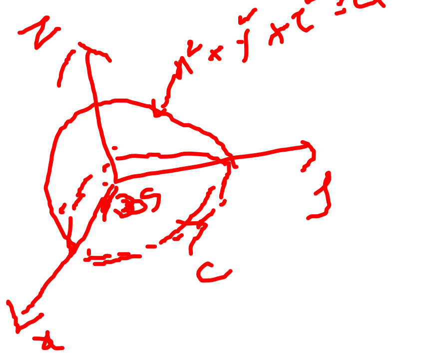
For the projection, $x: 0\to 2$
$y: 0\to 2$

$R.H.S=\iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,dS=\int_{y=0}^{2}\int_{x=0}^{2}-1\,dx\,dy=(-1)\,(x)_0^2\,(y)_0^2=-(2-0)(2-0)=-2(2)=-4$

$\therefore L.H.S=R.H.S=-4$

Hence stokes theorem is verified.

* verify stokes theorem for $\bar{F}=-y\bar{i}+2yz\bar{j}+y^2\bar{k}$, where S is the upper half of the sphere $\underline{x^2+y^2+z^2=a^2}$ and c is the boundary on the XOY-plane



G.T $\bar{F} = -y\bar{i} + 2yz\bar{j} + y^2\bar{k}$

By stokes theorem, we have

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$$

L.H.S $= \oint_C \bar{F}\cdot d\bar{r}$, where C is the circle in the xoy-plane ($\because z=0$) whose equation is $x^2+y^2=a^2$ and whose parametric equations are

$x = r\cos\theta = a\cos\theta$, $y = r\sin\theta = a\sin\theta$ and $\theta = 0 \to 2\pi$

$dx = a(-\sin\theta)d\theta$, $dy = a\cos\theta\, d\theta$

$\bar{F} = -y\bar{i} + 2yz\bar{j} + y^2\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\bar{F}\cdot d\bar{r} = -y\,dx + 2yz\,dy + y^2 dz$,

$\because$ C in the xoy-plane (i.e $z=0$, $dz=0$)

Now $\bar{F}\cdot d\bar{r} = -y\,dx$

L.H.S $= \oint_C \bar{F}\cdot d\bar{r} = \oint_C -y\,dx$

$$= \int_{\theta=0}^{2\pi} -a\sin\theta\,(-a\sin\theta)\,d\theta$$

$$= a^2 \int_{\theta=0}^{2\pi} \sin^2\theta\, d\theta \qquad \left[\because \cos 2\theta = 1 - 2\sin^2\theta,\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= a^2 \int_{\theta=0}^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_{\theta=0}^{2\pi} = \frac{a^2}{2}\left((2\pi - 0) - \frac{1}{2}\left(\sin 2(2\pi) - \sin 2(0)\right)\right)$$

$$= \frac{a^2}{2}\left(2\pi - \frac{1}{2}(0-0)\right) = a^2\pi$$

$L.H.S = a^2\pi$

$R.H.S = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$

$G.T\quad \bar{F} = -y\bar{i} + 2yz\bar{j} + y^2\bar{k}$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2yz & y^2 \end{vmatrix} \xrightarrow{R_1}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial z}(2yz)\right) - \bar{j}\left(\frac{\partial}{\partial x}(y^2) - \left(-\frac{\partial}{\partial z}y\right)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2yz) - \left(-\frac{\partial}{\partial y}y\right)\right)$$

$$= \bar{i}(2y-2y) - \bar{j}(0-0) + \bar{k}(0-(-1))$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k} \quad = +\bar{k}$$

let S be the surface of the sphere

i.e $S: x^2+y^2+z^2-a^2=0$

Normal to the surface S is $\nabla S = \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z} = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$

$$|\nabla S| = \sqrt{(2x)^2+(2y)^2+(2z)^2} = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{a^2} = 2a$$

$$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{2a} = \frac{x\bar{i}+y\bar{j}+z\bar{k}}{a}$$

let R be the projection of S in xoy-plane

$$\text{so } ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{\left|\frac{x\bar{i}+y\bar{j}+z\bar{k}}{a}\cdot\bar{k}\right|} = \frac{dx\,dy}{\frac{z}{a}} = \frac{a}{z}\,dx\,dy$$

$$R.H.S = \iint_S \bar{k}\cdot\left(\frac{x\bar{i}+y\bar{j}+z\bar{k}}{a}\right)\frac{a}{z}\,dx\,dy = \iint_S \frac{z}{a}\,\frac{a}{z}\,dx\,dy = \iint_{S:\,x^2+y^2=a^2} dx\,dy = A$$

, where A is the area of the circle $x^2+y^2=a^2$ is $\pi(a)^2$

Alternatively:

$\therefore$ R be the projection of surface S in xy plane (i,e $z=0$)

Given surface of the sphere becomes (i,e $x^2+y^2+z^2=a^2$) $[\because z=0]$

$$S: x^2+y^2=a^2$$

[$\because$ eqn of x-axis, $y=0$]

$y^2 = a^2-x^2$ & $x^2+0=a^2$

$y=\pm\sqrt{a^2-x^2}$ $\qquad x=\pm a$

i,e $y: -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x: -a$ to a ✓

Aliter:
parametric equations of the circle are
$x=r\cos\theta$, $y=r\sin\theta$, so that $dxdy = r\,dr\,d\theta$ and
$\theta: 0 \to 2\pi$

$$R.H.S = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\,dr\,d\theta = \left(\frac{r^2}{2}\right)_{r=0}^{a} (\theta)_0^{2\pi}$$

$$= \frac{1}{2}(a^2-0)(2\pi-0)$$

$$R.H.S = \pi a^2$$

$$L.H.S = R.H.S = \pi a^2$$

Hence Stokes theorem is verified.

* Evaluate by stokes theorem $\int_C \underbrace{e^x\,dx + 2y\,dy - dz}_{\bar{F}\cdot d\bar{r}}$, where C is the curve $x^2+y^2=4,\ z=2$

By stokes theorem, we have

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds \qquad [\because \bar{F}\cdot d\bar{r} = \text{scalar quantity}]$$

$$\bar{F}\cdot d\bar{r} = e^x\,dx + 2y\,dy - dz$$

$$(F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}) = e^x\,dx + 2y\,dy - dz$$

$$F_1\,dx + F_2\,dy + F_3\,dz = e^x\,dx + 2y\,dy - dz$$

Here $F_1 = e^x$, $F_2 = 2y$, $F_3 = -1$

set $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k} = e^x\,\bar{i} + 2y\,\bar{j} - \bar{k}$

$$\operatorname{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} \rightarrow R_1$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-1) - \frac{\partial}{\partial z}(2y)\right) - \bar{j}\left(\frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial z}(e^x)\right) + \bar{k}\left(\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(e^x)\right)$$

$$\operatorname{curl}\bar{F} = \nabla\times\bar{F} = \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(0-0) = \bar{0}$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \iint_S \bar{0}\cdot\bar{n}\,ds = 0$$

* Evaluate $\int_C \underbrace{\sin z\,dx - \cos x\,dy + \sin y\,dz}_{\bar{F}\cdot d\bar{r}}$ by using Stokes theorem, where C is the boundary of the rectangle defined by $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$

By Stokes theorem, we have

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$$\bar{F}\cdot d\bar{r} = \sin z\,dx - \cos x\,dy + \sin y\,dz$$

$$(F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}) = \sin z\,dx - \cos x\,dy + \sin y\,dz$$

$$F_1\,dx + F_2\,dy + F_3\,dz = \sin z\,dx - \cos x\,dy + \sin y\,dz$$

$$F_1 = \sin z \quad F_2 = -\cos x, \quad F_3 = \sin y$$

set $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k} = \sin z\,\bar{i} - \cos x\,\bar{j} + \sin y\,\bar{k}$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix} \xrightarrow{R_1}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(\sin y) - \left(-\frac{\partial}{\partial z}\cos x\right)\right) - \bar{j}\left(\frac{\partial}{\partial x}(\sin y) - \frac{\partial}{\partial z}(\sin z)\right) + \bar{k}\left(\frac{\partial}{\partial x}(-\cos x) - \frac{\partial}{\partial y}(\sin z)\right)$$

$$= \bar{i}(\cos y + 0) - \bar{j}(0 - \cos z) + \bar{k}(-(-\sin x) - 0)$$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \bar{i}\cos y + \bar{j}\cos z + \sin x\,\bar{k}$$

$\because$ S is the surface of the rectangle in the plane $z = 3$

$S: z - 3 = 0$

$$\nabla S = \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z} = \bar{i}\frac{\partial}{\partial x}(z-3) + \bar{j}\left(\frac{\partial}{\partial y}(z-3)\right) + \bar{k}\frac{\partial}{\partial z}(z-3)$$

$$= \bar{i}(0) + \bar{j}(0) + \bar{k}(1) = \bar{k}$$

$$|\nabla S| = |\bar{k}| = \sqrt{(1)^2} = 1$$

$$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{\bar{k}}{1} = \bar{k}$$

$$\text{curl}\,\bar{F}\cdot\bar{n} = (\cos y\,\bar{i} + \cos z\,\bar{j} + \sin x\,\bar{k})\cdot\bar{k} = \sin x$$

$\therefore$ R be the projection of surface S in xy-plane

$$dS = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{|\bar{k}\cdot\bar{k}|} = \frac{dx\,dy}{1} = dx\,dy$$

Given limits are $x: 0 \to \pi$

$y: 0 \to 1$

$$\iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,dS = \int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\,dx\,dy$$

$$= \left(\frac{-\cos x}{1}\right)_{x=0}^{\pi} (y)_0^1$$

$$= -(\cos\pi - \cos 0)\,(1-0)$$

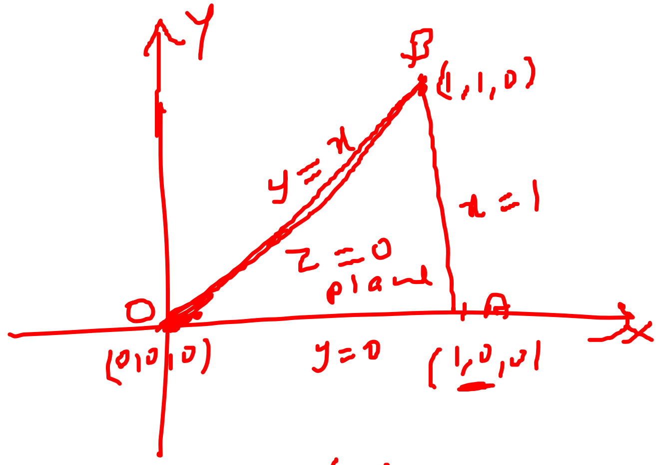
$$= -(-1-1)(1-0)$$

$$= -(-2)(1)$$

$$= +2$$

$$\therefore \oint_C \sin z\,dx - \cos x\,dy + \sin y\,dz = 2$$

* Evaluate by stokes theorem, $\oint_C \underbrace{(x+y)\,dx + (2x-z)\,dy + (y+z)\,dz}_{\bar F\cdot d\bar r}$, where C is the boundary of the triangle with vertices (0,0,0) (1,0,0) and (1,1,0)



By stokes theorem, we have

$$\oint_C \bar F\cdot d\bar r = \iint_S \text{curl}\,\bar F\cdot\bar n\,ds$$

$$\bar F\cdot d\bar r = (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

$$(F_1\bar i + F_2\bar j + F_3\bar k)\cdot(dx\,\bar i + dy\,\bar j + dz\,\bar k) = (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

$$F_1\,dx + F_2\,dy + F_3\,dz = (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

Here $F_1 = x+y$, $F_2 = (2x-z)$, $F_3 = (y+z)$

let $\bar F = F_1\bar i + F_2\bar j + F_3\bar k = (x+y)\bar i + (2x-z)\bar j + (y+z)\bar k$

$$\text{curl}\,\bar F = \nabla\times\bar F = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} \xrightarrow{R_1}$$

$$= \bar i\left(\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right) - \bar j\left(\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y)\right) + \bar k\left(\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y)\right)$$

$$= \bar i\,(1-(-1)) - \bar j\,(0-0) + \bar k\,(2-1)$$

$$= 2\bar i - \bar j(0) + \bar k$$

O (0,0) B (1,1)

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y - 0 = \frac{1-0}{1-0}(x-0)$$

$$y = x$$

let S be the surface of the plane $z=0$

i.e $S: z=0$

$\bar{n} = \bar{k}$ $\quad \left[S : z = 0, \ \nabla S = \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z} = \bar{i}\frac{\partial}{\partial x}(z) + \bar{j}\frac{\partial}{\partial y}z + \bar{k}\frac{\partial}{\partial z}(z) \right) = \bar{k} \right]$

$|\nabla S| = |\bar{k}| = \sqrt{1^2} = 1$

$\text{curl } \bar{F} \cdot \bar{n} = (2\bar{i} + \bar{k}) \cdot \bar{k} = 1$ and $ds = \dfrac{dx\,dy}{|\bar{n} \cdot \bar{k}|} = \dfrac{dx\,dy}{|\bar{k} \cdot \bar{k}|} = \dfrac{dx\,dy}{1}$

$\iint_S \text{curl } \bar{F} \cdot \bar{n} \, ds = \iint_S 1 \, dx\,dy = A$ (i.e area of the triangle $OAB = \frac{1}{2} \times OA \times AB$

$= \frac{1}{2} \times 1 \times 1$

$= \frac{1}{2}$

$O(0,0)$ $(1,0)$ $B(1,1)$

$OA = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{(1-0)^2 + (0-0)^2}$

$= \sqrt{1} = 1$

$AB = \sqrt{(1-1)^2 + (1-0)^2}$

$AB = \sqrt{0+1} = 1$

* **Gauss divergence theorem** (Transformation b/w surface integral and volume integral)

let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$\int_S \bar{F} \cdot d\bar{s}$, where $d\bar{s} = \bar{n}\, ds$

$\int_V \text{div } \bar{F} \, dv = \int_S \bar{F} \cdot \bar{n} \, ds$ where $\bar{n}$ is the unit outward drawn normal to the surface 'S' at any point.

(or) $\iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$

Note: Gauss divergence theorem is useful only for closed surfaces.

* using divergence theorem, evaluate $\iint_S \underbrace{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}$ where S is $x^2+y^2+z^2=a^2$

By Gauss divergence theorem, we have

$$\int_V \operatorname{div}\bar{F}\,dV = \iint_S \bar{F}\cdot\bar{n}\,ds$$

(or)

$$\iint_S \underbrace{F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy} = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

comparing with given data

$$F_1 = x,\ F_2 = y,\ F_3 = z$$

let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(x\bar{i} + y\bar{j} + z\bar{k}\right)$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_V 3\,dV$$

$= 3V$, where V is the volume of the sphere $x^2+y^2+z^2=a^2$ with radius 'a'

$\boxed{\because \text{ volume of the sphere } x^2+y^2+z^2=r^2 \text{ is } \frac{4}{3}\pi r^3}$

$= 3\left(\frac{4}{3}\pi a^3\right)$

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = 4\pi a^3$$

* Apply Gauss divergence theorem, to evaluate $\iint_S \underbrace{(x+z)}\,dy\,dz + \underbrace{(y+z)}\,dz\,dx + \underbrace{(x+y)}\,dx\,dy$, where S is the surface of the sphere $x^2+y^2+z^2=4$

By Gauss divergence theorem, we have

$$\int_V \operatorname{div}\bar{F}\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds$$

(or)

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

comparing with the given data, we have

$F_1 = x+z, \quad F_2 = y+z, \quad F_3 = x+y$

let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k} = (x+z)\bar{i} + (y+z)\bar{j} + (x+y)\bar{k}$

$$\text{div}\bar{F} = \nabla \cdot \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left((x+z)\bar{i} + (y+z)\bar{j} + (x+y)\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y) = 1+1+0 = 2$$

$$\therefore \iint_S (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy = \iiint_V 2\,dv$$

$= 2V$, where V is the volume of the sphere $x^2+y^2+z^2 = 2^2$ with radius $r=2$

$= 2\left(\frac{4}{3}\pi 2^3\right)$ $\quad$ [$\because$ volume of the sphere $x^2+y^2+z^2=r^2$ is $\frac{4}{3}\pi r^3$]

$= \frac{8}{3}\pi(8) = \frac{64\pi}{3}$

* Evaluate by using Gauss divergence theorem $\iint_S \bar{U}\cdot\bar{n}\,ds$, where $\bar{U} = \bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and S is the surface of the sphere $x^2+y^2+z^2=9$

By Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv$$

let $\bar{F} = \bar{U} = u_1\bar{i} + u_2\bar{j} + u_3\bar{k} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $\operatorname{div}\bar{F} = \operatorname{div}\bar{U} = \nabla\cdot\bar{U} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(x\bar{i} + y\bar{j} + z\bar{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$\operatorname{div}\bar{F} = \operatorname{div}\bar{U} = \nabla\cdot\bar{U} = 1+1+1 = 3$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\, ds = \iint_S \bar{U}\cdot\bar{n}\, ds = \iiint_V \operatorname{div}\bar{U}\, dv$$

$= \iiint_V 3\, dv = 3V$, where V is the volume of the sphere $x^2+y^2+z^2 = 3^2$ with radius $r = 3$

$$= 3\left(\frac{4}{3}\pi(3)^3\right) = 27(4)\pi$$

$$= 108\pi$$

* using Gauss divergence theorem to evaluate $\iint_S \bar{F}\cdot\bar{n}\, ds$, where $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

By Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv$$

G.T $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(x^3\bar{i} + y^3\bar{j} + z^3\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V 3(x^2+y^2+z^2)\, dx\,dy\,dz$$

To evaluate this triple integral by using spherical polar coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

and r is the radius

$\theta : 0 \to \pi$

$\phi : 0 \to 2\pi$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \int_{r=0}^{r}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 3(r^2)\, r^2\sin\theta\, dr\, d\theta\, d\phi \quad [\because x^2+y^2+z^2 = r^2]$$

$$= 3\left(\frac{r^5}{5}\right)_{r=0}^{r}\left(\frac{-\cos\theta}{1}\right)_{\theta=0}^{\pi}(\phi)_{\phi=0}^{2\pi} = \frac{3}{5}(r^5-0)\left(-(\cos\pi - \cos 0)\right)(2\pi - 0)$$

$$= \frac{3}{5}r^5(-(-1-1))(2\pi) = \frac{12r^5}{5}\pi$$

Alternatively:

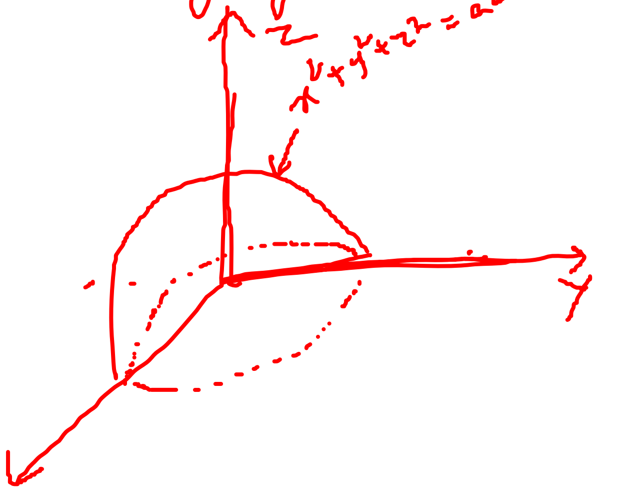
$z : -\sqrt{r^2-x^2-y^2}$ to $z = \sqrt{r^2-x^2-y^2}$

$y : -\sqrt{r^2-x^2}$ to $\sqrt{r^2-x^2}$

$x : -r$ to r also same

answer $\frac{12r^5}{5}\pi$

* use Gauss divergence theorem, to evaluate $\iint_S (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}) \cdot d\bar{s}$ where S is the closed surface bounded by the xy-plane and upper half of the sphere $x^2+y^2+z^2 = a^2$ above this plane.



By Gauss divergence theorem, we have

$$\iint_S \bar{F} \cdot d\bar{s} = \iint_S \bar{F} \cdot \bar{n}\, ds = \iiint_V \operatorname{div} \bar{F}\, dv \qquad [\because d\bar{s} = \bar{n}\, ds]$$

let $\bar{F} = yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}$

$$\operatorname{div}\bar{F} = \nabla \cdot \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k})$$

$$= \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2)$$

$$= 0 + 0 + 4z$$

$$\operatorname{div}\bar{F} = \nabla \cdot \bar{F} = 4z$$

$$\therefore \iint_S \bar{F} \cdot d\bar{s} = \iint_S \bar{F} \cdot \bar{n}\, ds = \iiint_V 4z\, dx\, dy\, dz$$

In spherical polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, Then

$$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$$

Alternatively

$z: 0 \to \sqrt{a^2-x^2-y^2}$

$y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$

$x: -a \to a$

[eqn of xy-plane $z=0$
$y^2 = a^2 - x^2$
$y = \pm\sqrt{a^2-x^2}$]

$$\iint_S \bar{F} \cdot \bar{n}\, ds = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} 4z\, dz\, dy\, dx = \pi a^4$$

and $r : 0 \to a$

$\theta : 0 \to \pi/2$ [$\because$ upper half of the sphere]

$\varphi : 0 \to 2\pi$

$$\iint_S \bar{F} \cdot d\bar{S} = \iint_S \bar{F} \cdot \hat{n}\, dS = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} 4\, r\cos\theta\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= 4\left(\frac{r^4}{4}\right)_{r=0}^{a} \left(\int_{\theta=0}^{\pi/2} \cos\theta \sin\theta\, d\theta\right) \left(\varphi\right)_{\varphi=0}^{2\pi}$$

$$= (a^4 - 0)\left(\frac{1}{2}\int_{\theta=0}^{\pi/2} \sin 2\theta\, d\theta\right)(2\pi - 0) \qquad [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$$= \frac{a^4}{2}\left(-\frac{\cos 2\theta}{2}\right)_{\theta=0}^{\pi/2} 2\pi$$

$$= -\frac{a^4}{4}\left(\cos 2(\pi/2) - \cos 2(0)\right) 2\pi$$

$$= -\frac{a^4}{4}\left(\cos\pi - \cos 0\right) 2\pi = -\frac{2\pi a^4}{4}(-1-1) = \frac{4}{4}\pi a^4$$

$$\iint_S \bar{F} \cdot d\bar{S} = \pi a^4$$

* compute $\int \underbrace{(ax^2+by^2+cz^2)}_{\bar{F}\cdot\bar{n}} ds$ over the surface of the sphere $x^2+y^2+z^2=1$

let S be the surface of the sphere $x^2+y^2+z^2=1$

ie $S: x^2+y^2+z^2-1=0$

normal to the surface S is $\nabla S = \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z}$

$$\nabla S = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)$$

$$|\nabla S| = \sqrt{(2x)^2+(2y)^2+(2z)^2}$$

$$= 2\sqrt{x^2+y^2+z^2} = 2\sqrt{1} = 2$$

let $\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{2} = \underbrace{x\bar{i}+y\bar{j}+z\bar{k}}_{\bar{n}}$

G.T $\int (ax^2+by^2+cz^2)ds = \int \bar{F}\cdot\bar{n}\, ds$

G.T $\bar{F}\cdot\bar{n} = ax^2+by^2+cz^2$ ✓

$\bar{F}\cdot\bar{n} = (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot(x\bar{i}+y\bar{j}+z\bar{k}) = ax^2+by^2+cz^2$

let $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$

By Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv$$

$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot(ax\bar{i}+by\bar{j}+cz\bar{k})$

$= \frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)$

$= a+b+c$

$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V (a+b+c)\,dv$

$=(a+b+c)\,V$, where V is the volume of the sphere $x^2+y^2+z^2=1$

$=(a+b+c)\,\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi(a+b+c)$

Note: volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4}{3}\pi(abc)$

* Use Gauss divergence theorem to evaluate $\iint_S \bar{F}\cdot d\bar{s}$ where $\bar{F}=4x\bar{i}-2y^2\bar{j}+z^2\bar{k}$ and S is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$

By Gauss divergence theorem, we have

$\iint_S \bar{F}\cdot d\bar{s} = \iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$

G.T $\bar{F}=4x\bar{i}-2y^2\bar{j}+z^2\bar{k}$

$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot(4x\bar{i}-2y^2\bar{j}+z^2\bar{k}) = \frac{\partial}{\partial x}(4x)-\frac{\partial}{\partial y}(2y^2)+\frac{\partial}{\partial z}(z^2)$

$= 4-4y+2z$

$$\therefore \iint_S \bar{F}\cdot d\bar{s} = \iiint_V \operatorname{div}\bar{F}\, dV$$

$$= \iiint_V (4-4y+2z)\, dV$$

$$= \iint_{x^2+y^2=4} \int_{z=0}^{3} (4-4y+2z)\, dx\,dy\,dz = \iint_{x^2+y^2=4} \left(\int_{z=0}^{3} \underbrace{(4-4y+2z)}_{x \,\&\, y \text{ are constants}} dz \right) dx\,dy$$

$$= \iint_{x^2+y^2=4} \left(4z-4yz+\frac{2z^2}{2}\right)_{z=0}^{3} dx\,dy$$

$$= \iint_{x^2+y^2=4} \left(4(3-0)-4y(3-0)+(3^2-0)\right) dx\,dy$$

$$= \iint_{x^2+y^2=4} (12-12y+9)\, dx\,dy$$

$$= \iint_{x^2+y^2=4} (21-12y)\, dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (21-12r\sin\theta)\, r\,dr\,d\theta$$

In polar coordinates
$x = r\cos\theta$, $y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$ and
$\theta : 0 \to 2\pi$, $r = 0 \to 2$

$$= \int_{\theta=0}^{2\pi}\left(\int_{r=0}^{2}(21r - \underbrace{12r^2\sin\theta)\,dr}_{\theta \text{ as constant}}\right)d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(21\left(\frac{r^2}{2}\right)_0^2 - 12\left(\frac{r^3}{3}\right)_{r=0}^{2}\sin\theta\right)d\theta = \int_{\theta=0}^{2\pi}\left(\frac{21}{2}(4-0) - \frac{12}{3}(2^3-0)\sin\theta\right)d\theta$$

$$= \int_{\theta=0}^{2\pi}(42 - 32\sin\theta)\,d\theta$$

$$= 42(\theta)_0^{2\pi} - 32\left(\frac{-\cos\theta}{1}\right)_{\theta=0}^{2\pi}$$

$$= 42(2\pi - 0) + 32(\cos 2\pi - \cos 0)$$

$$= 84\pi + 32(1-1) = \boxed{84\pi}$$

* Use divergence theorem to evaluate $\iint(x\bar{i} + y\bar{j} + z^2\bar{k})\cdot\bar{n}\,ds$, where S is the surface bounded by the cone $x^2+y^2=z^2$, $z=4$ in the first octant.

By Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$$

G.T $\bar{F} = x\bar{i} + y\bar{j} + z^2\bar{k}$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(x\bar{i} + y\bar{j} + z^2\bar{k}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}z^2 = 1+1+2z$$
$$= 2(z+1)$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,dS = \iint_S (x\bar{i}+y\bar{j}+z^2\bar{k})\cdot\bar{n}\,dS = \iiint_V 2(z+1)\,dV$$

$$= \iint_{x^2+y^2=4^2}\left(\int_{z=0}^{4} 2(z+1)\,dz\right)dx\,dy$$

$$\left[\because z=4,\ x^2+y^2=z^2 \quad x^2+y^2=4^2\right]$$

$$=2\iint_{x^2+y^2=4^2}\left(\frac{z^2}{2}+z\right)_{z=0}^{4} dx\,dy$$

$$=2\iint_{x^2+y^2=4^2}\left(\frac{1}{2}(16-0)+(4-0)\right)dx\,dy$$

$$=2\iint_{x^2+y^2=4^2}(8+4)\,dx\,dy = 2(12)\iint_{x^2+y^2=4} dx\,dy$$

$$=24\int_{x=0}^{4}\left(\int_{y=0}^{\sqrt{4^2-x^2}} dy\right)dx$$

$$=\int_{x=0}^{4}(y)_0^{\sqrt{4^2-x^2}}\,dx$$

$$=24\int_{x=0}^{4}\left(\sqrt{4^2-x^2}-0\right)dx = 24\int_{x=0}^{4}\sqrt{4^2-x^2}\,dx$$

$x^2+y^2=4^2$

In the first octant means lower limits must be zeros

$y^2=4^2-x^2 \Rightarrow y=\sqrt{4^2-x^2}$

$y: 0 \to \sqrt{4^2-x^2}$

$x: 0 \to 4$

$x^2+y^2=4^2$

eq^n of x-axis $y=0$

$x^2=4^2$

$x=4$

$$\iint_S (x\bar{i} + y\bar{j} + z^2\bar{k}) \cdot \bar{n}\, ds = 24\int_{x=0}^{4} \sqrt{4^2 - x^2}\, dx \qquad \left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c\right]$$

$$= 24\left(\frac{x}{2}\sqrt{4^2-x^2} + \frac{4^2}{2}\sin^{-1}\left(\frac{x}{4}\right)\right)_{x=0}^{4}$$

$$= 24\left(\frac{1}{2}(0-0) + 8\left(\sin^{-1}\left(\frac{4}{4}\right) - \sin^{-1}\left(\frac{0}{4}\right)\right)\right)$$

$$= 24\left(8\left(\sin^{-1}(\sin \pi/2) - \sin^{-1}(\sin 0)\right)\right)$$

$$= 24\left(8\left(\pi/2 - 0\right)\right) = 24(4\pi) = 96\pi$$

* verify Gauss divergence theorem for $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

By Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv.$$

$$R.H.S = \iiint_V \operatorname{div}\bar{F}\, dv$$

$$G.T\ \bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left((x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}\right) = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x + 2y + 2z = 2(x+y+z)$$

Given limits are $x : 0 \to a$
$y : 0 \to b$
$z : 0 \to c$

$$R.H.S = \iiint \operatorname{div} \vec{F}\, dv = \int_{z=0}^{c} \int_{y=0}^{b} \left(\int_{x=0}^{a} \underbrace{2(x+y+z)}_{y \,\&\, z \text{ are constants}} dx \right) dy\, dz$$

$$= 2 \int_{z=0}^{c} \left(\int_{y=0}^{b} \left(\frac{x^2}{2} + xy + xz \right)_{x=0}^{a} dy \right) dz$$

$$= 2 \int_{z=0}^{c} \left(\int_{y=0}^{b} \left(\underbrace{\left(\frac{a^2}{2} + ay + az \right)}_{z \text{ as constant}} - (0+0+0) \right) \right) dy\, dz$$

$$= 2 \int_{z=0}^{c} \left(\frac{a^2}{2} y + a \frac{y^2}{2} + azy \right)_{y=0}^{b} dz$$

$$= 2 \int_{z=0}^{c} \left(\frac{a^2}{2}(b-0) + \frac{a}{2}(b^2-0) + az(b-0) \right) dz$$

$$= 2 \int_{z=0}^{c} \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + azb \right) dz$$

$$= 2 \left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab \frac{z^2}{2} \right]_{z=0}^{c}$$

$$= 2 \left(\frac{a^2 b}{2}(c-0) + \frac{ab^2}{2}(c-0) + \frac{ab}{2}(c^2-0) \right] = 2 \left(\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right) = abc(a+b+c)$$

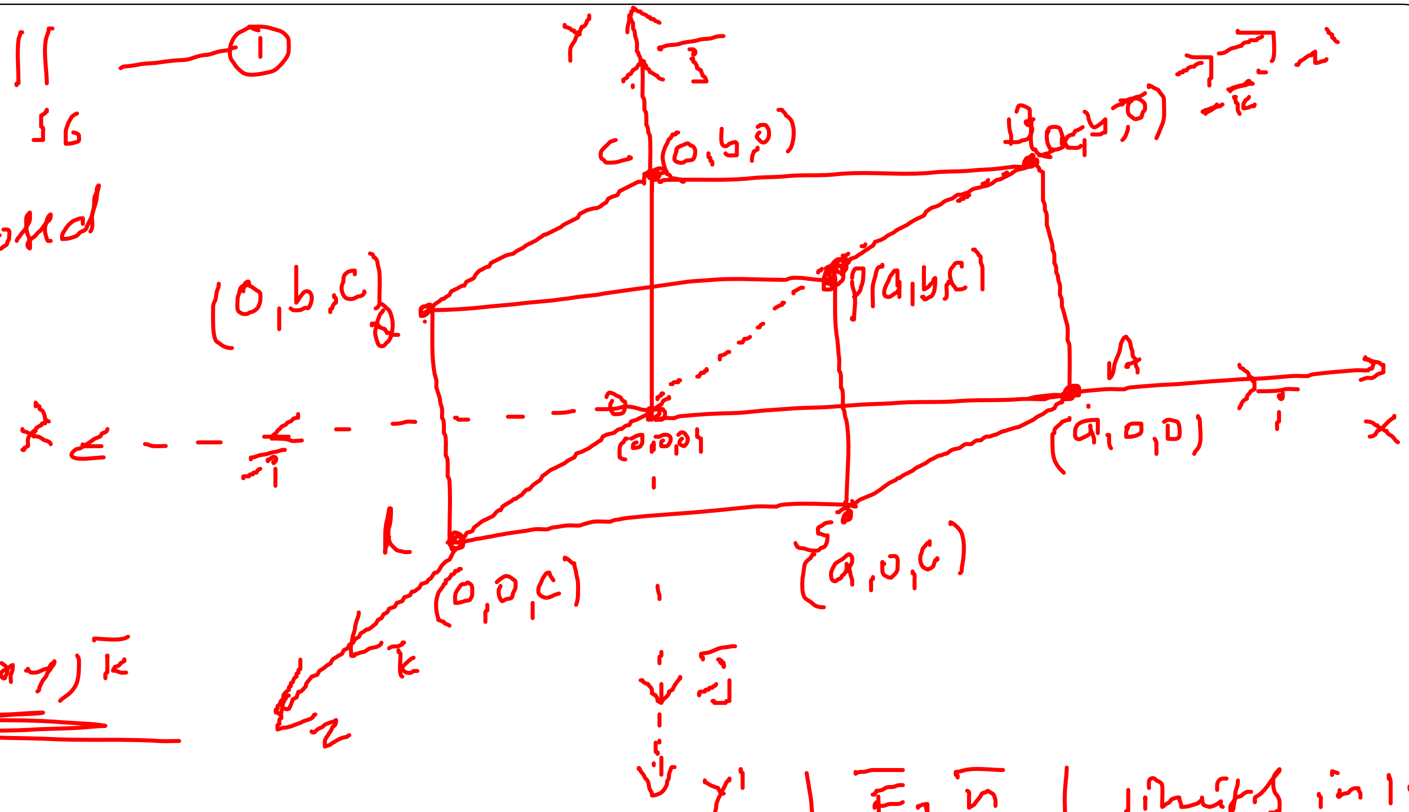
$$L.H.S = \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} \quad \text{—— (1)}$$

To evaluate the surface integral, divide the closed surface 'S' of the rectangular parallelopiped into 6 parts

S_1 : PBAS, S_2 : OCQR, S_3 : PBCQ, S_4 : ORSA

S_5 : PQRS, S_6 : OABC

[$\because$ $\bar{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$



Surface	$\bar{n}$	eqn	ds	$\bar{F}$	$\bar{F}\cdot\bar{n}$	limits in the projection
S_1 : PBAS	$\bar{i}$	$x=a$	$\frac{dydz}{\lvert\bar{n}\cdot\bar{i}\rvert} = \frac{dydz}{\lvert\bar{i}\cdot\bar{i}\rvert} = dydz$	$(a^2-yz)\bar{i} + (y^2-za)\bar{j} + (z^2-ay)\bar{k}$	a^2-yz	$y: 0\to b$, $z: 0\to c$
S_2 : OCQR	$-\bar{i}$	$x=0$	$\frac{dydz}{\lvert\bar{n}\cdot\bar{i}\rvert} = \frac{dydz}{\lvert-\bar{i}\cdot\bar{i}\rvert} = \frac{dydz}{\lvert-1\rvert} =$	$-yz\bar{i} + y^2\bar{j} + z^2\bar{k}$	yz	$y: 0\to b$, $z: 0\to c$
S_3 : PBCQ	$\bar{j}$	$y=b$	$\frac{dxdz}{\lvert\bar{n}\cdot\bar{j}\rvert} = \frac{dxdz}{\lvert\bar{j}\cdot\bar{j}\rvert} = \frac{dxdz}{1}$	$(x^2-bz)\bar{i} + (b^2-zx)\bar{j} + (z^2-bx)\bar{k}$	b^2-zx	$x: 0\to a$, $z: 0\to c$
S_4 : ORSA	$-\bar{j}$	$y=0$	$\frac{dxdz}{\lvert\bar{n}\cdot\bar{j}\rvert} = \frac{dxdz}{\lvert-\bar{j}\cdot\bar{j}\rvert} = \frac{dxdz}{\lvert-1\rvert}$	$x^2\bar{i} - zx\bar{j} + z^2\bar{k}$	zx	$x: 0\to a$, $z: 0\to c$
S_5 : PQRS	$\bar{k}$	$z=c$	$\frac{dxdy}{\lvert\bar{n}\cdot\bar{k}\rvert} = \frac{dxdy}{\lvert\bar{k}\cdot\bar{k}\rvert} = \frac{dxdy}{1}$	$(x^2-yc)\bar{i} + (y^2-cx)\bar{j} + (c^2-xy)\bar{k}$	c^2-xy	$x: 0\to a$, $y: 0\to b$
S_6 : OABC	$-\bar{k}$	$z=0$	$\frac{dxdy}{\lvert\bar{n}\cdot\bar{k}\rvert} = \frac{dxdy}{\lvert-\bar{k}\cdot\bar{k}\rvert} = \frac{dxdy}{\lvert-1\rvert}$	$x^2\bar{i} + y^2\bar{j} - xy\bar{k}$	xy	$x: 0\to a$, $y: 0\to b$

Along S_1:

$$\int_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} \underbrace{(a^2 - yz)\,dy}_{z \text{ as constant}}\,dz = \int_{z=0}^{c}\left(a^2y - z\frac{y^2}{2}\right)_{y=0}^{b} dz = \int_{z=0}^{c}\left(a^2b - \frac{zb^2}{2}\right)dz = \left(a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right)_{z=0}^{c}$$

$$= a^2bc - \frac{b^2}{4}c^2 \quad \text{——} \; (2)$$

Along S_2:

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} yz\,dy\,dz = \left(\frac{y^2}{2}\right)_0^b \left(\frac{z^2}{2}\right)_0^c = \left(\frac{b^2}{2}\right)\left(\frac{c^2}{2}\right) = \frac{b^2c^2}{4} \quad \text{——} \; (3)$$

Along S_3:

$$\int_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} \underbrace{(b^2 - zx)}_{z \text{ as constant}}\,dx\,dz = \int_{z=0}^{c}\left(b^2x - z\frac{x^2}{2}\right)_{x=0}^{a} dz = \int_{z=0}^{c}\left(b^2a - \frac{za^2}{2}\right)dz = \left(b^2a(z) - \frac{a^2}{2}\frac{z^2}{2}\right)_{z=0}^{c}$$

$$= b^2ac - \frac{a^2c^2}{4} \quad \text{——} \; (4)$$

Along S_4:

$$\int_{S_4} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} zx\,dx\,dz = \left(\frac{x^2}{2}\right)_0^a \left(\frac{z^2}{2}\right)_{z=0}^{c} = \frac{a^2}{2}\left(\frac{c^2}{2}\right) = \frac{a^2c^2}{4} \quad \text{——} \; (5)$$

Along S_5:

$$\int_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} \underbrace{(c^2 - xy)}_{y \text{ as constant}}\,dx\,dy = \int_{y=0}^{b}\left(c^2x - y\frac{x^2}{2}\right)_{x=0}^{a} dy = \int_{y=0}^{b}\left(c^2a - \frac{y}{2}c^2\right)dy = \left(c^2ay - \frac{c^2}{2}\frac{y^2}{2}\right)_{y=0}^{b}$$

$$= c^2ab - \frac{c^2b^2}{4} \quad \text{——} \; (6)$$

Along S_6:

$$\int_{S_6} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} xy\,dx\,dy = \left(\frac{x^2}{2}\right)_0^a \left(\frac{y^2}{2}\right)_0^b = \left(\frac{a^2}{2}\right)\left(\frac{b^2}{2}\right) = \frac{a^2b^2}{4} \quad \text{——} \; (7)$$

Sub (2), (3), (4), (5), (6), & (7) in (1), we get

$L.H.S = \iint_S \bar{F}\cdot\bar{n}\,ds = a^2bc + \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + b^2ca - \cancel{\frac{c^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}} + c^2ab - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$

$L.H.S = a^2bc + b^2ca + c^2ab = abc(a+b+c)$

$\therefore L.H.S = R.H.S = abc(a+b+c)$

$\therefore$ Gauss divergence theorem is verified.

* verify Gauss divergence theorem for $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ taken over the region bounded by the cylinder $x^2+y^2=4$, $z=0$, $z=3$

G.T $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$

By Gauss divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$$

$$R.H.S = \iiint_V \operatorname{div}\bar{F}\,dv$$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z$$

$$R.H.S = \iiint_V (4-4y+2z)\,dx\,dy\,dz$$

$$= \iint_{x^2+y^2=4} \left(\int_{z=0}^{3} \underbrace{(4-4y+2z)}_{y,\,x \text{ are constants}} dz \right) dx\,dy$$

$$= \iint_{x^2+y^2=4} \left(4z - 4yz + 2\frac{z^2}{2}\right)_{z=0}^{3} dx\,dy$$

$$= \iint_{x^2+y^2=4} \left(4(3-0) - 4y(3-0) + (3^2-0)\right) dx\,dy$$

$$= \iint_{x^2+y^2=4} (12 - 12y + 9)\,dx\,dy$$

$$= \iint_{x^2+y^2=4} (21 - 12y)\,dx\,dy$$

In polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

& $r : 0 \to 2$

$\theta : 0 \to 2\pi$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (21 - 12r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{2} \underbrace{(21r - 12r^2\sin\theta)}_{\theta \text{ as constant}} dr \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(21\left(\frac{r^2}{2}\right) - 12\sin\theta\left(\frac{r^3}{3}\right)\right)_{r=0}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{21}{2}(4-0) - 4\sin\theta(8-0) \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} (42 - 32\sin\theta)\,d\theta$$

$$= \left(42\theta - 32\left(\frac{-\cos\theta}{1}\right)\right)_{\theta=0}^{2\pi}$$

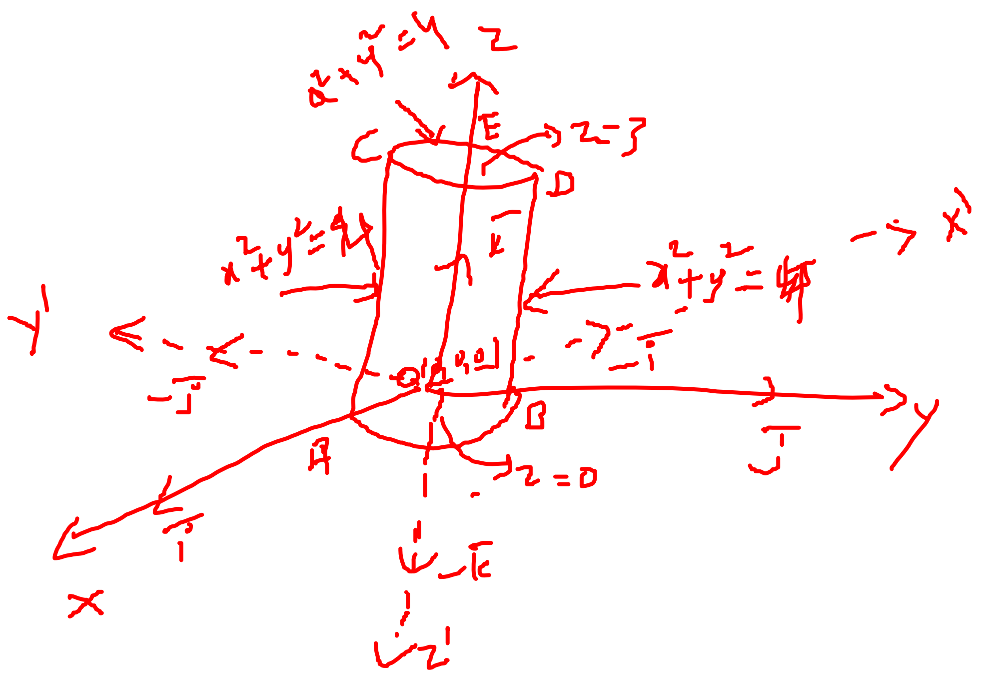
$$= 42(2\pi - 0) + 32(\cos 2\pi - \cos 0)$$

$$= 84\pi + 32(1-1)$$

$$= 84\pi$$

$$R.H.S = 84\pi$$

$$L.H.S = \iint_S \bar F\cdot\bar n\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} \quad \text{——— (1)}$$



where S_1: OAB, S_2: CDE, S_3: OBDE, S_4: OECA, S_5: ABDC

(i) Along S_1: $z=0$, $\bar n = -\bar k$, $\bar F\cdot\bar n = (4x\bar i - 2y^2\bar j + z^2\bar k)\cdot(-\bar k) = -z^2$

$\bar F\cdot\bar n = -z^2$ [∵ $z=0$]

$\bar F\cdot\bar n = 0$

$$\iint_{S_1} \bar F\cdot\bar n\,ds = \iint_{S_1} 0\,ds = 0 \quad \text{——— (2)}$$

ii) Along S_2: $z=3$, $\bar n = \bar k$, $\bar F\cdot\bar n = (4x\bar i - 2y^2\bar j + z^2\bar k)\cdot\bar k = z^2$

$\bar F\cdot\bar n = z^2 = 3^2 = 9$ [∵ $z=3$]

$\iint_{S_2} \bar F\cdot\bar n\,ds = \iint_{S_2} 9\,dx\,dy$ [∵ R is the projection of surface S_2 in xy-plane, so, $ds = \frac{dx\,dy}{|\bar n\cdot\bar k|} = \frac{dx\,dy}{|\bar k\cdot\bar k|} = \frac{dx\,dy}{1}$]

$= 9A$ where A is the Area of the circle S_2

$= 9\pi(2)^2 = 36\pi$ ——— (3)

iii) Along S_3: $x=0$, $\bar n = -\bar i$, $\bar F\cdot\bar n = (4x\bar i - 2y^2\bar j + z^2\bar k)\cdot(-\bar i) = -4x$

$\bar F\cdot\bar n = -4x = -4(0) = 0$ [∵ $x=0$]

$$\iint_{S_3} \bar F\cdot\bar n\,ds = \iint_{S_3} 0\,ds = 0 \quad \text{——— (4)}$$

(iv) Along S_4: $y=0$, $\bar{n} = -\bar{j}$, $\bar{F}\cdot\bar{n} = (4x\bar{i} - 2y^2\bar{j} + z^2\bar{k})\cdot -\bar{j} = 2y^2 = 0$

$$\iint_{S_4} \bar{F}\cdot\bar{n}\, ds = \iint_{S_4} 0\, ds = 0 \longrightarrow (5)$$

(v) Along S_5: $x^2+y^2=4$ be the curved surface of the cylinder

$S_5 : x^2+y^2-4=0$

$\nabla S_5 = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(0)$

$|\nabla S_5| = \sqrt{(2x)^2+(2y)^2+0^2} = 2\sqrt{x^2+y^2} = 2\sqrt{4} = 2(2) = 4$

$$\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\bar{i} + 2y\bar{j} + 0\bar{k}}{4} = \frac{1}{2}(x\bar{i} + y\bar{j} + 0\bar{k})$$

$$\bar{F}\cdot\bar{n} = (4x\bar{i} - 2y^2\bar{j} + z^2\bar{k})\cdot \frac{1}{2}(x\bar{i} + y\bar{j} + 0\bar{k}) = \frac{1}{2}(4x^2 - 2y^3 + 0) = 2x^2 - y^3$$

let R be the projection of surface S_5 in yz (or zx) - plane

$$ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{|\frac{1}{2}(x\bar{i}+y\bar{j}+0\bar{k})\cdot\bar{i}|} = \frac{dy\,dz}{\frac{x}{2}} = \frac{2}{x}\,dy\,dz$$

$\because$ R of S_5 in yz-plane (i.e $x=0$)

Given surface $S_5 : x^2+y^2-4=0$ becomes

$y^2=4$

$y = \pm 2$ & $z : 0 \to 3$

$$\iint_{S_5} \vec{F}\cdot\vec{n}\,ds = \int_{z=0}^{3}\int_{y=-2}^{2} (2x^2 - y^3)\,\frac{2}{x}\,dy\,dz$$

$\left[\because S_5 : x^2 + y^2 = 4 \quad x^2 = 4 - y^2 \quad x = \sqrt{4-y^2}\right]$

$$= \int_{z=0}^{3}\int_{y=-2}^{2} \left(2(4-y^2) - y^3\right)\frac{2}{\sqrt{4-y^2}}\,dy\,dz$$

$$= \int_{z=0}^{3}\left(\int_{y=-2}^{2}\left(\underbrace{4\sqrt{4-y^2}}_{\text{even}} - \underbrace{\frac{2y^3}{\sqrt{4-y^2}}}_{\text{odd}}\right)dy\right)dz$$

$\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is an even} \quad f(-x) = f(x)\right.$

$= 0, \text{ if } f(x) \text{ is an odd} \quad f(-x) = -f(x)$

$$= \int_{z=0}^{3}\left(2\int_{y=0}^{2} 4\sqrt{4-y^2}\,dy - 0\right)dz$$

$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c\right]$

$$= 8\int_{z=0}^{3}\left(\frac{y}{2}\sqrt{2^2-y^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{y}{2}\right)\right)_{y=0}^{2} dz$$

$$= 8\int_{z=0}^{3}\left(\frac{1}{2}(0-0) + 2\left(\sin^{-1}(2/2) - \sin^{-1}(0/2)\right)\right)dz$$

$$= 8\int_{z=0}^{3} 2\left(\sin^{-1}(\sin\pi/2) - \sin^{-1}(\sin 0)\right)dz$$

$$\iint_{S_5}\vec{F}\cdot\vec{n}\,dz = 8\int_{z=0}^{3} 2(\pi/2 - 0)\,dz = 8\pi\,(z)_0^3 = 8\pi(3-0) = 24\pi \quad \text{(Front view)}$$

$$= \underset{\text{Front view}}{24\pi} + \underset{\text{back view}}{24\pi} = 48\pi \quad \text{—— (6)}$$

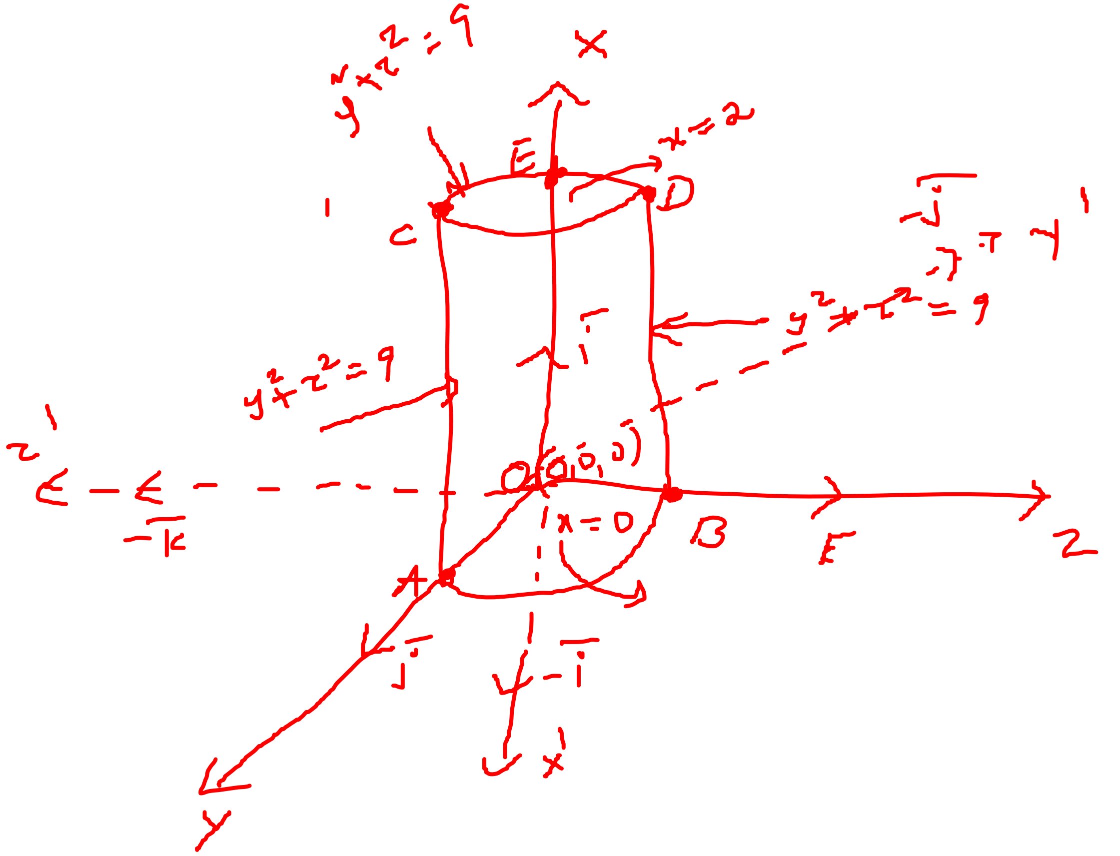
sub ②, ③, ④, ⑤ & ⑥ in ①, we get

$$\iint_S \bar{F} \cdot \bar{n}\, ds = 0 + 36\pi + 0 + 0 + 48\pi = 84\pi$$

$$L.H.S = R.H.S = 84\pi$$

$\therefore$ Gauss divergence theorem is verified.

* verify divergence theorem for $2x^2y\,\bar{i} - y^2\bar{j} + 4z^2x\,\bar{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$



By Gauss divergence Theorem, we have

$$\iint_S \bar{F} \cdot \bar{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv$$

let $\bar{F} = 2x^2y\,\bar{i} - y^2\bar{j} + 4z^2x\,\bar{k}$

$$\operatorname{div}\bar{F} = \nabla \cdot \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(2x^2y\,\bar{i} - y^2\bar{j} + 4z^2x\,\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4z^2x)$$

$$\operatorname{div}\bar{F} = 4xy - 2y + 8zx$$

$$R.H.S = \iiint_V \operatorname{div}\bar{F}\, dv$$

$$R.H.S = \iiint_V (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \iint_{y^2+z^2=9} \left(\underbrace{\int_{x=0}^{2} (4xy - 2y + 8xz)\, dx}_{y\ \&\ z\ \text{are constants}} \right) dy\, dz$$

[In its first octant means lower limits must be zeros] i.e $x=0,\ y=0,\ z=0$

$$= \iint_{y^2+z^2=9} \left(4y\left(\frac{x^2}{2}\right) - 2yx + 8z\left(\frac{x^2}{2}\right) \right)_{x=0}^{2} dy\, dz$$

$$= \iint_{y^2+z^2=9} \left(2y(4-0) - 2y(2-0) + 4z(4-0) \right) dy\, dz$$

$$= \iint_{y^2+z^2=9} (8y - 4y + 16z)\, dy\, dz$$

$$= \iint_{y^2+z^2=9} (4y + 16z)\, dy\, dz$$

$$= \int_{y=0}^{3} \left(\int_{z=0}^{\sqrt{9-y^2}} \underbrace{(4y + 16z)}_{y\ \text{as constant}}\, dz \right) dy$$

$$= \int_{y=0}^{3} \left(4yz + 16\frac{z^2}{2} \right)_{z=0}^{\sqrt{9-y^2}} dy$$

$$= \int_{y=0}^{3} \left(\left(4y\left(\sqrt{9-y^2} - 0\right) + 8\left(9 - y^2 - 0\right) \right) \right) dy$$

$$= \int_{y=0}^{3} \left(4y\sqrt{9-y^2} + 8(9-y^2) \right) dy$$

$$= \int_{y=0}^{3} 4y\sqrt{9-y^2}\, dy + 8\int_{y=0}^{3} (9-y^2)\, dy$$

$$= -2\int_{y=0}^{3} \sqrt{9-y^2}\,(-2y)\, dy + 8\left(9y - \frac{y^3}{3}\right)_{y=0}^{3}$$

$$= (-2)\left(\frac{(9-y^2)^{3/2}}{3/2} \right)_{y=0}^{3} + 8\left(9(3-0) - \frac{1}{3}(27-0)\right)$$

$$= -\frac{4}{3}\left(0 - 9^{3/2}\right) + 8\left(27 - \frac{27}{3}\right)$$

$$= -\frac{4}{3}\left(-(3^2)^{3/2}\right) + 8\left(\frac{81-27}{3}\right) = \cancel{27}^{\,9}\left(\frac{4}{\cancel{3}}\right) + \frac{8}{\cancel{3}}\left(\cancel{54}^{\,18}\right)$$

$$= 36 + 144 = 180$$

$R.H.S = 180$

$$L.H.S = \iint_S \bar{F} \cdot \bar{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} \longrightarrow (1)$$

where S_1 : OAB, S_2 : CDE, S_3 : OBDE, S_4 : OECA, S_5 : ABDC

Along S_1 : $x = 0$, $\bar{n} = -\bar{i}$, $\bar{F}\cdot\bar{n} = (2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot(-\bar{i})$

$$= -2x^2y = 0 \quad [\because S_1 : x = 0]$$

$$ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{|-\bar{i}\cdot\bar{i}|} = \frac{dy\,dz}{|-1|} = \frac{dy\,dz}{1}$$

$$\iint_{S_1} \bar{F}\cdot\bar{n}\, ds = \iint_{S_1} 0\, dy\,dz = 0 \longrightarrow (2)$$

Along S_2 : $x = 2$, $\bar{n} = \bar{i}$, $\bar{F}\cdot\bar{n} = (2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot\bar{i} = 2x^2y = 2(2)^2y = 8y$

$$ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = \frac{dy\,dz}{|\bar{i}\cdot\bar{i}|} = \frac{dy\,dz}{1} = dy\,dz$$

$z : 0 \to \sqrt{9-y^2}$ $\quad [\because y^2+z^2 = 9]$

$y : 0 \to 3$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\, ds = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} 8y\, dy\,dz = \int_{y=0}^{3}\left(\int_{z=0}^{\sqrt{9-y^2}} dz\right) 8y\,dy = \int_{y=0}^{3} (z)_{z=0}^{\sqrt{9-y^2}} 8y\,dy = \int_{y=0}^{3}(\sqrt{9-y^2} - 0)\, 8y\,dy$$

$$= -4\int_{y=0}^{3}\sqrt{9-y^2}\,(-2y)\,dy$$

$$= -4\left(\frac{(9-y^2)^{3/2}}{3/2}\right)^3_{y=0}$$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds = -\frac{8}{3}\left(0 - 9^{3/2}\right) = \frac{-8}{3}(-27) = 72 \quad \text{——— (3)}$$

Along S_3: $y=0$, $\bar{n} = -\bar{j}$, $\bar{F}\cdot\bar{n} = (2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot -\bar{j} = y^2 = 0 \quad [\because y=0]$

$$ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{j}|} = \frac{dx\,dy}{|-\bar{j}\cdot\bar{j}|} = \frac{dx\,dy}{|-1|} = dx\,dy$$

$$\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = \iint_{S_3} 0\,dx\,dy = 0 \quad \text{——— (4)}$$

Along S_4: $z=0$, $\bar{n} = -\bar{k}$, $\bar{F}\cdot\bar{n} = (2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot(-\bar{k}) = -4xz^2 = -4x(0) = 0 \quad [\because S_4 : z=0]$

$$ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{|-\bar{k}\cdot\bar{k}|} = \frac{dx\,dy}{|-1|} = \frac{dx\,dy}{1}$$

$$\iint_{S_4} \bar{F}\cdot\bar{n}\,ds = \iint_{S_4} 0\,dx\,dy = 0 \quad \text{——— (5)}$$

Along S_5: $y^2+z^2=9$

i.e $S_5 : y^2+z^2-9=0$

normal to the surface S_5 is $\nabla S_5 = \bar{i}\frac{\partial}{\partial x}S_5 + \bar{j}\frac{\partial}{\partial y}S_5 + \bar{k}\frac{\partial}{\partial z}S_5$

$$\nabla S_5 = \bar{i}\frac{\partial}{\partial x}(y^2+z^2-9)+\bar{j}\frac{\partial}{\partial y}(y^2+z^2-9)+\bar{k}\frac{\partial}{\partial z}(y^2+z^2-9)$$

$$\nabla S_5 = \bar{i}(0)+\bar{j}(2y)+\bar{k}(2z)$$

$$|\nabla S_5| = |0\bar{i}+2y\bar{j}+2z\bar{k}| = \sqrt{0^2+(2y)^2+(2z)^2} = 2\sqrt{y^2+z^2} = 2\sqrt{9} = 2(3)=6$$

$$\bar{n} = \frac{\nabla S_5}{|\nabla S_5|} = \frac{2(y\bar{j}+z\bar{k})}{6} = \frac{1}{3}(y\bar{j}+z\bar{k})$$

$$\bar{F}\cdot\bar{n} = (2x^2y\,\bar{i}-y^2\bar{j}+4xz^2\bar{k})\cdot\frac{1}{3}(y\bar{j}+z\bar{k}) = \frac{1}{3}[-y^3+4xz^3]$$

let R be the projection of surface S_5 in xy-plane (or zx plane)

$$ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{|\frac{1}{3}(y\bar{j}+z\bar{k})\cdot\bar{k}|} = \frac{dx\,dy}{\frac{z}{3}} = \frac{3}{z}\,dx\,dy$$

$\therefore$ R of S_5 in xy-plane (i.e $z=0$)

Given surface S_5: $y^2+z^2=9$

$y^2+0=9$

$y=3$

and y: 0 to 3

x: $0\to 2$

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \int_{x=0}^{2}\int_{y=0}^{3} \frac{1}{3}\left(-y^3+4xz^3\right)\frac{3\,dx\,dy}{z}$$

$$\left[\because y^2+z^2=9 \quad z^2 = 9-y^2 \quad z=\sqrt{9-y^2}\right]$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \frac{1}{3}\left(\frac{-y^3}{\sqrt{9-y^2}} + 4x(9-y^2)\right) 3\,dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \frac{-y^3}{\sqrt{9-y^2}}\,dy\,dx + \int_{x=0}^{2}\left(\int_{y=0}^{3}(9-y^2)\,dy\right)4x\,dx$$

$$= -\int_{x=0}^{2} 18\,dx + \int_{x=0}^{2}\left(\int_{y=0}^{3}(9-y^2)\,dy\right)4x\,dx$$

$$= -18(x)_0^2 + \int_{x=0}^{2}\left(9y-\frac{y^3}{3}\right)_{y=0}^{3} 4x\,dx$$

$$= -18(2-0) + \int_{x=0}^{2}\left(9(3-0)-\frac{1}{3}(27-0)\right)4x\,dx$$

$$= -36 + \int_{x=0}^{2}(27-9)\,4x\,dx = -36+(18)\,4\int_{x=0}^{2} x\,dx = -36+72\left(\frac{x^2}{2}\right)_0^2$$

$$= -36+36(4-0)$$

$$= -36+144 = 108 \longrightarrow (6)$$

$$\int_{y=0}^{3}\frac{y^3}{\sqrt{9-y^2}}\,dy = \int_{y=0}^{3}\frac{y^3}{3\sqrt{1-\frac{y^2}{9}}}\,dy$$

$$= \frac{1}{3}\int_{y=0}^{3}\frac{y^3}{\sqrt{1-\frac{y^2}{9}}}\,dy, \quad \text{Put } \frac{y^2}{9}=t$$

$$y^2 = 9t \qquad y = 3(t)^{1/2} \qquad dy = 3\,\frac{1}{2}\,t^{1/2-1}\,dt$$

$$\text{As } y: 0\to 3 \qquad t: 0\to 1$$

$$= \frac{1}{3}\int_{t=0}^{1}\frac{27\,t^{3/2}}{\sqrt{1-t}}\,\frac{3}{2}\,t^{1/2-1}\,dt$$

$$= \frac{27}{3}\cdot\frac{3}{2}\int_{t=0}^{1} t^{2-1}(1-t)^{-1/2}\,dt$$

$$= \frac{27}{2}\int_{t=0}^{1} t^{2-1}(1-t)^{1/2-1}\,dt$$

$$= \frac{27}{2}\,\beta(2, 1/2) = \frac{27}{2}\,\frac{\Gamma 2\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+2\right)}$$

$$= \frac{27}{2}\,\frac{1!\,\sqrt{\pi}}{\Gamma\frac{5}{2}}$$

$$= \frac{27}{2}\,\frac{\sqrt{\pi}}{\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}} = \frac{27}{2}\cdot\frac{4}{3}$$

$$= 18$$

$$\because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

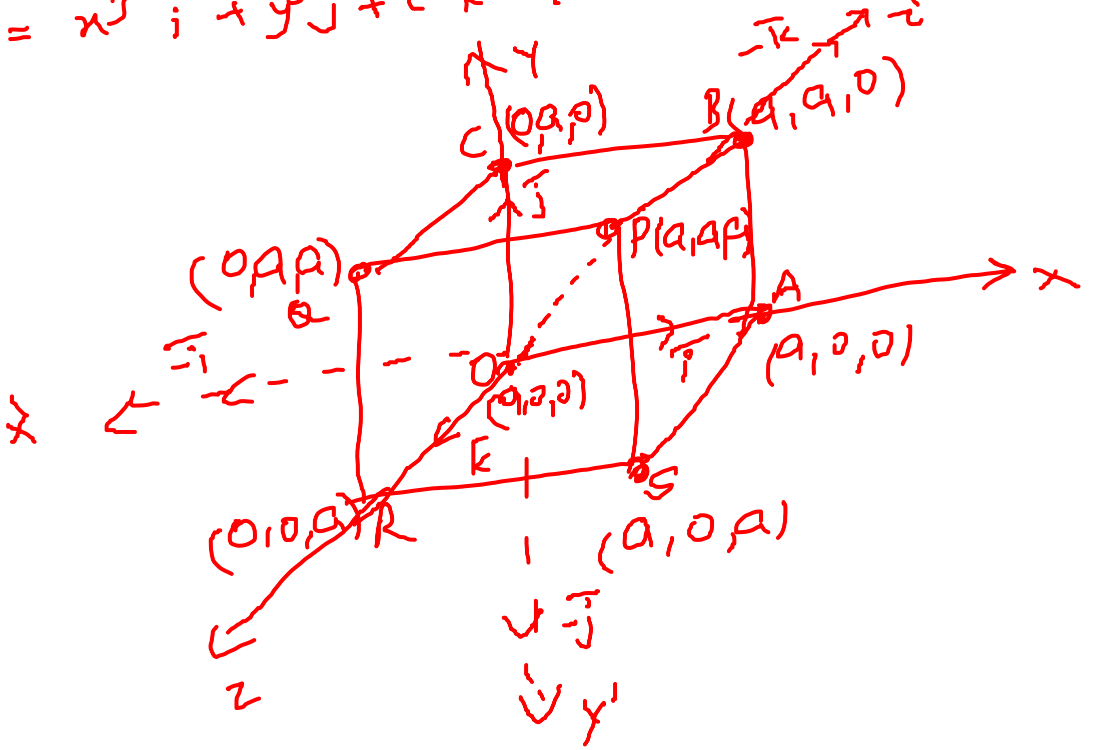
sub 2, 3, 4, 5 & 6 in 1, we get

$$L.H.S \iint_S \bar{F}\cdot\bar{n}\,ds = 0+72+0+0+108 = 180$$

$L.H.S = R.H.S$

Gauss divergence theorem is verified.

★ verify Gauss divergence theorem for $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ taken over the cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a$



Ans $\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V div\bar{F}\,dv = 3a^5$

$$L.H.S = \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1: PBAS} + \iint_{S_2: OCQR} + \iint_{S_3: PBCQ} + \iint_{S_4: OARS} + \iint_{S_5: PQRS} + \iint_{S_6: OABC}$$

$$R.H.S = \iiint_V div\bar{F}\,dv = \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} div\bar{F}\,.dv = 3a^5$$

$= 3a^5$