Euler-Cauchy Linear Eqn :- An Eqn of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \text{———} + P_n y = Q(x), \text{ where}$$

P_1, P_2 ——— P_n are real Constants and $Q(x)$ is a fun of x is called a homogeneous linear eqn or Euler-Cauchy's linear Eqn of order n.

The eqn in the operator form is $(x^n D^n + P_1 x^{n-1} D^{n-1} + \text{———} + P_n) y = Q(x)$. where $\frac{d}{dx} = D$. Cauchy's linear diff. Eqn can be transformed into a linear eqn with Constant Coefficients by the Change of independent Variable with the substitution,

$$x = e^z \text{ (or) } z = \log x, \quad x > 0 \longrightarrow (1)$$

$$\therefore \frac{dz}{dx} = \frac{1}{x}. \quad \text{Now } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \left(\frac{1}{x}\right)$$

$$\Rightarrow \boxed{x \frac{dy}{dx} = \frac{dy}{dz}} \longrightarrow (2)$$

$$\text{Again } \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x} \frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dz}\left(\frac{dy}{dz}\right) \frac{dz}{dx}$$

$$\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2 y}{dz^2}$$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} \longrightarrow (3)$$

Similarly, we can P.T $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} \longrightarrow (4)$

Let us denote $\frac{d}{dx} = D$ and $\frac{d}{dz} = \theta$. Then Eqn (2), (3), (4)

Can be written as $xD = \theta$, $x^2D^2 = \theta(\theta-1)$, $x^3D^3 = \theta(\theta-1)(\theta-2)$

etc. Using these in Eqn (1), the Euler-Cauchy eqn reduces to a diff Eqn with constant coefficients where y is dependent variable and z is independent variable.

Ex-1:- Solve $(x^2D^2 - 4xD + 6)y = x^2 \longrightarrow (1)$

Sol:- Let $x = e^z \Rightarrow z = \log x$ then

$xD = \theta$; $x^2D^2 = \theta(\theta-1)$

Sub. in Eqn (1), we get $(\theta(\theta-1) - 4\theta + 6)y = e^{2z}$

$$(\theta^2 - 5\theta + 6)y = e^{2z} \longrightarrow (2)$$

Eqn (2) is a linear diff. Eqn. with constant coefficients.

A. Eqn $f(m) = 0 \Rightarrow m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$,

$\therefore$ C.F is $y_c = c_1e^{3z} + c_2e^{2z}$

$$\text{P.I } y_p = \frac{1}{\theta^2 - 5\theta + 6}e^{2z} = \frac{1}{(\theta-2)(\theta-3)}e^{2z}$$

$$= \frac{1}{\theta-2}\left[\frac{1}{\theta-3}e^{2z}\right] = \frac{1}{\theta-2}\left[-e^{2z}\right]$$

$$y_p = -z\cdot e^{2z}$$

$\Rightarrow$ The g.s is $y = y_c + y_p$

$$y = c_1e^{3z} + c_2e^{2z} - ze^{2z}$$

$$y = c_1x^3 + c_2x^2 - \log x\,(x^2).$$

Ex-2:- Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol:- Given Eqn in the operator form is $(x^2D^2 - xD + 1)y = \log x \longrightarrow (1)$

Let $x = e^z \Rightarrow z = \log x$ then $xD = \theta$; $x^2D^2 = \theta(\theta-1)$, so that

Eqn (1) becomes $(\theta^2 - \theta + 1)y = \cancel{\log x}\ z$

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A.Eqn $f(m) = 0 \Rightarrow m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

$y_c = (c_1 + c_2 z) e^z$

$y_p = \dfrac{1}{\theta^2 - 2\theta + 1} z = \dfrac{1}{(\theta - 1)^2} z = (1-\theta)^{-2} z$

$= (1 + 2\theta + 3\theta^2) z = z + 2\theta z + 3\theta^2 z$

$= z + 2\dfrac{d}{dz}(z) + 3\dfrac{d^2}{dz^2}(z)$

$y_p = z + 2$

$\therefore$ The general Solution $y = y_c + y_p$

$y = (c_1 + c_2 z) e^z + z + 2$

$y = (c_1 + c_2 (\log x)) x + (\log x) + 2$ //

Ex-9 :- Solve $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$

Sol :- Given Eqn in the operator form is

$(x^3 D^3 + 2x^2 D^2 + 2) y = 10\left(x + \dfrac{1}{x}\right) \longrightarrow (1)$

Let $x = e^z$ ~~then~~, $\log x = z$ then $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

and $x^3 D^3 = \theta(\theta-1)(\theta-2)$. Sub in Eqn (1)

$(\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2) y = 10(e^z + e^{-z})$

$(\theta^3 - \theta^2 + 2) y = 10(e^z + e^{-z}) \longrightarrow (2)$

Eqn (2) is called Linear diff Eqn with Constant Coefficients.

A.Eqn $f(m) = 0 \Rightarrow m^3 - m^2 + 2 = 0 \Rightarrow m = -1, 1+i, 1-i$

One root is real and Two roots are Complex Conjugate.

$y_c = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$

P.I $y_p = \dfrac{1}{\theta^3 - \theta^2 + 2} 10(e^z + e^{-z})$

$= 10\left[\dfrac{1}{\theta^3 - \theta^2 + 2} e^z + \dfrac{1}{\theta^3 - \theta^2 + 2} e^{-z}\right]$

$= 10\left[\dfrac{1}{2} e^z + \dfrac{1}{(\theta+1)(\theta^2 - 2\theta + 2)} e^{-z}\right]$

$$= 10\left[\frac{1}{2}e^{z} + \frac{1}{\theta+1}\left(\frac{1}{5}e^{-z}\right)\right]$$

$$= 10\left(\frac{1}{2}e^{z} + \frac{1}{5}\frac{z}{1!}e^{-z}\right) = 5e^{z} + 2ze^{-z}$$

$$\therefore\ y = y_c + y_p$$

$$y = c_1 e^{-z} + e^{z}\left(c_2\cos z + c_3\sin z\right) + 5e^{z} + 2ze^{-z}$$

$$y = c_1\left(\frac{1}{x}\right) + x\left(c_2\cos(\log x) + c_3\sin(\log x)\right) + 5x + 2\log x\left(\frac{1}{x}\right)$$

Ex-4:- Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Sol:- Given Eqn in the operator is

$$(x^2D^2 - 3xD + 4)y = (1+x)^2 \longrightarrow (1)$$

Let $x = e^{z}$, $z = \log x$ then $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

Sub in Eqn (1) $(\theta(\theta-1) - 3\theta + 4)y = (1+e^{z})^2$

$$(\theta^2 - 4\theta + 4)y = 1 + e^{2z} + 2e^{z}$$

A.Eqn $f(m) = 0 \Rightarrow m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

$$y_c = (c_1 + c_2 z)e^{2z}$$

$$y_p = \frac{1}{\theta^2 - 4\theta + 4}\, 1 + e^{2z} + 2e^{z}$$

$$= \frac{1}{\theta^2 - 4\theta + 4}\, 1.e^{0z} + \frac{1}{\theta^2 - 4\theta + 4}e^{2z} + \frac{1}{\theta^2 - 4\theta + 4}2e^{z}$$

$$= \frac{1}{4} + \frac{1}{(\theta-2)^2}e^{2z} + \frac{1}{1}2e^{z}$$

$$= \frac{1}{4} + \frac{z^2}{2!}e^{2z} + 2e^{z}$$

$$= \cancel{\frac{1}{4} + \frac{e^{2z}}{2!}} \cancel{\frac{1}{4} + \frac{(\log x)^2}{2!}e^{2z} + 2z}$$

$$= \frac{1}{4} + \frac{(\log x)^2}{2!}.x^2 + 2x$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 z)e^{2z} + \frac{1}{4} + \frac{(\log x)^2}{2!}x^2 + 2x$$

Ex-5 :- Solve $\left(x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8\right)y=65\cos(\log x)$

Sol :- Given Eqn in the operator form is

$$(x^3D^3+3x^2D^2+xD+8)y=65\cos(\log x) \longrightarrow (1)$$

~~A Eqn f(m) = 0 ⇒~~ Let $x=e^z$, $z=\log x$ then

$$xD=\theta,\quad x^2D^2=\theta(\theta-1),\quad x^3D^3=\theta(\theta-1)(\theta-2).$$

$$(1)\Rightarrow \left(\theta(\theta-1)(\theta-2)+3\theta(\theta-1)+\theta+8\right)y=65\cos z$$

$$\Rightarrow \left(\theta(\theta^2-3\theta+2)+3\theta^2-3\theta+\theta+8\right)y=65\cos z$$

$$\Rightarrow \left(\theta^3-\cancel{3\theta^2}+\cancel{2\theta}+\cancel{3\theta^2}-\cancel{3\theta}+\cancel{\theta}+8\right)y=65\cos z$$

$$\Rightarrow (\theta^3+8)y=65\cos z \longrightarrow (2)$$

A Eqn $f(m)=0 \Rightarrow m^3+8=0$ ~~$\Rightarrow m^3=-8 \Rightarrow m=$~~

$$\Rightarrow (m+2)(m^2-2m+4)=0$$

$$\Rightarrow m=-2,\quad m=1\pm i\sqrt{3}$$

$$y_c=c_1e^{-2z}+e^{z}(c_2\cos\sqrt{3}z+c_3\sin\sqrt{3}z)$$

$$y_p=\frac{1}{\theta^3+8}65\cos z=\frac{1}{-\theta+8}65\cos z$$

$$=\frac{1}{8-\theta}65\cos z=\frac{8+\theta}{64-\theta^2}65\cos z$$

$$=\frac{8+\theta}{\cancel{65}}\cancel{65}\cos z$$

$$=(8+\theta)\cos z$$

$$=8\cos z+\theta(\cos z)$$

$$y_p=8\cos z+(-\sin z)$$

$$\therefore\ y=y_c+y_p$$

$$y=c_1e^{-2z}+e^{z}(c_2\cos\sqrt{3}z+c_3\sin\sqrt{3}z)+8\cos z-\sin z$$

$$y=c_1\left(\frac{1}{x^2}\right)+x\left(c_2\cos\sqrt{3}(\log x)+c_3\sin\sqrt{3}(\log x)\right)+8\cos(\log x)-\sin(\log x),$$

Ex - 6 :- Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$

Sol :- Given Eqn in the operator form is

$$(x^2D^2 - xD + 2)y = x\log x \longrightarrow (1)$$

Let $\cancel{xD = \theta}$, $x = e^z$, $z = \log x$ then $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

Sub in Eqn (1) $\Rightarrow (\theta(\theta-1) - \theta + 2)y = e^z.z$

$$\Rightarrow (\theta^2 - 2\theta + 2)y = ze^z \longrightarrow (2)$$

A.Eqn $f(m) = 0 \Rightarrow m^2 - 2m + 2 = 0 \Rightarrow m = 1 \pm i$

$$y_c = e^z(c_1\cos z + c_2\sin z)$$

$$y_p = \frac{1}{\theta^2 - 2\theta + 2} ze^z$$

$$= e^z\left[\frac{1}{(\theta+1)^2 - 2(\theta+1) + 2}\right]z$$

$$= e^z\left[\frac{1}{\theta^2 + \cancel{2\theta} + 1 - \cancel{2\theta} - \cancel{2} + \cancel{2}}\right]z$$

$$= e^z\left[\frac{1}{\theta^2+1}\right]z \quad = e^z(1+\theta^2)^{-1}z$$

$$= e^z(1-\theta^2)z$$

$$= e^z(z - \theta^2(z))$$

$$= e^z\left(z - \frac{d^2}{dz^2}(z)\right)$$

$$y_p = e^z.z.$$

The g.s is $y = y_c + y_p$

$$y = e^z(c_1\cos z + c_2\sin z) + ze^z$$

$$y = x(c_1\cos(\log x) + c_2\sin(\log x)) + (\log x)x$$

Ex-7 :- Solve $(x^2D^2 + 4xD + 2)y = e^x$

Sol :- Given $(x^2D^2 + 4xD + 2)y = e^x$

Let $x = e^z$, $z = \log x$ then $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

$\Rightarrow (\theta(\theta-1) + 4\theta + 2)y = e^{e^z} \Rightarrow (\theta^2 + 3\theta + 2)y = e^{e^z} \longrightarrow (1)$

$A \cdot Eqn \; f(m) = 0 \Rightarrow m^2 + 3m + 2 = 0$

$\Rightarrow m = -1, -2$

$y_c = c_1 e^{-z} + c_2 e^{-2z}$

$y_p = \dfrac{1}{\theta^2 + 3\theta + 2} e^{e^z}$

$\dfrac{1}{D-\alpha} Q = e^{\alpha x} \int Q e^{-\alpha x} dx$

$\dfrac{1}{D+\alpha} Q = e^{-\alpha x} \int Q e^{\alpha x} dx$

$y_p = \dfrac{1}{(\theta+1)(\theta+2)} e^{e^z} = \dfrac{1}{\theta+2}\left[\dfrac{1}{\theta+1} e^{e^z}\right]$

$= \dfrac{1}{\theta+2}\left[e^{-z} \int e^{e^z} \cdot e^z dz\right]$

$= \dfrac{1}{\theta+2}\left[e^{-z} e^{e^z}\right]$

$= e^{-2z} \int e^{-z} e^{e^z} \cdot e^{2z} dz$

$= e^{-2z} \int e^{e^z} \cdot e^z dz$

$y_p = e^{-2z} e^{e^z}$

The G.S is $y = y_c + y_p$

$y = c_1 e^{-z} + c_2 e^{-2z} + e^{-2z} e^{e^z}$

$y = c_1 x^{-1} + c_2 x^{-2} + x^{-2} e^x$

$y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{e^x}{x^2}$

Ex-8 :- Solve $(x^3D^3 + 2x^2D^2 + xD - 1)y = \cos(\log x)$

Sol :- Let $x = e^z$, $z = \log x$ then $xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$x^3D^3 = \theta(\theta-1)(\theta-2)$

$\Rightarrow [\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + \theta - 1]y = \cos z$

$\Rightarrow (\theta(\theta^2 - 3\theta + 2) + 2\theta^2 - 2\theta + \theta - 1) y = \cos z$

$\Rightarrow (\theta^3 - 3\theta^2 + \cancel{2\theta} + 2\theta^2 - \cancel{2\theta} + \theta - 1) y = \cos z$

$\Rightarrow (\theta^3 - \theta^2 + \theta - 1) y = \cos z$

$\Rightarrow (m-1)(m^2+1) = 0 \Rightarrow m - 1 = 0, \; m^2 + 1 = 0$

$\Rightarrow m = 1, \; m = \pm i$

$y_c = c_1 e^z + c_2 \cos z + c_2 \sin z$

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$$y_p = \frac{1}{D^3 - D^2 + D - 1}\cos z = \frac{1}{(D-1)(D^2+1)}\cos z$$

$$\cancel{= \frac{1}{-D+1}} \qquad = \frac{1}{D-1}\left[\frac{1}{D^2+1}\cos z\right]$$

$$= \frac{1}{D-1}\left[\frac{z}{2D}\cos z\right]$$

$$= \frac{1}{D-1}\left[\frac{z}{2}\int \cos z\,dz\right] = \frac{1}{D-1}\left[\frac{z}{2}\sin z\right]$$

$$= \frac{1}{2}\left[\frac{1}{D-1} z\sin z\right]$$

$$= \frac{1}{2}\left[z - \frac{1}{D-1}\right]\frac{1}{D-1}\sin z$$

$$= \frac{1}{2}\left[z - \frac{1}{D-1}\right]\frac{D+1}{D^2-1}\sin z$$

$$= \frac{1}{2}\left[z - \frac{1}{D-1}\right]\frac{D+1}{(-2)}\sin z$$

$$= -\frac{1}{4}\left[z - \frac{1}{D-1}\right](D+1)\sin z$$

$$= -\frac{1}{4}\left[z - \frac{1}{D-1}\right](\cos z + \sin z)$$

$$= -\frac{1}{4}\left[z(\cos z + \sin z) - \frac{1}{D-1}(\cos z + \sin z)\right]$$

$$= -\frac{1}{4}\left[z(\cos z + \sin z) - \frac{D+1}{D^2-1}(\cos z + \sin z)\right]$$

$$= -\frac{1}{4}\left[z(\cos z + \sin z) - \frac{D+1}{-2}(\cos z + \sin z)\right]$$

$$= -\frac{1}{4}\left[z(\cos z + \sin z) + (D+1)(\cos z + \sin z)\right]$$

$$\cancel{= -\frac{1}{4}\left[z(\cos z + \sin z) + \sin z - \cos z + \cos z + \sin z\right.}$$

$$= -\frac{1}{4}\left[z(\cos z + \sin z) + (-\cancel{\sin z} + \cos z + \cos z + \cancel{\sin z})\right]$$

$$y_p = -\frac{1}{4}\left[z(\cos z + \sin z) + 2\cos z\right]$$

The G.S is $y = y_c + y_p$

$$y = c_1 e^{z} + c_2\cos z + c_3 \sin z - \frac{1}{4}\left[z(\cos z + \sin z) + 2\cos z\right]$$

$$y = c_1 x + c_2\cos(\log x) + c_3\sin(\log x) - \frac{1}{4}\left[\log x\left(\cos(\log x) + \sin(\log x)\right) + 2\cos(\log x)\right]$$

Ex-9 :- Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Ex-10 :- Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

Legendre's Linear Equation :-

An Eqn of the form $(a+bx)^n\frac{d^ny}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n y = Q(x)$. where $P_1, P_2 \ldots P_n$ are real consts and $Q(x)$ is a fun of x is called Legendre's linear Eqn.

This can be solved by the substitution
$a+bx = e^z$ or $z = \log(a+bx)$ and $\theta = \frac{d}{dz}$ then

$(a+bx)Dy = b\theta y$

$(a+bx)^2D^2y = b^2\theta(\theta-1)y$

$(a+bx)^3D^3y = b^3\theta(\theta-1)(\theta-2)y$ and so on ---

Ex-1 :- Solve $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2+x+1 \longrightarrow (1)$

Sol :- Let $\boxed{x+1=e^z} \Rightarrow z = \log(x+1)$

$\Downarrow$

$\boxed{x = e^z - 1}$ Then $(x+1)D = \theta$

$(x+1)^2D^2 = \theta(\theta-1)$

$(1) \Rightarrow [\theta(\theta-1)y - 3\theta y + 4]y = (e^z-1)^2 + e^z - 1 + 1$

$(\theta^2 - 4\theta + 4)y = e^{2z} + 1 - 2e^z + e^z - \cancel{1} + \cancel{1}$

$(\theta^2 - 4\theta + 4)y = e^{2z} - e^z + 1 \longrightarrow (2)$

A.Eqn $f(m) = 0 \Rightarrow m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

$y_c = (C_1 + C_2 z)e^{2z}$

$y_p = \frac{1}{\theta^2 - 4\theta + 4}\, e^{2z} - e^z + 1$

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$$y_p = \frac{1}{\theta^2-4\theta+4} e^{2z} - \frac{1}{\theta^2-4\theta+4} e^{z} + \frac{1}{\theta^2-4\theta+4} 1.e^{0z}$$

$$= \frac{z}{2\theta-4} e^{2z} - \frac{1}{1-4+4} e^{z} + \frac{1}{4}$$

$$y_p = \frac{z^2}{2} e^{2z} - e^{z} + \frac{1}{4}$$

$\therefore$ the G.S is $y = y_c + y_p$

$$y = (c_1 + c_2 z)e^{2z} + \frac{z^2}{2} e^{2z} - e^{z} + \frac{1}{4}$$

$$y = (c_1 + c_2 \log(x+1))(x+1)^2 + \frac{(\log(x+1))^2}{2}(x+1)^2 - (x+1) + \frac{1}{4}$$

Ex-2 :- Solve $(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = x$

Sol :- $[(2x-1)^3 D^3 + (2x-1)D - 2]y = x \longrightarrow (1)$

Let $\boxed{2x-1 = e^z} \Rightarrow z = \log(2x-1)$

$\Downarrow$

$2x = e^z + 1 \Rightarrow \boxed{x = \frac{e^z+1}{2}}$ then

$(2x-1)D = 2\theta$

$(2x-1)^2 D^2 = 2^2\theta(\theta-1)$; $(2x-1)^3 D^3 = 2^3\theta(\theta-1)(\theta-2)$

$(1) \Rightarrow [8\theta(\theta-1)(\theta-2) + 2\theta - 2]y = \frac{e^z+1}{2}$

$[8\theta(\theta^2 - 3\theta + 2) + 2\theta - 2]y = \frac{e^z+1}{2}$

$(8\theta^3 - 24\theta^2 + 16\theta + 2\theta - 2)y = \frac{e^z+1}{2}$

$(8\theta^3 - 24\theta^2 + 18\theta - 2)y = \frac{e^z+1}{2} \longrightarrow (2)$

A.Eqn $f(m) = 0 \Rightarrow 8m^3 - 24m^2 + 18m - 2 = 0 \Rightarrow m=1, m = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

$$y_c = c_1 e^{z} + e^{\frac{z}{2}}\left(c_2 \cos\frac{\sqrt{3}}{2}z + c_3 \sin\frac{\sqrt{3}}{2}z\right)$$

$$y_p = \frac{1}{8\theta^3-24\theta^2+18\theta-2}\left(\frac{e^z+1}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{8\theta^3-24\theta^2+18\theta-2} e^{z} + \frac{1}{8\theta^3-24\theta^2+18\theta-2} 1.e^{0z}\right]$$

$$= \frac{1}{2}\left[\frac{z}{24\theta^2-45\theta+18} e^{z} + \frac{1}{-2}\right]$$

$$= \frac{1}{2}\left[\frac{z}{-24+18} e^{z} - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{-z}{6} e^{z} - \frac{1}{2}\right] \Rightarrow \frac{-ze^{z}}{12} - \frac{1}{4}$$

$$y = y_c + y_p$$

$$y = C_1 e^{z} + e^{\frac{z}{2}}\left[C_2 \cos\frac{\sqrt{3}}{2} z + C_3 \sin\frac{\sqrt{3}}{2} z\right] - \frac{ze^{z}}{12} - \frac{1}{4}$$

$$y = C_1 (2x-1) + e^{\frac{\log(2x-1)}{2}}\left[C_2 \cos\frac{\sqrt{3}}{2} \log(2x-1) + C_3 \sin\frac{\sqrt{3}}{2} \log(2x-1)\right] - \frac{\log(2x-1)(2x-1)}{12} - \frac{1}{4}$$

Ex-3 :- Solve $\left[(x+1)^2 D^2 + (x+1) D\right] y = (2x+3)(2x+4) \longrightarrow (1)$

Sol :- Let $x+1 = e^{z} \Rightarrow z = \log(x+1)$

$$\Downarrow$$

$$\boxed{x = e^{z}-1}$$

Then $(x+1)D = \theta$

$(x+1)^2 D^2 = \theta(\theta-1)$

$$(1) \Rightarrow \left[\theta(\theta-1) + \theta\right] y = \left(2(e^{z}-1)+3\right)\left(2(e^{z}-1)+4\right)$$

$$\theta^2 y = (2e^{z} - 2 + 3)(2e^{z} - 2 + 4)$$

$$\theta^2 y = (2e^{z}+1)(2e^{z}+2)$$

$$\theta^2 y = 4e^{2z} + 4e^{z} + 2e^{z} + 2$$

$$\theta^2 y = 4e^{2z} + 6e^{z} + 2 \longrightarrow (2)$$

A.Eqn $f(m) = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0, 0$

$$y_c = (C_1 + C_2 z) e^{0z} \Rightarrow y_c = C_1 + C_2 z$$

$$y_p = \frac{1}{\theta^2}\left(4e^{2z} + 6e^{z} + 2\right)$$

$$= \frac{1}{\theta^2} 4e^{2z} + \frac{1}{\theta^2} 6e^{z} + \frac{1}{\theta^2} 2 \cdot e^{0z}$$

$$= \frac{1}{4} 4e^{2z} + 1 \cdot 6e^{z} + \frac{1}{(\theta-0)^2} 2e^{0z}$$

$$y_p = e^{2z} + 6e^{z} + \frac{z^2}{2} 2 \Rightarrow y_p = e^{2z} + 6e^{z} + z^2$$

The general Sol $y = y_c + y_p$

$$y = (C_1 + C_2 z) + e^{2z} + 6e^{z} + z^2$$

$$y = C_1 + C_2 \log(x+1) + (x+1)^2 + 6(x+1) + (\log(x+1))^2$$

Ex-4 :- Solve $[(1+x)^2 D^2 + (1+x) D + 1] y = 4 \cos\log(x+1) \longrightarrow (1)$

Sol :- Let $\boxed{x + 1 = e^{z}} \Rightarrow z = \log(x+1)$ then

$$(x+1)D = \theta, \quad (x+1)^2 D^2 = \theta(\theta - 1)$$

$$(1) \Rightarrow (\theta(\theta-1) + \theta + 1) y = 4 \cos z$$

$$\Rightarrow (\theta^2 + 1) y = 4 \cos z \longrightarrow (2)$$

A.Eqn $f(m) = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$

$$y_c = C_1 \cos z + C_2 \sin z$$

$$y_p = \frac{1}{\theta^2 + 1} 4 \cos z \Rightarrow 4 \cdot \frac{z}{2\theta} \cos z$$

$$= 2z \int \cos z \, dz$$

$$= 2z \sin z$$

$\therefore$ The general Solution is $y = y_c + y_p$

$$y = C_1 \cos z + C_2 \sin z + 2z \sin z$$

$$y = C_1 \cos\log(x+1) + C_2 \sin\log(x+1) + 2\log(x+1) \sin\log(x+1)$$

Electrical Circuits :-

LCR Circuits :- Consider the discharge of a condenser C through an induction L and the resistance R. Since the voltage drop across L, C and R are resp. $L \frac{d^2q}{dt^2}$, $\frac{q}{C}$ and $R \frac{dq}{dt}$.

$\therefore$ By Kirchhoff's law, $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$

on dividing by L and writing $\frac{R}{L} = 2\lambda$ and $\frac{1}{LC} = \mu^2$

$$\frac{d^2q}{dt^2} + 2\lambda \frac{dq}{dt} + \mu^2 q = 0 \quad \text{or} \quad (D^2 + 2\lambda D + \mu^2) q = 0$$

$$\therefore m = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\mu^2}}{2} \Rightarrow m = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$$

Now three cases arise

Case (i) :- when $\lambda > \mu$, the roots are real and distinct. (Say m_1, m_2)

$\therefore$ $q = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

Case (ii) :- when $\lambda = \mu$, the roots are real and equal

$\therefore$ $q = (c_1 + c_2 t) e^{-\lambda t}$

Case (iii) :- when $\lambda < \mu$, the roots are complex conjugate (Say $-\lambda \pm i\alpha$)

$\therefore$ $q = e^{-\lambda t}(c_1 \cos\alpha t + c_2 \sin\alpha t)$

Ex-1 :- A condenser of a capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the eqn $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given that $L = 0.25h$, $R = 250\Omega$, $C = 2\times10^{-6}F$, and that when $t = 0$, charge q is 0.002 Coulombs and the current $\frac{dq}{dt} = 0$. Find the value of q interms of t.

Sol :- Given diff. Eqn $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$

or $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0 \longrightarrow (1)$

Sub the given values in Eqn (1)

$$\frac{d^2q}{dt^2} + \frac{250}{0.25}\frac{dq}{dt} + \frac{q}{0.25\times10^{-6}} = 0$$

$$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + 2\times10^6 q = 0$$

(or) $(D^2 + 1000D + 2\times10^6)q = 0$

A. Eqn $f(m) = 0 \Rightarrow m^2 + 1000m + 2\times10^6 = 0$

$\Rightarrow m = \frac{-1000 \pm \sqrt{10^6 - 8\times10^6}}{2} \Rightarrow m = \frac{-1000 \pm 1000\sqrt{7}i}{2}$

$\Rightarrow m = -500 \pm 1323i$

The Solution $q = e^{-500t}(c_1 \cos 1323t + c_2 \sin 1323t) \rightarrow (2)$

when $t=0$, $q = 0.002$

$0.002q = c_1(1) + c_2(0) \Rightarrow \boxed{c_1 = 0.002}$

Diff Eqn (1) w.r.t. t.

$$\frac{dq}{dt} = -500e^{-500t}\left(c_1\cos 1323t + c_2\sin 1323t\right) + e^{-500t}\left(-1323c_1\sin 1323t + 1323c_2\cos 1323t\right)$$

when $t=0$, $\frac{dq}{dt} = 0$

$$0 = -500\left((0.002)(1) + c_2(0)\right) + (-1323c_1(0)) + 1323c_2(1)$$

$$0 = -1 + 1323c_2 \Rightarrow c_2 = \frac{1}{1323} \Rightarrow \boxed{c_2 = 0.0008}$$

Sub c_1, c_2 values in (2)

$$q = e^{-500t}\left(0.002\cos 1323t + 0.0008\sin 1323t\right)$$

Ex-2:- An uncharged condenser of capacity c is charged by applying an e.m.f $E\sin\left(\frac{t}{\sqrt{LC}}\right)$, through leads of self inductance L and negligible resistance. P.T at any time t, the charge on one of the plate is $\frac{EC}{2}\left\{\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right\}$

Sol:- If q be the charge on the condenser, the diff. Eqn of the circuit is $L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\frac{t}{\sqrt{LC}}$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow \left(D^2 + \frac{1}{LC}\right)q = \frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right) \longrightarrow (1)$$

A.Eqn $f(m) = 0 \Rightarrow m^2 + \frac{1}{LC} = 0 \Rightarrow m = \pm\frac{1}{\sqrt{LC}}i$

C.F $q_c = c_1\cos\frac{t}{\sqrt{LC}} + c_2\sin\frac{t}{\sqrt{LC}}$

P.I $\quad q_p = \dfrac{1}{D^2 + 1/LC} \dfrac{E}{L} \sin\left(\dfrac{t}{\sqrt{LC}}\right)$

$$\cancel{= \frac{1}{-\frac{1}{LC} + 1/LC} \frac{E}{L} \sin\left(\frac{t}{\sqrt{LC}}\right)}$$

$$= \frac{t}{2D} \frac{E}{L} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$= \frac{Et}{2L}\left[\frac{-\cos(t/\sqrt{LC})}{1/\sqrt{LC}}\right]$$

$$= \frac{Et}{2L}\left[-\sqrt{LC}\cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

$$q_p = \frac{-Et}{2}\sqrt{\frac{C}{L}}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

The g.s of Eqn (1) is $q = q_c + q_p$

$$q = C_1\cos\left(\frac{t}{\sqrt{LC}}\right) + C_2\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{Et}{2}\sqrt{\frac{C}{L}}\cos\left(\frac{t}{\sqrt{LC}}\right) \longrightarrow (2)$$

when $t=0$, $q=0$ in (1)

$$0 = C_1(1) + C_2(0) - 0 \Rightarrow \boxed{C_1 = 0}$$

Diff Eqn (1) w.r.t t.

$$\frac{dq}{dt} = -C_1\frac{1}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) + C_2\frac{1}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{E}{2}\sqrt{\frac{C}{L}}\left\{-\frac{t}{\sqrt{LC}}\sin\left(\frac{1}{\sqrt{LC}}\right) + \cos\left(\frac{t}{\sqrt{LC}}\right)\right\}$$

when $t=0$, $\dfrac{dq}{dt}=0$

$$0 = -C_1(0) + C_2\cdot\frac{1}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}}\left\{0+1\right\}$$

$$0 = \frac{C_2}{\sqrt{LC}} - \frac{E}{2}\frac{\sqrt{C}}{\sqrt{L}} \Rightarrow C_2 = \frac{E}{2}\frac{\sqrt{C}}{\cancel{\sqrt{L}}} \times \cancel{\sqrt{L}}\sqrt{C}$$

$$\boxed{C_2 = \frac{EC}{2}}$$

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Sub C_1, C_2 Values in (1)

$$q = \frac{EC}{2}\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{Et}{2}\sqrt{\frac{C}{L}}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$= \frac{EC}{2}\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{Et}{2}\frac{\sqrt{C}}{\sqrt{C}}\cdot\frac{\sqrt{C}}{\sqrt{L}}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$= \frac{EC}{2}\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{EtC}{2}\cdot\frac{1}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$q = \frac{EC}{2}\left\{\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{t}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)\right\}$$

Ex-3:- The charge $q(t)$ on the capacitor is given by D.E, $10\frac{d^2q}{dt^2} + 120\frac{dq}{dt} + 1000q = 17\sin(2t)$. At time zero the current is zero and the charge on the capacitor is $\frac{1}{2000}$ Coulomb. Find the charge on the capacitor for $t > 0$.

Sol:- The diff Eqn can be written as

$$\frac{d^2q}{dt^2} + 12\frac{dq}{dt} + 100q = \frac{17}{10}\sin 2t$$

$$(D^2+12D+100)q = \frac{17}{10}\sin 2t \longrightarrow (1)$$

A. Eqn $f(m) = 0 \Rightarrow m^2 + 12m + 100 = 0 \Rightarrow m = -6 \pm 8i$

C.F $\quad q_c = e^{-6t}(c_1\cos 8t + c_2\sin 8t)$

P.I $\quad q_p = \frac{1}{D^2+12D+100}\,\frac{17}{10}\sin 2t$

$$= \frac{17}{10}\left[\frac{1}{12D+96}\sin 2t\right]$$

$$= \frac{17}{120}\left[\frac{D-8}{D^2-64}\right]\sin 2t$$

$$= \frac{17}{120}\left[\frac{D-8}{-\cancel{68}_{4}}\right]\sin 2t$$

$$= -\frac{1}{480}(2\cos 2t - 8\sin 2t)$$

$$q_p = \frac{1}{240}(4\sin 2t + \cos 2t)$$

The g.s of Eqn ① is $q = q_c + q_p$

$$q = e^{-6t}\left(c_1 \cos 8t + c_2 \sin 8t\right) + \frac{1}{240}\left(4\sin 2t + \cos 2t\right) \longrightarrow ②$$

At $t=0$, $q = \frac{1}{2000}$ in ①

$$\frac{1}{2000} = c_1(1) + c_2(0) + \frac{1}{240}(1) \Rightarrow c_1 =$$

$$\frac{1}{2000} - \frac{1}{240} = c_1 \Rightarrow \boxed{c_1 = \frac{7}{1500}}$$

Diff Eqn ① w.r.t t.

$$\frac{dq}{dt} = e^{-6t}\left(-8c_1 \sin 8t + 8c_2 \cos 8t\right) - 6e^{-6t}\left(c_1 \cos 8t + c_2 \sin 8t\right)$$
$$+ \frac{1}{240}\left(8\cos 2t - 2\sin 2t\right)$$

At, $t=0$, $\frac{dq}{dt} = 0$ in ①

$$0 = c_1(0) + 8c_2(1) - 6\left(c_1(1) + c_2(0)\right) + \frac{1}{\cancel{240}_{30}}(\cancel{8})$$

$$0 = 8c_2 - 6c_1 + \frac{\cancel{8}1}{30}$$

$$0 = 8c_2 - \frac{42}{1500} + \frac{1}{30} \Rightarrow \boxed{c_2 = -\frac{1}{1500}}$$

Sub c_1 & c_2 values in ①

$$q = e^{-6t}\left(\frac{7}{1500}\cos 8t - \frac{1}{1500}\sin 8t\right) + \frac{1}{240}\left(4\sin 2t + \cos 2t\right)$$

System of Simultaneous Linear Differential Eqns with Constant Coefficients :-

Let the given Simultaneous eqns be

$$\left.\begin{aligned} f_1(D)x + f_2(D)y &= f(t) \\ g_1(D)x + g_2(D)y &= g(t), \quad \text{where } D = \frac{d}{dt} \end{aligned}\right\} \longrightarrow ①$$

in x and y are functions of t and $f_1(D)$, $f_2(D)$, $g_1(D)$, $g_2(D)$ are rational integral functions with constant coefficients and $f(t)$ and $g(t)$ are the functions of the independent variable t.

Now let Δ be the determinant obtained from ① as given below.

Clearly Δ involves the operator "Coefficients" of x & y in (1)

$$\Delta = \begin{vmatrix} f_1(D) & f_2(D) \\ g_1(D) & g_2(D) \end{vmatrix} \longrightarrow (2)$$

then the number of arbitrary constants appearing in the general solution of the system of Eqn (1) is equal to the degree in D of the determinant Δ given by Eqn (2), provided that this determinant is not zero.

Ex 5.1 Solve $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -5x - 3y$.

Sol:- In operator notation, the eqns are

$$(D-3)x - 2y = 0 \longrightarrow (1)$$

$$5x + (D+3)y = 0 \longrightarrow (2)$$

Multiplying Eqn (1) by $(D+3)$ and Eqn (2) by -2, we get

$$(D+3)(D-3)x - (D+3)2y = 0 \longrightarrow (3)$$

$$-10x - 2(D+3)y = 0 \longrightarrow (4)$$

Solve Eqns (3) & (4) $\Rightarrow$ $(D^2-9)x + 10x = 0$

$\Rightarrow$ $(D^2+1)x = 0$

A. Eqn $f(m) = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F is $\boxed{x = C_1 \cos t + C_2 \sin t} \longrightarrow (5)$

Eqn (5) sub. in Eqn (1)

$$y = \frac{1}{2}(D-3)(C_1 \cos t + C_2 \sin t)$$

$$y = \frac{1}{2}\left[-C_1 \sin t + C_2 \cos t - C_1 3 \cos t - C_2 3 \sin t\right]$$

$$= \frac{1}{2}\left[(C_2 - 3C_1)\cos t + (-C_1 - 3C_2)\sin t\right]$$

$$y = \frac{1}{2}\left[A \cos t - B \sin t\right], \text{ where } A = \frac{C_2 - 3C_1}{2}$$

$$B = \frac{-C_1 - 3C_2}{2}$$

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Ex-2 :- Solve $\frac{dx}{dt} = 5x + y$; $\frac{dy}{dt} = y - 4x$.

Sol :- The given system of Eqns is

$$(D-5)x - y = 0 \longrightarrow (1)$$
$$4x + (D-1)y = 0 \longrightarrow (2)$$

Multiplying Eqn (1) by (D-1) and add to Eqn (2)

$$(D-1)(D-5)x \cancel{(D-1)y} + 4x + \cancel{(D-1)y} = 0$$
$$[(D-1)(D-5)+4]x = 0$$
$$(D^2 - 6D + 5 + 4)x = 0 \Rightarrow (D^2 - 6D + 9)x = 0$$
$$\Rightarrow (D-3)^2 x = 0$$

A Eqn $f(m) = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$.

$$\therefore \boxed{x = (c_1 + c_2 t)e^{3t}} \longrightarrow (3)$$

Sub. Eqn (3) in (1)

$$(D-5)(c_1 + c_2 t)e^{3t} - y = 0$$
$$(D-5)(c_1 e^{3t} + c_2 t e^{3t}) = y$$
$$D(c_1 e^{3t} + c_2 t e^{3t}) - 5c_1 e^{3t} - 5t c_2 e^{3t} = y$$
$$\Rightarrow y = c_1 3 e^{3t} + c_2 (3t e^{3t} + e^{3t}) - 5c_1 e^{3t} - 5t c_2 e^{3t}$$
$$y = 3c_1 e^{3t} + 3c_2 t e^{3t} + c_2 e^{3t} - 5c_1 e^{3t} - 5c_2 t e^{3t}$$
$$y = -2c_1 e^{3t} + c_2 e^{3t} - 2c_2 t e^{3t}$$
$$\boxed{y = (-2c_1 + c_2 - 2c_2 t)e^{3t}}$$

Ex-3 :- Solve $D^2 x + y = \sin t$; $x + D^2 y = \cos t$.

Sol :- Given Eqns

$$D^2 x + y = \sin t \longrightarrow (1)$$
$$x + D^2 y = \cos t \longrightarrow (2)$$

Eqn (2) Multiplying D^2 obs, $D^2 x + D^4 y = -\cos t \longrightarrow (3)$

$$\text{Eqn}(3) - \text{Eqn}(4) = (D^2x + D^4y - D^2x - y) = -\cos t - \sin t$$

$$\Rightarrow (D^4 - 1)y = -(\cos t + \sin t)$$

$$\text{A Eqn } f(m) = 0 \Rightarrow m^4 - 1 = 0 \Rightarrow (m^2)^2 - 1^2 = 0$$

$$\Rightarrow (m^2+1)(m^2-1) = 0$$

$$\Rightarrow m = \pm i,\ m = \pm 1$$

$$\boxed{y_c = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t}$$

$$y_p = \frac{1}{D^4 - 1}\left(-(\cos t + \sin t)\right)$$

$$= -\frac{t}{4D^3}(\cos t + \sin t)$$

$$= -\frac{t}{4(-1)D}(\cos t + \sin t)$$

$$y_p = \frac{t}{4D}(\cos t + \sin t) = \frac{t}{4}(\sin t - \cos t)$$

$$y = y_c + y_p$$

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t) \rightarrow (3)$$

sub Eqn (3) in Eqn (2)

$$x + D^2\left(c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t)\right) = \cos t$$

$$x + D\left(c_1 e^t - c_2 e^{-t} - c_3 \sin t + c_4 \cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)\right) = \cos t$$

$$x + \left(c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)\right) = \cos t$$

$$x + \left(c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t\right) + \frac{t}{4}(\cos t - \sin t) + \frac{1}{2}(\cos t + \sin t) = \cos t$$

$$y = -c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t - \frac{t}{4}(\cos t - \sin t)$$
$$-\frac{1}{2}\cos t + \frac{\sin t}{2} + \cos t$$

$$y = -c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t) +$$
$$\frac{\sin t}{2} + \frac{\cos t}{2}$$

$$= -c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t + \cos t)$$

$$= -c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4}(t \sin t - t\cos t + 2\sin t + 2\cos t)$$

$$= -c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4}\left[(2+t)\sin t + (2-t)\cos t\right]$$

Ex-4 :- Solve $\frac{dx}{dt} + y = e^{t}$, $\frac{dy}{dt} - x = e^{-t}$

Sol :- Given Eqns are $Dx + y = e^{t} \longrightarrow (1)$

$Dy - x = e^{-t} \longrightarrow (2)$

Eqn (1) multiplying with D and subtract from Eqn (2)

$$D^2x + Dy - \cancel{Dy} + x = e^{t} - e^{-t}$$
$$(D^2+1)x = e^{t} - e^{-t}$$

A Eqn $f(m) = 0 \Rightarrow m^2+1 = 0 \Rightarrow m = \pm i$

$$x_c = c_1 \cos t + c_2 \sin t$$

$$x_p = \frac{1}{D^2+1}(e^{t} - e^{-t})$$

$$x_p = \frac{1}{D^2+1} e^{t} - \frac{1}{D^2+1} e^{-t} = \frac{e^{t} - e^{-t}}{2} = \sinh t$$

$\therefore x = x_c + x_p$

$$\boxed{x = c_1 \cos t + c_2 \sin t + \sinh t} \longrightarrow (3)$$

Sub Eqn (3) in (1)

$D(c_1 \cos t + c_2 \sin t + \sinh t) + y = e^t$

$-c_1 \sin t + c_2 \cos t + \cosh t + y = e^t$

$y = e^t + c_1 \sin t + c_2 \cos t + \cosh t$

$\cancel{= e^t + c_1 \sin t + c_2 \cos t + \frac{e^t - e^{-t}}{2}}$

Ex-5 :- Solve $(D+2)x + (D+1)y = t$

$5x + (D+3)y = t^2$

Sol :- Given Eqns are $(D+2)x + (D+1)y = t \longrightarrow (1)$

$5x + (D+3)y = t^2 \longrightarrow (2)$

Eqn (1) multiply with 5 and Eqn (2) Eqn (2) multiply with $(D+2)$ and subtract Eqn (1) - Eqn (2)

$5\cancel{(D+2)x} + 5(D+1)y - \cancel{5(D+2)x} - (D+2)(D+3)y = 5t - (D+2)t^2$

$5(D+1)y - (D^2+5D+6)y = 5t - 2t - 2t^2$

$(5\cancel{D}+5 - D^2 - \cancel{5D} - 6)y = 3t - 2t^2$

$(-D^2-1)y = 3t - 2t^2$

$(D^2+1)y = 2t^2 - 3t \longrightarrow (3)$

A Eqn $f(m) = 0 \Rightarrow m^2+1 = 0 \Rightarrow m = \pm i$

C.F $y_c = c_1 \cos t + c_2 \sin t$

P.I $y_p = \frac{1}{D^2+1} 2t^2 - 3t$

$= (1+D^2)^{-1}(2t^2 - 3t)$

$= (1-D^2)(2t^2-3t)$

$= 2t^2 - 3t - D^2(2t^2 - 3t)$

$= 2t^2 - 3t - 4$

$\therefore y = y_c + y_p \Rightarrow \boxed{y = c_1 \cos t + c_2 \sin t + 2t^2 - 3t - 4} \rightarrow (4)$

Sub Eqn (4) in (2)

$\Rightarrow 5x + (D+3)(c_1\cos t + c_2\sin t + 2t^2 - 3t - 4) = t^2$

$\Rightarrow 5x + (-c_1\sin t + c_2\cos t + 4t - 3) + 3c_1\cos t + 3c_2\sin t + 3(2t^2 - 3t - 4) = t^2$

$\Rightarrow 5x - c_1\sin t + c_2\cos t + 4t - 3 + 3c_1\cos t + 3c_2\sin t + 6t^2 - 9t - 12 - t^2 = 0$

$\Rightarrow 5x = c_1\sin t - c_2\cos t - 4t + 3 - 3c_1\cos t - 3c_2\sin t - 6t^2 + 9t + 12 + t^2$

$x = \frac{1}{5}\left[(c_1 - 3c_2)\sin t + (-c_2 - 3c_1)\cos t - 5t^2 + 5t + 15\right]$

$x = \left(\frac{c_1 - 3c_2}{5}\right)\sin t - \left(\frac{c_2 + 3c_1}{5}\right)\cos t - t^2 + t + 3$ //

Ex-6 :- $(D+6)y - Dx = 0$; $(3-D)x - 2Dy = 0$ with $x=2, y=3$ when $t=0$

Sol :- Given Eqns are

$(D+6)y - Dx = 0 \longrightarrow (1)$

$(3-D)x - 2Dy = 0 \longrightarrow (2)$

Eqn (1) Multiply with $(2D)$ and Eqn (2) Multiply with $(D+6)$ and Add from Eqn (1) to Eqn (2)

$\cancel{(3-D)(D+6)y - (3-D)Dx - (D+6)(3-D)}$

$2D(D+6)y - 2D^2x + (D+6)(3-D)x - 2D(D+6)y = 0$

$\left[-2D^2 + (D+6)(3-D)\right]x = 0$

$(-2D^2 + 3D - D^2 + 18 - 6D)x = 0$

$(-3D^2 - 3D + 18)x = 0 \Rightarrow (D^2 + D - 6)x = 0 \longrightarrow (3)$

A.Eqn $m^2 + m - 6 = 0 \Rightarrow (m-2)(m+3) = 0$

$\Rightarrow m = 2, -3$

$\boxed{x = c_1e^{2t} + c_2e^{-3t}} \longrightarrow (4)$

Sub Eqn (4) in (2)

$(3-D)(c_1e^{2t} + c_2e^{-3t}) - 2Dy = 0$

$$\Rightarrow 3c_1e^{2t} + 3c_2 e^{-3t} - D(c_1e^{2t} + c_2 e^{-3t}) - 2Dy = 0$$

$$\Rightarrow \underline{3c_1e^{2t}} + 3c_2e^{-3t} - \underline{c_1 2e^{2t}} + 3c_2 e^{-3t} = 2Dy$$

$$\Rightarrow c_1e^{2t} + 6c_2e^{-3t} = 2Dy$$

$$\Rightarrow y = \frac{1}{2D}\left[c_1e^{2t} + 6c_2e^{-3t}\right]$$

$$= \frac{1}{2}\left(c_1\frac{e^{2t}}{2} - \cancel{6}^{\,2} c_2 \frac{e^{-3t}}{\cancel{3}}\right)$$

$$\boxed{y = \frac{c_1}{4}e^{2t} - c_2e^{-3t}} \longrightarrow (5)$$

Given $x = 2$, $y = 3$, when $t = 0$

from Eqn (4), $x = 2$, $t = 0$

$$\boxed{2 = c_1 + c_2} \longrightarrow (6)$$

from Eqn (5), $y = 3$, $t = 0$

$$\boxed{3 = \frac{c_1}{4} - c_2} \longrightarrow (7)$$

Solving Eqn's (6) & (7), we get $\boxed{c_1 = 4, \; c_2 = -2}$

Sub c_1 & c_2 values in Eqn's (4), (5)

$$x = 4e^{2t} - 2e^{-3t}$$

$$y = e^{2t} + 2e^{-3t} \;//$$