UNIT-5

① Line integral: $\oint \bar{F} \cdot d\bar{r}$ $\quad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

② Surface integral: $\iint_S \bar{F} \cdot \hat{n} \, ds$

③ Green theorem: $\oint_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \partial x \, \partial y$

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$

④ Gauss divergence theorem: $\iint_S \bar{F} \cdot \hat{n} \, ds = \iiint_V \operatorname{div} \bar{F} \, dv$

⑤ Stokes theorem:

$\oint \bar{F} \cdot d\bar{r} = \iint_S \hat{n} \cdot \operatorname{curl} \bar{F} \, ds$

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(1) $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$

Evaluate $\int_C \bar{F} \cdot d\bar{r}$

$y = x^3$ from $(1,1)$ to $(2,8)$

given

$\bar{F} = (5xy - 6x^2)i + (2y - 4x)\bar{j}$ and $y = x^3$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\int_C \bar{F} \cdot d\bar{r} = \int_C (5xy - 6x^2) dx + \int (2y - 4x) dy$

given plane $y = x^3$; diff $dy = 3x^2$; put $y = x^3$, $dy = 3x^2$

$(1,1)$ to $(2,8)$; limits = 1 → 2

from ① $\int_1^2 5xy - 6x^2 + [2(x^3) - 4x]\, 3x^2$

$= \int_1^2 (5x \cdot x^3 - 6x^2 + 6x^5 - 12x^3) dx$

$= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3) dx$

$\left[\frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4} \right]_1^2$

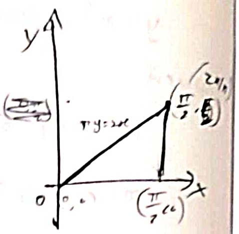
$\left[x^5 - 2x^3 + x^6 - 3x^4 \right]_1^2$

$= [(32 - 16 + 64 - 3(16)) - (1 - 2 + 1 - 3)] = 32 + 3 = 35$

Green's theorem:

1(B) $\int_C (y - \sin x)\,dx + \cos x\,dy$; where C is the triangle enclosed by the lines

$y = 0$, $x = \frac{\pi}{2}$ and $\pi y = 2x$



given $\int_C (y - \sin x)\,dx + \cos x\,dy$

compare with $M\,dx + N\,dy$

$M = y - \sin x$ $\quad N = \cos x$

$\frac{\partial M}{\partial y} = 1 - 0$ $\quad \frac{\partial N}{\partial x} = -\sin x$

$\iint_C \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ $\quad \iint_C [-\sin x - 1] = -\iint (\sin x + 1)\,\partial x\,\partial y$

given $x = \frac{\pi}{2}$; $y = 0$ and $\pi y = 2x$, $y = \frac{2x}{\pi}$

$\therefore$ limit is $\quad 0 = \frac{2x}{\pi}$

$\therefore x \to 0 \to \frac{\pi}{2}$ $\quad [x = 0]$ $[y = 0]$

$y \to 0 \to \frac{2x}{\pi}$

$- \int_0^{\pi/2} \int_0^{2x/\pi} [\sin x + 1]\,dy\,dx$

$- (\sin x + 1)\,(y)_0^{2x/\pi}$

$- (\sin x + 1)\,(\frac{2x}{\pi} - 0]$

$\Rightarrow \frac{-2}{\pi} \int_0^{\pi/2} (x\sin x + x)\,dx$

$\Rightarrow \frac{-2}{\pi} \left[x(-\cos x) + \sin x + \frac{x^2}{2}\right]_0^{\pi/2}$

$= \frac{-2}{\pi} \left[-x\cos x + \sin x + \frac{x^2}{2}\right]_0^{\pi/2}$

$\frac{-2}{\pi} \left[\left[-\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) + \sin\frac{\pi}{2} + \frac{\pi^2}{8}\right] - \left[0 + \sin 0 + 0\right]\right]$

$\Rightarrow 0 + 1 + \frac{\pi^2}{8}$

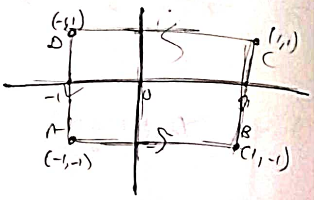
$\frac{-2}{\pi} \left[0 + 1 + \frac{\pi^2}{8}\right]$

$\Rightarrow \frac{-2}{\pi} \cdot \frac{-2\pi ...}{8\pi}$ 4

$\Rightarrow \frac{-2}{\pi} - \frac{\pi}{4}$

2 (a) given $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

lines are $x=\pm1$, $y=\pm1$



$\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

$\int M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\oint (x^2+xy)\,dx + (x^2+y^2)\,dy$

(AB) limits -1 to 1

$y=-1$, $dy=0$

$\int_{-1}^{1} (x^2+xy)\,dx + \int_{-1}^{1} (x^2+y^2)\,dy$

$dy=0$ $\int_{-1}^{1} x^2+xy\,(dy) = 0$

$\Rightarrow \int_{-1}^{1} (x^2+xy)\,dx$

put $y=-1$

$\int_{-1}^{1} (x^2-x)\,dx$ — (1)

(BC) $x=1$ $y \to -1, 1$

$dx=0$

$BC = \int_{-1}^{1} (x^2+y^2)\,dy$

put $x=1$

$\int_{-1}^{1} (1+y^2)\,dy = 0$ — (2)

(CD) $y=1$ $x \to 1 \to -1$

$dy=0$

$\int_{1}^{-1} (x^2+xy)\,dx$ put $y=1$

$CD = \int_{1}^{-1} (x^2+x)\,dx$ — (3)

DA $\to$ $x=-1$ $y \to 1 \to -1$

$dx=0$

$\int_{1}^{-1} (x^2+y^2)\,dy$

$\Rightarrow \int_{1}^{-1} (1+y^2)\,dy$ — (4)

$\Rightarrow \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

(1) + (2) + (3) + (4)

$\int_{-1}^{1} (x^2-x)\,dx + \int_{-1}^{1} (1+y^2)\,dy + \int_{1}^{-1} (x^2+x)\,dx + \int_{1}^{-1} (1+y^2)\,dy$

$\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} + \left[y + \frac{y^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{1}^{-1} + \left[y + \frac{y^3}{3}\right]_{1}^{-1}$

$\left[\left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{-1}{3} + \frac{1}{2}\right)\right] + \left[\left(1+\frac{1}{3}\right) - \left(-1-\frac{1}{3}\right)\right] + \left[\frac{-1}{3} \frac{-1}{2} - \left(\frac{1}{3} + \frac{1}{2}\right)\right]$

$+ \left[-1-\frac{1}{3} - \left(1+\frac{1}{3}\right)\right]$

$\Rightarrow \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{1}{2}\right] + \left[1 + \frac{1}{3} + 1 + \frac{1}{3}\right] + \left[\frac{-1}{3} - \frac{1}{2} - \frac{1}{3} - \frac{1}{2}\right]$

$+ \left[-1 - \frac{1}{3} - 1 - \frac{1}{3}\right]$

$\Rightarrow \left[\frac{2}{3} - \frac{2}{2}\right] + \left[2 + \frac{2}{3}\right] + \left[\frac{-2}{3} - \frac{2}{2}\right] + \left[-2 - \frac{2}{3}\right]$

$\frac{2}{3} - 1 + 2 + \frac{2}{3} - \frac{2}{3} - 1 - 2 - \frac{2}{3}$

$\Rightarrow -2$

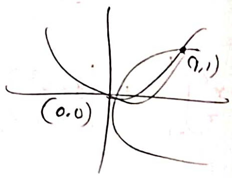
Verify only RHS; Evaluate LHS

(8) given $\int (3x^2 - 8y^2)dx + (4y - 6xy)dy$

where region is $y = \sqrt{x}$ and $y = x^2$

given $\int (3x^2 - 8y^2)dx + (4y - 6xy)\,dy$

M N



$y = \sqrt{x}$ | $y = x^2$

$y^2 = x$ | $y = x^2$

$\int (3x^2 - 8y^2)dx + (4y - 6xy)\,dy$

M N

$\frac{\partial M}{\partial y} = 0 - 16y \quad \frac{\partial N}{\partial x} = -6y$

$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\Rightarrow \int_{C_1}\int_{C_2} (-6y - 16y)\,dx\,dy$

x → 0 → 1
y → $x^2 \to \sqrt{x}$

$\int_0^1 \int_{x^2}^{\sqrt{x}} (-6y - 16y)\,dy\,dx$

$\int_0^1 \left[\frac{-6y^2}{2} - \frac{16y^2}{2}\right]_{x^2}^{\sqrt{x}} \Rightarrow \int_0^1 (-3y^2 - 8y^2)_{x^2}^{\sqrt{x}}$

$\Rightarrow \int_0^1 -3(x) - 8(x) - (3x^4 - 8x^4)$

$\Rightarrow \int_0^1 -3x - 8x - (3x^4 - 8x^4)$

$\Rightarrow \int_0^1 -11x^4 - 11x$

$\Rightarrow \int_0^1 \int_{x^2}^{\sqrt{x}} (-22y)\,dy\,dx$

$\int_0^1 -22\left(\frac{y^2}{2}\right) dx$

$\int_0^1 \left[11y^2\right]_{x^2}^{\sqrt{x}}$

$\Rightarrow -\int_0^1 (11x - 11x^4)$

$\Rightarrow -\left[\frac{11x^2}{2} - \frac{11x^5}{5}\right]$

$\Rightarrow -\left[\frac{55x^2 - 22x^5}{10}\right]_0^1$

$\Rightarrow -\left[\frac{55 - 22}{10}\right] - [0] = -\frac{33}{10}$

$-\frac{3}{2}$

$\int M\,dx + N\,dy = -\frac{3}{2}$

A curve (1) i.e. $y = x^2$ ⇒ $dy = 2x\,dx$

$x = y^{1/2}$

$$\int_0^1 (3x^2 - 8y^2)\,dx + \int_0^1 (4y - 6xy)\,dy$$

$3x^2 - 8y^2$	$4y - 6xy$
$3x^2 - 8(x^2)^2$	$4y - 6(y^{1/2})y$
⇒ $3x^2 - 8x^4$	$4y - 6y^{3/2}$

$$\Rightarrow \int_0^1 (3x^2 - 8x^4)\,dx + \int_0^1 (4y - 6y^{3/2})\,dy$$

$$\left[\frac{3x^3}{3} - \frac{8x^5}{5}\right]_0^1 + \left[\frac{4y^2}{2} - \frac{6y^{5/2}}{5/2}\right]_0^1$$

$$\Rightarrow \left[x^3 - \frac{8}{5}x^5\right]_0^1 + \left[2y^2 - \frac{12}{5}y^{5/2}\right]_0^1$$

$$\Rightarrow \left[1 - \frac{8}{5}\right] + \left[2 - \frac{12}{5}\right]$$

$$\Rightarrow \frac{-3}{5} - \frac{2}{5} = \frac{-5}{5} = -1$$

A curve (2): $y = \sqrt{x} = x^{1/2}$

$x = y^2$

$M = (3x^2 - 8y^2)\,dx$ | $N = 4y - 6xy$

$M \Rightarrow (3x^2 - 8x)\,dx$ | $N = 4y - 6y^2 y$

$N = (4y - 6y^3)\,dy$

$$\int_1^0 (3x^2 - 8x)\,dx + \int_1^0 (4y - 6y^3)\,dy$$

$$\Rightarrow \left[\frac{3x^3}{3} - \frac{8x^2}{2}\right]_1^0 + \left[\frac{4y^2}{2} - \frac{6y^4}{4}\right]_1^0$$

$$= (x^3 - 4x^2)_1^0 + \left(2y^2 - \frac{3}{2}y^4\right)_1^0$$

$$0 - [1 - 4] + \left[0 - \left[2 - \frac{3}{2}\right]\right]$$

$$\Rightarrow 0 - (-3) + 0 - \left[2 - \frac{3}{2}\right]$$

$$+3 - 2 + \frac{3}{2}$$

curve (2) ⇒ $1 + \frac{3}{2}$

∴ curve (1) + curve (2)

$$\Rightarrow -1 + 1 + \frac{3}{2}$$

$$\Rightarrow \frac{3}{2}$$

∴ L.H.S = R.H.S

i.e.

$$\int M\,dx + N\,dy = \iint_C \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy = \frac{3}{2}$$

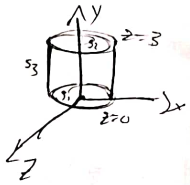
Hence green theorem is verified.

Divergence theorem: $\iint_S F \cdot \hat{n}\, ds = \iiint_V \operatorname{div} F \, dv$

2(B) given $F = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and surface bounded by the region is $x^2+y^2=4$

$z=0$

$z=3$



$F = 4x\hat{i} - 2y^2\hat{j} + z^2 k$

$\operatorname{div} \bar{F} = \nabla \bar{F}$

$\Rightarrow \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(4x i - 2y^2 j + z^2 k)$

$\Rightarrow 4x \cdot \frac{\partial}{\partial x} - 2y^2 \cdot \frac{\partial}{\partial y} + z^2 \cdot \frac{\partial}{\partial z}$

$\nabla f = 4 - 4y + 2z$

$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

$y = -\sqrt{4-x^2}, \sqrt{4-x^2}$

$z = 0, 3$

$x = \pm 2$ (radius)

$x = -2, 2$

$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4 - 4y + 2z)\, dz \cdot dy \cdot dx$

$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + \frac{2z^2}{2}\right]_0^3 dy \cdot dx$

$\Rightarrow \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4(3) - 4y(3) + (3)^2 - 0\right]$

$\Rightarrow \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) \cdot dy \cdot dx$

$\Rightarrow \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dy\, dx$

$\Rightarrow \int_{-2}^{2} \left(21y - \frac{12y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$

$\Rightarrow \int_{-2}^{2} (21y - 6y^2)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$

$\Rightarrow \int_{-2}^{2} \left[21(\sqrt{4-x^2}) - 6(4-x^2)\right] - \left[-21(\sqrt{4-x^2}) + 6(4-x^2)\right]$

$\Rightarrow \int_{-2}^{2} 21\sqrt{4-x^2} - 24 + 6x^2 + 21\sqrt{4-x^2} + 24 - 6x^2$

$\int_{-2}^{2}\left(42\sqrt{4-x^2}\right)dx$

$\Rightarrow 42\int_{-2}^{2}\sqrt{4-x^2}\,dx$ $\left(\because \int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)$

$\Rightarrow 42\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$

$= 42\left[\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$

$\Rightarrow 42\left[\left[\frac{2}{2}\sqrt{4-4}+2\sin^{-1}\frac{2}{2}\right]-\left[\frac{-2}{2}\sqrt{4-4}+2\sin^{-1}\frac{-2}{2}\right]\right]$

$\Rightarrow 42\left[0+2\left(\frac{\pi}{2}\right)+0-2\sin^{-1}(-1)\right]$

$\Rightarrow 42\left[\pi-2\left(\frac{-\pi}{2}\right)\right]$

$\Rightarrow 42[2\pi] = 84\pi$

Divergence

5 (B) given $\iint_S \bar{v}\cdot\bar{n}\,ds = \iiint_V \text{div}\,\bar{v}\,dv$

$\bar{v} = \bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and surface of sphere is $x^2+y^2+z^2=9 \Rightarrow r=3$

$\bar{F}=u_1\bar{i}+u_2\bar{j}+u_3\bar{k}$

$\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$

$\nabla\cdot\bar{F} = \text{div}\,\bar{F} = \left(\frac{\partial}{\partial x}\bar{i}+\frac{\partial}{\partial y}\bar{j}+\frac{\partial}{\partial z}\bar{k}\right)(x\bar{i}+y\bar{j}+z\bar{k})$

$\Rightarrow \left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)\right]$

$\nabla F \Rightarrow (1+1+1)=3$

$\Rightarrow \iiint 3\,dv$

volume of sphere $= \frac{4}{3}\pi r^3$

~~$\Rightarrow \iiint 3\left(\frac{4}{3}\pi r^3\right)$~~

$\iiint 3\,dv \Rightarrow 3\left[\frac{4}{3}\pi(r^3)\right]$

$\Rightarrow 4\pi(3)^3 \Rightarrow 4(27)\pi$

$\Rightarrow 108\pi$

⑧ given $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ where

$S = x^2+y^2+z^2 = a^2$

$f_1 = x,\ f_2 = y;\ f_3 = z$

$\iint f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy$

$\operatorname{div} F = \nabla f$

$\Rightarrow \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\left[x\bar{i}+y\bar{j}+z\bar{k}\right]$

$\Rightarrow \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right]$

$\Rightarrow 1+1+1 = 3$

$\nabla f = 3$

$\iiint \operatorname{div} f\,dV = \iiint 3 \cdot dv$

$\Rightarrow 3(v)$

$\Rightarrow 3\left[\frac{4}{3}\pi r^3\right]$

$\Rightarrow 4\pi r^3$

$\Rightarrow 4\pi a^3$

volume of sphere $= \frac{4}{3}\pi r^3$

given $x^2+y^2+z^2 = a^2$

$\therefore$ radius $r = a$

⑨ $\iint x^2\,dy\,dz + y^2\,dz\,dx + 2z(xy-x-y)\,dx\,dy$

where $S =$ $0<x<1,\ 0<y<1,\ 0<z<1$

$F = \iint x^2\,dy\,dz + y^2\,dz\,dx + 2z(xy-x-y)$

$\Rightarrow f_1 = x^2,\ f_2 = y^2;\ F_3 = 2z(xy-x-y)$

$f_3 \Rightarrow 2xyz - 2xz - 2yz$

$\operatorname{div} f = \nabla f = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}(2xyz - 2xz - 2yz)$

$= 2x + 2y + 2xy - 2x - 2y$

$\operatorname{div} \bar{F} = 2xy$

$\int_0^1\int_0^1\int_0^1 (2xy)\,dx\,dy\,dz$

$\Rightarrow \int_0^1\int_0^1 (2xyz)_0^1$

$\Rightarrow \int_0^1\int_0^1 (2xy)\,dy\,dx$

$\Rightarrow \int_0^1 \frac{2xy^2}{2}\,dx \quad \Rightarrow \int_0^1 (xy^2)_0^1$

$\Rightarrow \int_0^1 (x-0)\,dx$

$\Rightarrow \left[\frac{x^2}{2}\right]_0^1 \Rightarrow \frac{1}{2} - 0$

$\Rightarrow \therefore = \frac{1}{2}$

Divergence:

4(B) $\iint_S \bar{F}\cdot d\bar{s}$; $\bar{F} = yz\,\bar{i} + 2y^2\,\bar{j} + xz^2\,\bar{k}$

$x^2+y^2=9$

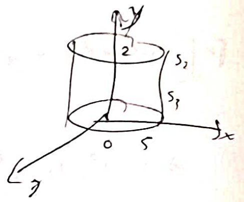
$z=0$; $z=2$

$z = 0 \to 2$

$y^2 = \sqrt{9-x^2}$

$y = -\sqrt{9-x^2},\ \sqrt{9-x^2}$

$x = -3, 3$



limits are $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{2}$

$\nabla f = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(xz^2)$

$\nabla f = 0 + 4y + 2xz$

$\Rightarrow \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{2}(4y+2xz)\,dz\,dy\,dx$

$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\left[4yz + \frac{2xz^2}{2}\right]_0^2$

$\Rightarrow \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4yz+2xz^2)_0^2$ $\Rightarrow \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 8y + 8x$

$\Rightarrow \int_{-3}^{3}\left(\frac{8y^2}{2} + 8xy\right)_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}$

$\int_{-3}^{3}(4y^2 + 8xy)_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}$

$\Rightarrow 4\left(9-x^2 - (9-x^2)\right) + 8x\left[\sqrt{9-x^2} + \sqrt{9-x^2}\right]$

$\Rightarrow 4\left[9-x^2-9+x^2\right] + 8x\left[2\sqrt{9-x^2}\right]$

$\Rightarrow \int_{-3}^{3} 16x\sqrt{9-x^2}$

$\Rightarrow 16\int_{-3}^{3} x\cdot\sqrt{9-x^2}$

$\int\sqrt{a^2-x^2} = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$\Rightarrow x\cdot\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + \sqrt{9-x^2}\cdot(1)$

$\Rightarrow \left[\frac{x^2}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + \sqrt{9-x^2}\right]_{-3}^{3}$ (0, 0)

$\Rightarrow \frac{9}{2}\left[\sin^{-1}\frac{3}{3} - \sin^{-1}\frac{-3}{3}\right]$

$\Rightarrow \frac{9}{2}\left[\frac{\pi}{2} - \left[\frac{-\pi}{2}\right]\right] \Rightarrow \frac{9}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$

$\Rightarrow \frac{9}{2}\left[\frac{2\pi}{2}\right]$

$\Rightarrow \frac{9\pi}{2}$

Stokes Theorem: $\oint_C F \cdot dr = \iint_S n \cdot curl \ldots$

(1) Verify

$F = -y^3 \bar{i} + x^3 \bar{j}$; 'S' is circular disc

$x^2+y^2 \le 1,\ z=0$

$r = x\bar{i} + y\bar{j} + z\bar{k}$

$dr = dx\,i + dy\,j + dz\,k$

$\int_C \bar{F} \cdot dr = (dx\,i + dy\,j + dz\,k)(-y^3 i + x^3 j)$

$\oint_C \bar{F} \cdot dr \Rightarrow \oint_C -y^3 \cdot dx + x^3 \cdot dy$ (1)

Since circle limit $(0 \to 2\pi)$

~~$x = x\cos\theta + y\sin\theta$~~

$x = \cos\theta$; $y = \sin\theta$

$dx = -\sin\theta$; $dy = \cos\theta$

Put in eqn (1)

$$\int_0^{2\pi} -\sin^3\theta\,[-\sin\theta] + (\cos^3\theta)(\cos\theta) \cdot d\theta$$

$$\Rightarrow \int_0^{2\pi} (\sin^4\theta + \cos^4\theta)\, d\theta$$

$$2\int_0^{\pi} (\sin^4\theta + \cos^4\theta)\, d\theta$$

$$4\int_0^{\pi/2} (\sin^4\theta + \cos^4\theta)\, d\theta$$

$$\Rightarrow 4\left[\int_0^{\pi/2} \sin^4\theta \cdot d\theta + \int_0^{\pi/2} \cos^4\theta \cdot d\theta\right]$$

$$\int_0^{\pi/2} \sin^n\theta = \frac{(n-1)(n-3)\ldots}{n(n-2)(n-4)} \frac{\pi}{2}$$

$$\Rightarrow 4\left[\frac{(4-1)(4-3)}{4(4-2)} \cdot \frac{\pi}{2} + \frac{(4-1)(4-3)}{4(4-2)} \frac{\pi}{2}\right]$$

$$\Rightarrow 4\left[\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\right]$$

$$\Rightarrow 4\left[\frac{3\pi}{16} + \frac{3\pi}{16}\right] = \frac{6\pi}{16} \times 4 \Rightarrow \frac{3\pi}{2}$$

$\therefore \oint_C \bar{F} \cdot dr = \frac{3\pi}{2}$ = L.H.S

R.H.S $\Rightarrow \iint_S \hat{n} \cdot curl\,\bar{F} \cdot ds$

$F = -y^3 i + x^3 j + 0k$

$$curl\,\bar{F} = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$i\left[0 - \frac{\partial}{\partial z} x^3\right] - j\left[0 + \frac{\partial}{\partial z} y^3\right] + k\left[\frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} y^3\right]$$

$$\Rightarrow -\frac{\partial}{\partial z} x^3 i - \frac{\partial}{\partial z} y^3 j + (3x^2 + 3y^2) k$$

$$0 + 0 + 3(x^2+y^2) k$$

$\nabla \times F = 3(x^2+y^2)$

$3\int x^2+y^2 \cdot dx \cdot dy$ $\left[\because \text{sphere } x^2+y^2=r^2\right]$

$x = r\cos\theta, \; y = r\sin\theta$

$x^2+y^2 = r^2$

$dx \cdot dy = r \cdot dr \cdot d\theta$

$r = 0 \to 1$

$\theta = 0 \to 2\pi$

$\Rightarrow \int_0^1 \int_0^{2\pi} 3(r^2)\, r \cdot dr\, d\theta$

$\int_0^1 \int_0^{2\pi} 3r^3 \cdot dr \cdot d\theta$

$\int_0^{2\pi} \left[\frac{3r^4}{4}\right]_0^1 \cdot d\theta$

$\Rightarrow \frac{3}{4}\int_0^{2\pi} 1$

$\Rightarrow \frac{3}{4}\left[\theta\right]_0^{2\pi} \Rightarrow \frac{3}{4}\left[2\pi - 0\right]$

$\Rightarrow \frac{3}{4}[2\pi]$ $\Rightarrow \frac{3\pi}{2}$

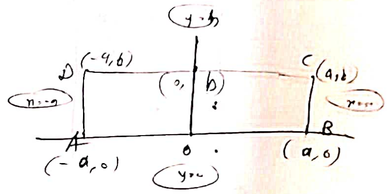
$\therefore \oint_C f \cdot dr = \int_S \text{curl} f \cdot ds = \frac{3\pi}{2}$

Hence proved

Stokes theorem verify

$F = (x^2+y^2)i - 2xyj$ taken around the rectangle. $x = \pm a;\; y = 0,\; y = b$

$\oint_C f \cdot dr = \iint_S K \cdot \text{curl} f \cdot ds$



$\oint f \cdot dr = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

$dr = dx\, i + dy\, j + dz\, k$

$\int f \cdot dr = \int dx(x^2+y^2) + (-2xy)\,dy$

AB $\Rightarrow \int_{-a}^{a} x^2 \cdot dx$, $y=0$, $dy=0$

BC $x = a \Rightarrow dx = 0$: $\int_0^b -2ay \cdot dy$, $-2xy = -2ay$

CD $\Rightarrow y = b$, $dy = 0$: $\int_a^{-a} (x^2+b^2)\,dx$, $x^2+y^2 \Rightarrow x^2+b^2$

DA $x = -a$, $dx = 0$: $\int_b^0 2ay \cdot dy$, $-2xy = -2(-a)y = 2ay$

$\Rightarrow \oint f \cdot dr = \int_{-a}^{a} x^2 \cdot dx + \int_0^b -2ay \cdot dy + \int_a^{-a} (x^2+b^2)\,dx + \int_b^0 2ay \cdot dy$

$\Rightarrow \left[\frac{x^3}{3}\right]_{-a}^{a} + \left[\frac{-2ay^2}{2}\right]_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^{-a} + \left[\frac{2ay^2}{2}\right]_b^0$

$\Rightarrow \left[\frac{a^3}{3} + \frac{a^3}{3}\right] + \left[\frac{-2ab^2}{2} + c\right] + \left[\frac{-a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2\right] + \left[0 - \frac{2ab^2}{2}\right]$

$\Rightarrow \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 \Rightarrow -4ab^2$

$L.H.S = -4ab^2$

RHS $= \iint_S \operatorname{curl} \vec{F} \cdot d\vec{s}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\text{[illegible]})\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(\text{[illegible]})\right] + \hat{k}\left[\ldots\right]$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2xy)\right) - \hat{j}\left(0 - \frac{\partial}{\partial z}(x^2 + y^2)\right)$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 + y^2)\right]$$

$$= 0 + 0 + \hat{k}\left[-2y - 2y\right]$$

$$= -4y\hat{k}$$

$\hat{n} = \hat{k}$

$= \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, ds$

$= \iint_S -4y\, ds$

$= \int \int -4y\, dx\, dy$

$= \text{[illegible]}$

$= -4a^3$

$\therefore$ LHS = RHS $= -4a^3$

$\therefore$ Stokes theorem is verified.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{s}$$