IMPORTANT QUESTIONS UNIT - I Part - I

(1) (A) $y^2 dx + (x^2 - xy - y^2) dy = 0$

Sol: given differential eqn is

$$y^2 dx + (x^2 - xy - y^2) dy = 0$$

compare with $M dx + N dy = 0$

$M = y^2$ $\quad\quad$ $N = x^2 - xy - y^2$

$\frac{\partial M}{\partial y} = 2y$ $\quad\quad$ $\frac{\partial N}{\partial x} = 2x - y$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ eqn (1) is not exact differential eqn

let M and N are homogeneous functions of same degree

i.e eqn (1) is homogeneous diff eqn.

$$\therefore Mx + Ny = y^2(x) + (x^2 - xy - y^2)y =$$

$$\Rightarrow \cancel{xy^2} + x^2y - \cancel{xy^2} - y^3$$

$$\Rightarrow x^2y - y^3$$

$$I.F = \frac{1}{Mx + Ny} = \frac{1}{x^2y - y^3}$$

Multiply I.F $= \frac{1}{Mx+Ny}$ to eqn ① on b.s

$$\frac{1}{x^2y-y^3}\left[(y^2)dx+(x^2-xy-y^2)dy\right]=\frac{1}{x^2y-y^3}\times(0)$$

$$\Rightarrow \underbrace{\frac{y^2}{x^2y-y^3}}_{M_1}dx+\underbrace{\frac{x^2-xy-y^2}{x^2y-y^3}}_{N_1}dy=0$$

~~comp with $M_1x+N_1y=0$~~

$$\cancel{\therefore M_1=\frac{y^2}{x^2y-y^3}=\frac{y}{x^2-y^2}\quad N_1=\frac{x^2-xy-y^2}{x^2y-y^3}}$$

$$\cancel{\frac{\partial M_1}{\partial y}=\frac{\partial}{\partial y}\left[\frac{y^2}{(x^2y-y^3)}\right]}$$

$$\left[\because \frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{uv'-u'v}{v^2}\right]$$

$$\Rightarrow \frac{y^2}{y(x^2-y^2)}dx+\frac{x^2-xy-y^2}{y(x^2-y^2)}dy=0$$

$$\Rightarrow \frac{y}{x^2-y^2}dx+\frac{\cancel{(x^2-y^2)}}{y\cancel{(x^2-y^2)}}-\frac{xy}{y(x^2-y^2)}$$

$$\underbrace{\frac{y}{x^2-y^2}}_{M_1}dx+\underbrace{\frac{1}{y}-\frac{x}{(x^2-y^2)}}_{N_1}dy=0 \quad \rightarrow ②$$

comp with $M_1x+N_1y=0$

$$\therefore M_1=\frac{y}{x^2-y^2}\ ;\ N_1=\frac{1}{y}-\frac{x}{x^2-y^2}$$

$$\therefore \frac{\partial M_1}{\partial y}=\frac{\partial}{\partial y}\left[\frac{y}{x^2-y^2}\right] \qquad \left[\because \frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}\right]$$

$$\Rightarrow \frac{(x^2-y^2)(1)-y(-2y)}{(x^2-y^2)^2}$$

$$\Rightarrow \frac{x^2-y^2+2y^2}{(x^2-y^2)^2}$$

$$\frac{\partial M_1}{\partial y}=\frac{x^2+y^2}{(x^2-y^2)^2}$$

$$\frac{\partial N_1}{\partial x}=0-\left[\frac{(x^2-y^2)(1)-x(2x)}{(x^2-y^2)^2}\right]$$

$$=0-\frac{x^2-y^2-2x^2}{(x^2-y^2)^2}$$

$$\Rightarrow \frac{-(-y^2-x^2)}{(x^2-y^2)^2}$$

$$=\frac{(x^2+y^2)}{(x^2-y^2)^2}$$

$$\therefore \frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

∴ eqn ② is Exact diff eqn.

G.S is $\int M\,dx + \int N$ terms not containing x $dy=c$

$$\int \frac{y}{x^2-y^2}\,dx + \int \frac{1}{y}\,dy = c$$

$$y\int\left(\frac{1}{x^2-y^2}\,dx + \log y = c\right.$$

$$y\left[\frac{1}{2y}\log\left|\frac{x-y}{x+y}\right|\right] + \log y = \log c$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{x-y}{x+y}\right) + \log y = \log c$$

$$\log\left(\frac{x-y}{x+y}\right)^{1/2} + \log y = \log c$$

$$\Rightarrow \log\left(\frac{x-y}{x+y}\right)^{1/2} \cdot y = \log c$$

$$y\left(\frac{x-y}{x+y}\right)^{1/2} = c$$

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② (A) $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy$ — (1)

compare with $f_1(x,y)y\,dx + f_2(x,y)x\,dy = 0$

$$M = xy^2\sin xy + y\cos xy$$

$$N = x^2y\sin xy - x\cos xy$$

$$\frac{\partial M}{\partial y} = \left[xy^2(\cos xy)(x) + \sin xy\,(2xy)\right] + \left[y(-\sin xy)(x) + \cos xy\,(1)\right]$$

$$\boxed{\frac{\partial}{\partial x}(uv) = uv' + vu'}$$

$$\Rightarrow x^2y^2\cos xy + 2xy\sin xy - xy\sin xy + \cos xy$$

$$\frac{\partial M}{\partial y} \Rightarrow x^2y^2\cos xy + xy\sin xy + \cos xy$$

$$\frac{\partial N}{\partial x} = \left[x^2y(\cos xy)(y) + \sin xy\,(2xy)\right] - \left[x(-\sin xy)y + \cos xy\,(1)\right]$$

$$\Rightarrow x^2y^2\cos xy + 2xy\sin xy + xy\sin xy - \cos xy$$

$$\frac{\partial N}{\partial x} \Rightarrow x^2y^2\cos xy + 3xy\sin xy - \cos xy$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

given eqn (1) is not exact differential eqn

∴ consider $M_x - N_y$

$\Rightarrow (xy^2\sin xy + y\cos xy)_x - (x^2y\sin xy - x\cos xy)_y$

$\Rightarrow x^2y^2\sin xy + xy\cos xy - [x^2y^2\sin xy - xy\cos xy]$

$\Rightarrow \cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy$

$\Rightarrow 2xy\cos xy \neq 0$

$\therefore I.F = \frac{1}{M_x - N_y} = \frac{1}{2xy\cos xy}$

Multiply $I.F = \frac{1}{2xy\cos xy}$ on b.s of eqn (1)

$\frac{1}{2xy\cos xy}[xy^2\sin xy + y\cos xy]dx + \frac{1}{2xy\cos xy}(x^2y\sin xy - x\cos xy)dy = 0$

$\Rightarrow \left[\frac{xy^2\sin xy}{2xy\cos xy} + \frac{y\cos xy}{2xy\cos xy}\right]dx + \left[\frac{x^2y\sin xy}{2xy\cos xy} - \frac{x\cos xy}{2xy\cos xy}\right]dy = 0$

$\Rightarrow \underbrace{\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)}_{M_1}dx + \underbrace{\left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)}_{N_1}dy = 0$ — (2)

Compare with $M_1dx + N_1dy$

$M_1 = \frac{y}{2}\tan xy + \frac{1}{2x}$

$\frac{\partial M_1}{\partial y} = \frac{y}{2}[\sec^2 xy\,(x)] + \tan xy\left(\frac{1}{2}\right) + 0$

$\Rightarrow \frac{xy}{2}\sec^2(xy) + \frac{\tan xy}{2}$

$N_1 = \frac{x}{2}\tan xy + \frac{1}{2y}$

$\frac{x}{2}[\sec^2 xy\,(y)] + \tan xy\left(\frac{1}{2}\right) + c$

$\frac{xy}{2}(\sec^2 xy) + \frac{\tan xy}{2}$

$\therefore \frac{\partial M_1}{\partial x} = \frac{\partial N_1}{\partial x}$

∴ eqn (2) is exact diff eqn

∴ the solution is

$\int M dx + \int N$ (terms not containing x) dy

$\int \frac{y}{2}\tan xy + \frac{1}{2x}\,dx + \int -\frac{1}{2y}\,dy = c$

$\Rightarrow \frac{y}{2}\int \tan xy\,dx + \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\int -\frac{1}{y}\,dy = \log c$

$\Rightarrow \frac{y}{2}\frac{\log(\sec xy)}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log c$

$\Rightarrow \frac{1}{2}[\log(\sec xy) + \log x - \log y] = \log c$

$\log(\sec xy) + \log\frac{x}{y} = 2\log c$

$\log(\sec xy) \cdot \frac{x}{y} = \log c^2$

$\Rightarrow \frac{x}{y}(\sec xy) = c^2 = c'$

$\Rightarrow \boxed{c' = \frac{x(\sec xy)}{y}}$

3(A) $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ — (1)

given eqn (1) compr with $Mdx + Ndy = 0$

$M = 3xy - 2ay^2$ | $N = x^2 - 2axy$

$\frac{\partial M}{\partial y} = 3x - 4ay$ | $\frac{\partial N}{\partial x} = 2x - 2ay$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

eqn is not exact diff eqn

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x - 4ay - 2x - 2ay$

$\Rightarrow x - 2ay$

$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{x - 2ay}{x^2 - 2axy}$

$\Rightarrow \frac{\cancel{x - 2ay}}{x(x - 2ay)} = \frac{1}{x}$

$\therefore \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x} \Rightarrow f(x)$

type (3)

$\therefore I.F = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$I.F = x$

Mult I.F = x on both sides of (1)

$\therefore (3xy - 2ay^2)x\,dx + (x^2 - 2axy)x\,dy = 0(x)$

$\Rightarrow \underbrace{(3x^2y - 2axy^2)}_{M_1}dx + \underbrace{(x^3 - 2ax^2y)}_{N_1}dy = 0$ — (2)

$\therefore M_1 = 3x^2y - 2axy^2$ | $N_1 = x^3 - 2ax^2y$

$\frac{\partial M_1}{\partial y} = 3x^2 - 4axy$ | $\frac{\partial N_1}{\partial x} = 3x^2 - 4axy$

$\boxed{\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$

$\therefore$ eqn (2) is exact differential eqn.

$\therefore \int M_1\, \partial x + \int N$ of terms having no x.

$\int 3x^2y - 2axy^2 \cdot \partial x + \int 0 \cdot dy = c$

$\Rightarrow \dfrac{3x^3y}{3} - \dfrac{2ay^2x^2}{2} = c$

$\Rightarrow \cancel{3}y\left(\dfrac{x^3}{\cancel{3}}\right) - \cancel{2}ay^2\left(\dfrac{x^2}{\cancel{2}}\right) = c.$

$\Rightarrow x^3y - x^2y^2a = c$ is solution

(4) (A) $(y^4+2y)\partial x + (xy^3+2y^4-4x)\partial y = 0$ —(1)

compare with $M\partial x + N dy = 0$

$M = y^4+2y$; $N = xy^3+2y^4-4x$

$\dfrac{\partial M}{\partial y} = 4y^3+2$ $\quad \dfrac{\partial N}{\partial x} = y^3 - 4$

$y^3 - 4$

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 4y^3+2 - y^3 + 4$

$\Rightarrow 3y^3 + 6$

$\dfrac{1}{M}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] = \dfrac{1}{y^4+2y}\left[3y^3+6\right]$

$\Rightarrow \dfrac{3(\cancel{y^3+2})}{y(\cancel{y^3+2})}$

$\Rightarrow \dfrac{3}{y}$

$I.F = e^{\int f(y)dy} = e^{3\int \frac{1}{y} \cdot dy}$

$= e^{3\log y} = e^{\log y^3}$

$= 3y \quad = y^3$

$I.F = y^3$

$\therefore$ Mul I.F $= y^3$ to eqn (1) on b.s

$y^3(y^4+2y)\partial x + y^3(xy^3+2y^4-4x)\partial y = 0(y^3)$.

$\Rightarrow \underbrace{(y^7+2y^4)}_{M_1}\partial x + \underbrace{(xy^6+2y^7-4xy^3)}_{N_1}\partial y = 0.$

$\dfrac{\partial M_1}{\partial y} = 7y^6 + 8y^3$ | $\dfrac{\partial N_1}{\partial x} = y^6 + 0 - 12xy$

Part - II Important questions:

1(B): $(D^2-3D+2)y = \cosh x$

Sol: given eqn $(D^2-3D+2)y = \cosh x$ $\rightarrow (1)$

compare with

$$f(D)y = \phi(x)$$

$$f(D) = D^2-3D+2 \; ; \quad \phi(x) = \cosh x = \frac{e^x+e^{-x}}{2}$$

C.F $= m^2-3m+2$

$\rightarrow m^2-2m-m+2$

$m(m-2)-1(m-2)$

$(m-1)(m-2)$

$m = 1, 2$

C.F $= c_1e^x + c_2e^{2x}$

$$P.I = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2-3D+2}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-3D+2}(e^x) + \frac{1}{D^2-3D+2}(e^{-x})\right]$$

$$\left(\because \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}\right) = \frac{1}{2}\left[\frac{e^x}{\underbrace{[D-1]}_{=0}\underbrace{[D-2]}_{\neq 0}} + \frac{e^{-x}}{(-1)^2-3(-1)+2}\right]$$

$$\rightarrow \frac{1}{2}\left[\frac{1}{1-2}\left[\frac{1}{[D-1]}e^x\right] + \left[\frac{1}{6}e^{-x}\right]\right]$$

$\because \left[\because \dfrac{1}{(D-a)^k} \cdot e^{ax} = \dfrac{x^k}{k!} e^{ax}\right]$

$\Rightarrow \dfrac{1}{2}\left[\dfrac{1}{(1-2)}\left[\dfrac{x^1}{1!} e^{1x}\right] + \dfrac{1}{6} e^{-x}\right]$

$\Rightarrow \dfrac{1}{2}\left[\dfrac{1}{-1}\left[\dfrac{x}{1} e^{x}\right] + \dfrac{1}{6} e^{-x}\right]$

$P.I \Rightarrow \dfrac{1}{2}\left[(-xe^{x}) + \dfrac{1}{6} e^{-x}\right]$

$\therefore y = CF + PI$

$= c_1 e^{x} + c_2 e^{2x} + \dfrac{1}{2}\left[-xe^{x} + \dfrac{1}{6} e^{-x}\right]$

2(B) $(D^2+4)y = e^x + \sin 2x + \cos 2x$

given eqn is $(D^2+4)y = e^x + \sin 2x + \cos 2x$

$C.F = m^2+4$

$\Rightarrow \cancel{m^2+2^2=0}$

$m^2 = -4$

$m = \sqrt{-4}$

$m = 2\sqrt{-1}$

$\boxed{m \Rightarrow 0 \pm 2i}$

$\cancel{C.F = (c_1 + c_2 x)}$ $C.F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$= e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$\boxed{C.F \Rightarrow c_1 \cos 2x + c_2 \sin 2x}$

$P.I \; y = \dfrac{1}{D^2+4} e^x + \dfrac{1}{D^2+4} \cos 2x + \dfrac{1}{D^2+4} \sin 2x$

type I   type 2   type 2

$\boxed{\because \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}}$ // $\left[\because \dfrac{1}{f(D)} \sin ax / \cos ax \right.$

$= \dfrac{1}{f(-a^2)} \sin ax/\cos ax$

$\cancel{\Rightarrow y = \dfrac{1}{(1)^2+4} e^x} +$ type 2 get '0' $f(a) = 0$

$\therefore \dfrac{x}{f'(a)} \cdot \sin ax$

$\therefore f = \dfrac{1}{(1)^2+4} e^x + \dfrac{x}{2D} \sin 2x + \dfrac{x}{2D} \cos 2x$

$\Rightarrow \dfrac{1}{5} e^x + \dfrac{x}{2(2)} \sin 2x - \dfrac{x}{2(2)} \cos 2x$

$P.I \Rightarrow \dfrac{e^x}{5} + \dfrac{x}{4} \sin 2x - \dfrac{x}{4} \cos 2x$

$\therefore G.S \; y = CF + PI$

$\Rightarrow c_1 \cos 2x + c_2 \sin 2x + \dfrac{e^x}{5} + \dfrac{x \sin 2x}{4} - \dfrac{x \cos 2x}{4}$

3(B) solve $[D^3 - 7D^2 + 14D - 8]y = e^{x} + \sin^3 x + 2$

$AE = m^3 - 7m^2 + 14m - 8$

upon solving we get

$m = 4, 2, 1$

$\therefore CF = c_1 e^{4x} + c_2 e^{2x} + c_3 e^{x}$

$y_p = \frac{1}{f(D)} \phi(x)$

$\phi(x) = e^{x} + \sin^3 x + 2$

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$

consider y_1

$y_p = \frac{e^x}{D^3 - 7D^2 + 14D - 8}$ $\quad \left(\because \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \right)$

$\because f(a) = 0$, diff w.r.t x

$\cancel{\frac{1}{(1) - 7(1) + 14 - 8} (e^x) = e^x}$

$\Rightarrow \frac{x}{3D^2 - 14D + 14} \cdot e^x \Rightarrow \frac{x \cdot e^x}{3(1) - 14 + 14}$

$\boxed{y_{p_1} \Rightarrow \frac{x}{3} \cdot e^x}$

$y_{p_2} = \frac{\sin^3 x}{D^3 - 7D^2 + 14D - 8}$ $\quad \left[\because \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \right]$

$\Rightarrow \frac{3}{4} \frac{1}{D^3 - 7D^2 + 14D - 8} \sin x$

$- \frac{1}{4} \frac{1}{D^3 - 7D^2 + 14D - 8} \times \sin 3x$

$\Rightarrow \frac{3}{4} \frac{1}{D(-1)^2 - 7(-1)^2 + 14D - 8} - \frac{1}{4} \frac{1}{D(-3^2) - 7(-3^2) + 14D - 8} \sin 3x$

$\Rightarrow \frac{3}{4} \frac{1}{-D + 7 + 14D - 8} - \frac{1}{4} \frac{1}{9D + 63 + 14D - 8} \sin 3x$

$\Rightarrow \frac{3}{4} \left(\frac{1}{13D - 1} \right) \sin x - \frac{1}{4} \frac{1}{5D + 55} \sin 3x$

$\Rightarrow \frac{3}{4} \frac{1}{(13D - 1)} \frac{(13D + 1)}{(13D + 1)} \sin x - \frac{1}{4} \frac{1}{5D + 55} \frac{5D - 55}{5D - 55} \sin 3x$

$\Rightarrow \frac{3}{4} \frac{(13D + 1) \sin x}{169D^2 - 1} - \frac{1}{4} \frac{(5D - 55) \sin 3x}{25D^2 - (55)^2}$

$\Rightarrow \frac{3}{4} \frac{13D \sin x + \sin x}{-169 - 1} - \frac{1}{4} \frac{5D \sin 3x - 55 \sin 3x}{-225 - 3025}$

$\Rightarrow \frac{3}{4} \frac{13 \cos x + \sin x}{-170} - \frac{1}{4} \frac{15 \cos 3x - 55 \sin 3x}{-3250}$

$y_{p_2} = \frac{-3}{680} (13 \cos x + \sin x) + \frac{1}{13000} (15 \cos 3x - 55 \sin 3x)$

$y_{p_3} = \frac{1}{D^3 - 7D^2 + 14D - 8} \cdot 2 e^{0x}$

$\Rightarrow \frac{1}{(0)^3 - (7)^0 + 14(0) - 8} \cdot 2$

$y_{p_3} \Rightarrow -2$ ..

$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3}$

$= \frac{x}{3} e^x - \frac{3}{680} (13 \cos x + \sin x) + \frac{1}{13000} (15 \cos 3x - 55 \sin 3x) - \frac{1}{4}$

$\therefore G.S = CF + PI$

$\Rightarrow c_1 e^{4x} + c_2 e^{2x} + c_3 e^{x} + PI$

4(B) solve $(D^3+2D^2+D)y = e^{2x}+x^2+x+\sin 2x$

comp eqn $f(D)y=\phi(x)$

$f(D) = D^3+2D^2+D$

$A \cdot E = m^3+2m^2+m$

upon solving

$m = 0, -1, -1$

$\therefore C \cdot F = c_1 e^{0x} + (c_2+c_3 x)e^{-x}$

$\Rightarrow c_1 + (c_2+c_3 x)e^{-x}$

$P \cdot I = y = \left[e^{2x} + x^2+x + \sin 2x\right]\dfrac{1}{D^3+2D^2+D}$

$\Rightarrow \dfrac{1}{D^3+2D^2+D} e^{2x} + \dfrac{1}{D^3+2D^2+D}(x^2+x) + \dfrac{1}{D^3+2D^2+D}\sin 2x$

Type I   Typ III   Type II

$y_{p_1} \Rightarrow \dfrac{1}{D^3+2D^2+D}(e^{2x})$ $\left[\because \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}\right]$

$\Rightarrow \dfrac{1}{(2)^3+2(2)^2+2} e^{2x} \Rightarrow \dfrac{1}{8+8+2} e^{2x} = \dfrac{e^{2x}}{18}$

$y_{p_1} = \dfrac{1}{18} e^{2x}$

consider $y_{p_2} \Rightarrow \dfrac{1}{D^3+2D^2+D}(x^2+x)$

$\Rightarrow \dfrac{1}{D\left(\dfrac{D^3+2D^2}{D}+1\right)}$

$\Rightarrow \dfrac{1}{D}\left[1+(D^2+2D)\right]^{-1}(x^2+x)$

$\left(\because (1+x)^{-1} = 1+x+x^2+x^3+\dots\right]$

$\Rightarrow \dfrac{1}{D}\left[1-(D^2+2D)+(D^2+2D)^2 - \dots\right](x^2+x)$

$\Rightarrow \dfrac{1}{D}\left[1-(D^2+2D)+(D^4+4D^2+4D^3)+\dots\right](x^2+x)$

neglect $D^3, D^4, D^5 \dots$

$\Rightarrow \dfrac{1}{D}\left[1-(D^2+2D)+(4D^2+4D^3)+(-8D^3)\right]$

diff

$\Rightarrow \cancel{\dfrac{1}{D}\left[1-(D+2)+(4+8D)+8D^2\right]}$

$\Rightarrow \left[\dfrac{1}{D} - (D+2) + (4D+4D^2) - 8D^2\right](x^2+x)$

$\Rightarrow \int x^2+x - D(x^2+x)+2(x^2+x) + 4D(x^2+x) + 4D^2(x^2+x)$
$- 8(x^2+x)D^2$

$$\frac{x^3}{3}+\frac{x^2}{2}-(2x+1)+4(2x+1)+4(2)-8(2\ldots$$

$$\Rightarrow \frac{x^3}{3}+\frac{x^2}{2}-(2x+1)+8x+4\cancel{+8}\cancel{-8}$$

$$\Rightarrow \frac{x^3}{3}+\frac{x^2}{2}-2x-1+8x+4$$

$$y_{p_2}=\frac{x^3}{3}+\frac{3}{2}x^2+4x-5$$

$$y_{p_3}=\frac{1}{D^3+3D^2+D}\sin 2x$$

$$\left(\because \frac{1}{f(D)}\sin ax=\frac{1}{f(-a^2)}\sin ax\right)$$

$$\Rightarrow \frac{1}{D^2\cdot D+3D^2+D}\sin 2x$$

$$\frac{1}{-4D+3(-4)+D}\sin 2x$$

$$\Rightarrow \frac{1}{-8-3D}\cdot\sin 2x$$

$$\Rightarrow -\left[\frac{1}{3D+8}\sin 2x\right]$$

$$y_{p_3}=-\left[\frac{1}{3D+8}\cdot\frac{3D-8}{3D-8}\sin 2x\right]$$

$$\Rightarrow -\left[\frac{(3D-8)\sin 2x}{9D^2-64}\right] \quad \left(\because \frac{1}{f(D)}\sin ax \Rightarrow \frac{1}{f(-a^2)}\sin ax\right)$$

$$\Rightarrow -\left[\frac{(3D-8)\sin 2x}{9(-4)-64}\right]$$

$$\Rightarrow -\left[\frac{(3D-8)\sin 2x}{-100}\right]$$

$$\Rightarrow \frac{1}{100}\left[3(2\cos 2x)-8\sin 2x\right]$$

$$y_{p_3}=\frac{1}{100}(6\cos 2x-8\sin 2x)$$

$$\therefore y=CF+y_{p_1}+y_{p_2}+y_{p_3}$$

$$y=c_1+(c_2+c_3x)e^{-x}+\frac{e^{2x}}{18}+\frac{x^3}{3}+\frac{3}{2}x^2+4x-5+\frac{1}{100}(6\cos 2x-8\sin 2x)$$

6(A) given $(D^2-6D+13)y = 8e^x \sin 2x$

compare with $F(D)y = 8e^x \sin 2x$

$F(D) = D^2-6D+13$ and $\phi(x) = 8e^x \sin 2x$

$\Rightarrow m^2-6m+13$

$\Rightarrow m = 3+2i, 3-2i$

$C.F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$\Rightarrow e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

$P.I \Rightarrow (y_p) = 8e^x \sin 2x$

$= \frac{1}{F(D)} \phi(x)$

$\Rightarrow 8\left[\frac{1}{D^2-6D+13}\right] e^x \sin 2x$ → $e^{ax}V$

Type IV

$\left[\because y = e^{ax}\left(\frac{V}{F(D+a)}\right)\right]$

$\Rightarrow 8\left[e^{ax}\left[\frac{1}{(D+3)^2-6(D+3)+13} \sin 2x\right]\right]$

$\Rightarrow 8\left[e^{ax}\left[\frac{1}{D^2+9+6D-6D-18+13} \sin 2x\right]\right]$

$\Rightarrow 8\left[e^{ax}\left[\frac{1}{D^2+4} \sin 2x\right]\right]$ → Type II

$a = c$; so spl condition is applied.

$\left[\because \frac{1}{D^2+b^2} \sin bx = \frac{-x}{2b} \cos bx\right]$

(or) $\left[\frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax\right]$

$y_p = \overset{2}{\cancel{8}}\left[e^{ax}\left[\frac{-x}{\cancel{2}(2)} \cos 2x\right]\right]$

$\Rightarrow 2e^{ax}[-x\cos 2x]$

$y_p = -2e^{ax} x \cos 2x$

$\therefore y = CF + PI$

$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - 2e^{ax} x \cos 2x.$

6(B) $D^2(D^2+4)y = 320(x^2+2x^2)$

comp with $f(D)y = \phi(x)$

$f(D) = D^2(D^2+4)$

$AE = m^2(m^2+4)$

$m^2 = 0$ | m^2+4

$m = 0, 0$ | $m = \pm 2i$

$\therefore m = 0, 0, 0+2i, 0-2i$

$C.F = (C_1 + C_2 x)e^{0x} + e^{0x}(C_3 \cos 2x + C_4 \sin 2x)$

$C.F = C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x$

$y_p = \frac{1}{F(D)} \phi(x)$

$= \frac{1}{D^2(D^2+4)} 320 (x^3 + 2x^2)$

$\Rightarrow 320 \left[\frac{1}{D^4 + 4D^2} (x^3 + 2x^2) \right]$

Type III

lowest degree take out as common

$\Rightarrow 320 \left[\frac{1}{4D^2\left[\frac{D^4}{4D^2} + 1\right]} \right] (x^3 + 2x^2)$

$\Rightarrow 320 \left[\frac{1}{4D^2 \left[\frac{D^2}{4} + 1\right]} \right] (x^3 + 2x^2)$

$\Rightarrow 320 \left[\frac{1}{4D^2} \left[1 + \frac{D^2}{4} \right]^{-1} \right] (x^3 + 2x^2)$

$\because (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \ldots$

$\Rightarrow 320 \left[\frac{1}{4D^2} \left[1 - \frac{D^2}{4} + \left(\frac{D^2}{4}\right)^2 - \left(\frac{D^2}{4}\right)^3 \right] \right] (x^3 + 2x^2)$

$\Rightarrow \overset{80}{\cancel{320}} \left[\frac{1}{\cancel{4}D^2} \left[1 - \frac{D^2}{4} + \frac{D^4}{16} - \frac{D^8}{64} + \ldots \right] (x^3 + 2x^2) \right.$

Neglect 'D' terms higher than D^3.

$\therefore 80 \left[\frac{1}{D^2} \left[1 - \frac{D^2}{4} + \frac{D^4}{16} \right] \right] (x^3 + 2x^2)$

$\Rightarrow 80 \left[\frac{1}{D^2} - \frac{1}{D^2}\left[\frac{D^2}{4}\right] + \frac{1}{D^2}\left[\frac{D^4}{16}\right] \right] (x^3 + 2x^2)$

$\Rightarrow 80 \left[\frac{1}{D^2} - \frac{1}{4} + \frac{1}{16}(D^2) \right] (x^3 + 2x^2)$

$D(x^3 + 2x^2) = 3x^2 + 4x$

$D^2(x^3 + 2x^2) = 6x + 4$

$\int (x^3 + 2x^2) = \frac{x^4}{4} + \frac{2x^3}{3}$

$\frac{1}{D^2} = \iint x^3 + 2x^2 = \frac{1}{4}\left(\frac{x^5}{5}\right) + \frac{2}{3}\left(\frac{x^4}{4}\right)$

$\Rightarrow \frac{x^5}{5} + \frac{\cancel{2}}{\cancel{12}_6} x^4$

$$\boxed{\frac{1}{D^2} = \frac{x^5}{20} + \frac{x^4}{6}}$$

$\therefore 80 \left[\frac{x^5}{20} + \frac{x^4}{6} - \frac{1}{4}(x^3 + 2x^2) + \frac{1}{16}(6x + 4) \right]$

$\Rightarrow \cancel{80\left[\frac{x^5}{20} + \frac{x^4}{6}\right.}\quad \frac{\overset{4}{\cancel{80}}x^5}{\cancel{20}} + \frac{\overset{40}{\cancel{80}}}{\cancel{6}_3} x^4 - \frac{\overset{20}{\cancel{80}}}{\cancel{4}}(x^3 + 2x^2) + \frac{\overset{5}{\cancel{80}}}{\cancel{16}}(6x + 4)$

$P.I \Rightarrow 4x^5 + \frac{40}{3} x^4 - 20(x^3 + 2x^2) + 5(6x + 4)$

$y = CF + PI$

$\Rightarrow C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x$

$+ 4x^5 + \frac{40}{3} x^4 - \cancel{\frac{1}{4}[x^2+2x]^2}$

$-20(x^3+2x^2) + 5(6x+4)$

7 (A) $(D^2+2D-3)y = x^3 e^{-3x}$

comp $f(D)y = \phi(x)$

$f(D) = D^2+2D-3$

$A\cdot E = m^2+2m-3$

$= m^2+3m-m-3$

$\Rightarrow m(m+3)-1(m+3)$

$(m-1)(m+3)$

$m = -1, 3$

$\therefore CF = C_1 e^{-x} + C_2 e^{3x}$

$P.I = y_p = \frac{1}{f(D)} \phi(x)$

$= \frac{1}{D^2+2D-3} (x^2 \cdot e^{-3x})$

$\Downarrow$

Type (4)

$\frac{1}{f(D)} e^{ax} \cdot V = e^{ax}\left(\frac{V}{f(D+a)}\right)$

$\Rightarrow e^{-3x}\left[\frac{x^3}{(D-3)^2+2(D-3)-3}\right]$

$\Rightarrow e^{-3x}\left[\frac{1}{D^2+9-6D-2D+6-3} \cdot x^3\right]$

$\Rightarrow e^{-3x}\left[\frac{1}{D^2-8D+12} x^3\right]$

$\uparrow$

Ty III

$\Rightarrow e^{-3x}\left[\frac{1}{12\left[\frac{D^2-8D}{12}+1\right]}\right]x^3$

$e^{-3x}\left[\frac{1}{12\left(\frac{D^2-8D}{12}+1\right)}\right]x^3 = e^{-3x}\left[\frac{1}{12}\left[1+\left(\frac{D^2-8D}{12}\right)\right]^{-1}\right]x^3$

$(\because (1+x)^{-1} = 1-x+x^2-x^3+\ldots$

$\Rightarrow e^{-3x}\left[\frac{1}{12}\left[1+\left(\frac{D^2-8D}{12}\right)+\left(\frac{D^2-8D}{12}\right)^2+\ldots\right]x^3\right.$

$\Rightarrow e^{-3x}\left[\frac{1}{12}\left[1-\frac{D^2}{12}-\frac{8D}{12}+\frac{D^4}{144}-\frac{64D^2}{144}-\frac{16D^3}{144}\right.\right.$

$\left.-\frac{1}{1728}(-512D^3)\right]$

$$\frac{e^{-3x}}{12}\left[1-\frac{1}{12}\left[D^2-8D\right]+\frac{1}{144}\left(64D^2-16D^3\right)-\frac{1}{1728}\left(-512D^3\right)\right]x^3$$

$$D = 3x^2$$

$$D^2 = 6x$$

$$D^3 = 6$$

$$\Rightarrow \frac{e^{-3x}}{12}\left[x^3-\frac{1}{12}\left[6x-8(3x^2)\right]+\frac{1}{144}\left(64(6x)-16(6)\right)-\frac{1}{178}\left(-512(6)\right)\right]$$

$$\Rightarrow \frac{e^{-3x}}{12}\left[x^3-\frac{1}{2}x+2x^2+\frac{8}{3}x-\frac{2}{3}+\frac{16}{9}\right]$$

$$\Rightarrow \frac{e^{-3x}}{12}\left[x^3+2x^2+\left(\frac{8}{3}-\frac{1}{2}\right)x+\left(\frac{16}{9}-\frac{2}{3}\right)\right]$$

$$PI = y_p \Rightarrow \frac{e^{-3x}}{12}\left[x^3+2x^2+\frac{13}{6}x+\frac{10}{9}\right]$$

$\therefore y = CF + PI$

$$= c_1e^{-x}+c_2e^{3x}+\frac{e^{-3x}}{12}\left[x^3+2x^2+\frac{13}{6}x+\frac{10}{9}\right]$$

Method of variation of parameters:-

To solve $D^2 + 4D + 2y = 0$ by method of variation

Steps

(1) Reduce given $D^2+4D+2=0$ to standard form if required.

(2) Then find C.F $\boxed{CF = c_1y_1+c_2y_2}$

(3) Find y_1' and y_2'

(4) Find $\boxed{W(y_1, y_2) = y_1y_2' - y_2y_1'}$

(5) Find $\boxed{P.I = v_1y_1 + v_2y_2}$

where $v_1 = -\int \frac{Ry_2}{W}\,dx$

$v_2 = -\int \frac{Ry_1}{W}\,dx$

(6) General solution is $\boxed{y = y_{C.F} + y_{P.I}}$

~~Q1 (A) Solve $(D^3-7D^2+14D-8)y = e^x \cos 2x$~~

9 (a) $(D^2-4)y = x\sinh x + 54x + 8$

$$(D^2-4)y = x\left[\frac{e^x-e^{-x}}{2}\right] + 54x + 8 \qquad \left(\because \sinh x = \frac{e^x-e^{-x}}{2}\right)$$

CF $= m^2-4$

$\Rightarrow m^2 = 4$

$m = \pm 2$

$\therefore m = 2, -2$

$CF = c_1 e^{2x} + c_2 e^{-2x}$

$f(D)\, y_p = x\sinh x + 54x + 8$

$$y_p = \frac{1}{f(D)} x\left(\frac{e^x+e^{-x}}{2}\right) + \frac{1}{f(D)} 54x + \frac{1}{f(D)} 8$$

$$\Rightarrow \underbrace{\frac{1}{D^2-4}\, x\left[\frac{e^x+e^{-x}}{2}\right]}_{y_{p_1}} + \underbrace{\frac{1}{D^2-4}[54x]}_{y_{p_2}} + \underbrace{\frac{8}{D^2-4}}_{y_{p_3}}$$

$\Rightarrow$ consider $y_p = \frac{1}{D^2-4}\, x\left[\frac{e^x+e^{-x}}{2}\right]$

$$= \frac{1}{D^2-4}\left[\frac12 x e^x + \frac12 x\cdot e^{-x}\right]$$

$$y_p = -\frac12\left[\frac{1}{D^2-4} e^x \cdot x\right] - \frac13\left[\frac{1}{D^2-4} e^{-x}\cdot x\right]$$

$$\left[\because \frac{1}{f(D)} e^{ax} V = e^{ax}\frac{1}{f(D+a)} V\right]$$

$$\Rightarrow \frac12\left[e^x \frac{1}{(D+1)^2-4} x\right] - \frac12\left[e^{-x}\left[\frac{1}{(D-1)^2-4} x\right]\right]$$

$$\Rightarrow \frac12\left[e^x \frac{x}{D^2+1+2D-4}\right] - \frac12\left[e^{-x}\frac{x}{D^2+1-2D-4}\right]$$

$$\Rightarrow \frac{e^x}{2}\left[\frac{x}{D^2+2D-3}\right] - \frac{e^{-x}}{2}\left[\frac{x}{D^2-2D-3}\right]$$

$$\Rightarrow \frac{e^x}{2}\left[\frac{x}{-3\left[\frac{D^2+2D}{3}\right]}\right] - \frac{e^{-x}}{2}\left[\frac{x}{3\left[\frac{D^2-2D}{3}-1\right]}\right]$$

$$\Rightarrow -\frac{e^x}{6}\left[1-\left[\frac{D^2+2D}{3}\right]^{-1} x\right] + \frac{e^{-x}}{6}\left[1-\left[\frac{D^2-2D}{3}\right]\right] x$$

$$\Rightarrow -\frac{e^x}{6}\left[1+\frac{D^2+2D}{3}\right] x + \frac{e^{-x}}{6}\left[1+\frac{D^2-2D}{3}\right] x$$

$\cancel{\boxed{y_p = \frac{e^{x}}{6}}}$ $y_p = \frac{-e^{x}}{6}\left(x+\frac{1}{3}D^2x+\frac{2}{3}Dx\right)$

$+\frac{e^{-x}}{6}\left(x+\frac{1}{3}D^2x-\frac{2}{3}Dx\right)$

$\boxed{D=1 \quad D^2=0}$

$\Rightarrow \frac{-e^{x}}{6}\left(x+0+\frac{2}{3}\right)+\frac{e^{-x}}{6}\left(x+0-\frac{2}{3}\right)$

$\Rightarrow \frac{-e^{x}}{6}\left(x+\frac{2}{3}\right)+\frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$

$\Rightarrow -\frac{1}{6}xe^{x} - \frac{1}{9}e^{x}+\frac{1}{6}xe^{-x}-\frac{1}{9}e^{-x}$

$\Rightarrow -\frac{1}{3}x\left(\frac{e^{x}-e^{-x}}{2}\right)-\frac{1}{9}\times 2\left(\frac{e^{x}+e^{-x}}{2}\right)$

$y_p = \frac{1}{3}x\sinh x - \frac{2}{9}\cosh x$

$y_{p_3} = 54x\left(\frac{1}{D^2+4}\right)$

$\Rightarrow 54\left(\frac{1}{D^2+4}\right)x$

$\Rightarrow 54\left[\frac{1}{4\left(\frac{D^2}{4}+1\right)}\right]x$

$\Rightarrow 54\left[\frac{1}{4}\left[1+\frac{D^2}{4}\right]^{-1}\right]x$

$(1+x)^{-1} = 1-x+x^2+x^3+\dots$

$54\left[\frac{1}{4}\left[1-\frac{D^2}{4}+\frac{D^4}{16}\dots\right]\right]x$

$\Rightarrow 54\left[\frac{1}{4}-\frac{1}{16}D^2\right]x$

$D=1$, $x=0$ $\quad\Rightarrow 54\left(\frac{x}{4}-\frac{1}{16}D^2x\right)$

$\Rightarrow 54\times\frac{1}{4}x \Rightarrow \cancel{\frac{54}{4}}\ 27$

$y_{p_2} = \frac{27x}{2}$

$y_{p_3} = \frac{8\cdot e^{0x}}{D^2+4}$

$\Rightarrow \frac{8}{0+4} = 2$

$y_{p_3} = 2$

$\Rightarrow \frac{8}{D^2}+\frac{8}{4}$

$\Rightarrow \iint 8 + 2$

$\Rightarrow \int 8x$

$\Rightarrow \int \frac{8x^2}{2} \Rightarrow 4x^2+2$

$\cancel{y_{p_3} = 4x^2+2}$ $\quad y_{p_3} = 2$

$\therefore y = CF + y_{p_1} + y_2 + y_{p_3}$

$= c_1e^{2x}+c_2e^{-2x}+\frac{1}{3}x\sinh x-\frac{2}{9}\cosh x+\frac{27}{2}x+2$

7(A) Solve $(D^3-7D^2+14D-8)y = e^x\cos 2x$

$f(D) = D^3-7D^2+14D-8$

AE: $m^3-7m^2+14m-8$

upon solving we get

$m = 4, 2, 1$

$\Rightarrow CF = c_1e^{4x}+c_2e^{2x}+c_3e^x$

$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^3-7D^2+14D-8}(e^x\cos 2x)$

$\downarrow$ Type ④

$\frac{1}{f(D)} e^{x}\phi(x) = e^{x}\frac{1}{f(D+a)}\phi(x)$

$\Rightarrow e^x\left[\frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8}\cos 2x\right]$

$\Rightarrow e^x\left[\frac{1 \cdot \cos 2x}{D^3+1^3+3D(D+1)-7(D^2+1+2D)+14D+14-8}\right]$

$\Rightarrow e^x\left[\frac{\cos 2x}{D^3+3D^2+3D+1-7D^2-7-14D+14D+14-8}\right]$

$\Rightarrow e^x\left[\frac{\cos 2x}{D^3-4D^2+3D}\right]$

$\downarrow$ Type ②

$\Rightarrow e^x\left[\frac{\cos 2x}{D^2\cdot D-4D^2+3D}\right]$

$\Rightarrow e^x\left[\frac{\cos 2x}{-4D-4(-4)+3D}\right]$

$\Rightarrow e^x\left[\frac{\cos 2x}{16-D}\right]$

$\Rightarrow e^x\left[\frac{16+D}{16+D}\cdot\frac{\cos 2x}{16-D}\right]$

$\Rightarrow e^x\left[\frac{16\cos 2x+(-\sin 2x)(2)}{(16)^2-D^2}\right]$

$\Rightarrow e^x\left[\frac{16\cos 2x-2\sin 2x}{(16)^2-(-4)^2}\right]$

$\Rightarrow e^x\left[\frac{16\cos 2x-2\sin 2x}{256-16}\right]$

$\Rightarrow \frac{16e^x\cos 2x-2e^x\sin 2x}{240}$

$PI \Rightarrow \frac{e^x\cos 2x}{15} - \frac{e^x\sin 2x}{120}$

$\therefore y = CF+PI \Rightarrow c_1e^{4x}+c_2e^{2x}+c_3e^x+\frac{e^x\cos 2x}{15}-\frac{e^x\sin 2x}{120}$