Vector Differential formulae

$\bar{r}(P) = \bar{r}(x,y,z) = F_1(x,y,z)\bar{i} + F_2(x,y,z)\bar{j} + F_3(x,y,z)\bar{k}$

$$\boxed{\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k}$$

$$\boxed{\nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\phi}$$ where $\phi = ax^2 + bx + c$ (or) any eqn.

positional vector:-

$P(x,y,z)$ and $O(0,0,0)$

$\Rightarrow \overline{OP} = (x-0)\,i + (y-0)\,\bar{j} + (z-0)\,\bar{k}$

$\bar{r} = \overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$

magnitude $|\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\therefore \hat{r} = \frac{\bar{r}}{|\bar{r}|} = \frac{x\bar{i} + y\bar{j} + z\bar{k}}{\sqrt{x^2+y^2+z^2}}$$

vector differential operator (∇):- (del) / nabla

$$\boxed{\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}}$$

gradient of a scalar function:

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\phi = \boxed{\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}}$$

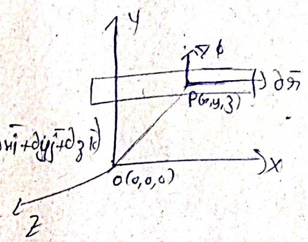
Note: If $\phi(x,y,z)$ ~~denotes~~ define scalar field then

$\nabla\phi$ defines a vector field.

Physical significance of grad ϕ:-

If $\phi(x,y,z) = c$ (constant) represents a surface then $\nabla\phi$ represents normal to the surface at point $\phi(x,y,z)$ on surface we have.

$d\bar{r} = \partial x\,\bar{i} + \partial y\,\bar{j} + \partial z\,\bar{k}$. This is the tangent to the surface at $P(x,y,z)$.



Note:

$$\nabla\phi \cdot \partial\bar{r} = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)(\partial x\,\bar{i} + \partial y\,\bar{j} + \partial z\,\bar{k})$$

$$\Rightarrow \frac{\partial\phi}{\partial x}\partial x + \frac{\partial\phi}{\partial y}\partial y + \frac{\partial\phi}{\partial z}\partial z$$

$$\Rightarrow \partial\phi = 0$$

Directional derivatives.

The directional derivative of $\phi(x,y,z)$ in the direction of vector $\bar{a}$ at any point P is $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$i.e.\ \boxed{\nabla\phi \cdot \hat{a}} \ / \ \boxed{\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}}$$

Unit normal to the surfaces

Unit normal to the surface $\phi(x,y,z)$ is defined as $\frac{\nabla\phi}{|\nabla\phi|}$

Angle b/w the surfaces:

Angle b/w the 2 surfaces $\phi = c_1$ and $\phi = c_2$ at a common point 'P' is the angle b/w the normals $\nabla\phi_1 + \nabla\phi_2$ to the surface at point is

$$\cos\theta = \frac{\nabla\phi_1 + \nabla\phi_2}{|\nabla\phi_1| + |\nabla\phi_2|} = \frac{\bar{n}_1 \cdot \bar{n}_2}{|n_1||n_2|}$$

Curl of a vector point function:-

curl of $\bar{F} = \nabla\times\bar{F}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times\bar{F}$$

$$= i\times\frac{\partial\bar{F}}{\partial x} + j\times\frac{\partial\bar{F}}{\partial y} + k\times\frac{\partial\bar{F}}{\partial z}$$

If $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then $f = \nabla\times\bar{F}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(f_1 i + f_2 j + f_3 k)$$

$$(\nabla\times\bar{F}) = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector:-

A vector point function $\bar{F}$ is said to be irrotational if

$$\boxed{\text{curl}\,\bar{F} = \nabla\times\bar{F} = 0}$$

If $\bar{F}$ is irrotational then there always exists a scalar function $\phi(x,y,z)$, such that

$\boxed{\bar{F} = \nabla\phi}$, this point is called scalar quantity of $\bar{F}$

In this case $\boxed{\text{curl}\,\bar{F} = \nabla\times\bar{F} = 0}$

Degree of vector:-

Divergence of a vector function $\bar{F}(x,y,z)$ is written as divergence of $\bar{F}$ (or) div $\bar{F}$ and denoted by $\boxed{\nabla \cdot \bar{f}}$ is defined as:

$$\text{div}\bar{F} = \nabla\cdot\bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\bar{f}$$

$$\boxed{\nabla\cdot\bar{f} \Rightarrow i\cdot\frac{\partial \bar{f}}{\partial x} + j\cdot\frac{\partial \bar{F}}{\partial y} + k\cdot\frac{\partial \bar{f}}{\partial z}}$$

grad(any scalar) = ∇(scalar) $\to$ vector quantity

div(any vector) = $\nabla\cdot$ Vector $\to$ scalar quantity

$$\boxed{\text{div}\bar{F} = \nabla\cdot\bar{f}}$$

Note: (i) $(\bar{a}\cdot\nabla)\phi = a\cdot\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$

$$\to (\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial\phi}{\partial z}$$

$$\boxed{\bar{a}\cdot\nabla\phi \Rightarrow \sum \bar{a}\cdot\bar{i}\cdot\frac{\partial\phi}{\partial x}}$$

(ii) $\nabla\cdot\bar{F} = \bar{F}\cdot\nabla$

If A and B are vector functions and 'f' is scalar function then.

(i) $\text{div}(\bar{A}+\bar{B}) = \text{div}\bar{A} + \text{div}\bar{B}$ (or)

$$\nabla\cdot(\bar{A}+\bar{B}) = \nabla\cdot\bar{A} + \nabla\cdot\bar{B}$$

(ii) $\text{div}(f\bar{A}) = \text{grad} f\cdot\bar{A} + f(\text{div}\bar{A})$ (or)

$$\nabla\cdot f\bar{A} = \nabla f\cdot\bar{A} + f(\nabla\bar{A})$$

Solenoidal vector:-

A vector point function $\bar{F}$ is said to be solenoidal, if $\boxed{\text{div}\bar{F} = \nabla\cdot\bar{f} = 0}$

This eqn is also called the eqn of continuity (or) conservation of mass.

Problems:

(1) Find div $\bar{F}$, where $\bar{F} = r^n\bar{r}$.

Find 'n' if it is solenoidal (or)

Prove that $(r^n\bar{r})$ is solenoidal

Hence show that $\frac{\bar{r}}{r^3}$ is solenoidal

Sol: $f = r^n\bar{r}$. div $\bar{F}$

div $f = \nabla\cdot f$

$$\nabla = \left(\frac{\partial}{\partial x}\right)i + \left(\frac{\partial}{\partial y}\right)j + \left(\frac{\partial}{\partial z}\right)k$$

$f = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

diff

$2r\cdot\frac{\partial r}{\partial x} = 2x$ | $2r\frac{\partial r}{\partial y} = 2y$ | $2r\frac{\partial r}{\partial z} = 2z$

$\frac{\partial r}{\partial x} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = \frac{y}{r}$ | $\frac{\partial r}{\partial z} = \frac{z}{r}$ $\to$ (1)

$F = r^n\bar{r}$

$= r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$\bar{F} = r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$

$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

div $\bar{F} = \nabla\bar{F}$

$\nabla = \frac{\partial}{\partial x}f_1 + \frac{\partial}{\partial y}f_2 + \frac{\partial}{\partial z}f_3$

$\frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= \left(r^n(1) + x\cdot n r^{n-1}\frac{\partial r}{\partial x}\right)$

$+ \left(r^n + y n r^{n-1}\frac{\partial r}{\partial y}\right) + \left(r^n + z n r^{n-1}\frac{\partial r}{\partial z}\right)$

(∵ from ①)

$r^n + x n r^{n-1} \frac{x}{r} + r^n + y\, n\, r^{n-1} \frac{y}{r}$

$+ r^n + z\, n\, r^{n-1} \frac{z}{r}$

$\Rightarrow 3r^n + \frac{n r^{n-1}}{r}(x^2+y^2+z^2)$

$(\because x^2+y^2+z^2 = r^2)$

$\Rightarrow 3r^n + \frac{n r^{n-1}}{\cancel{r}} (\cancel{r^2})\, r$

$\Rightarrow 3r^n + n\, r^{n-1+1}$

$\Rightarrow 3r^n + n \cdot r^{n-1+1}$

$\Rightarrow 3r^n + n r^n$

$r^n (3+n) = 0$

$n+3 = \frac{0}{r^n}$

$\boxed{n = -3}$

$\text{div}\, \bar{F} = \text{div}\,(r^n \cdot \bar{r}) = (n+3)\, r$

$r^n\, \bar{r} = 0$

$r^{-3} \cdot \bar{r} = 0$

$\boxed{\frac{\bar{r}}{r^3} = 0}$

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① (B) given $\phi = x^2 - y^2 + 2z^2$

and point $P = (1,2,3)$ and $\phi = (5,0,4)$

$P\phi = P = i + 2j + 3k$

$\phi = 5i + 0j + 4k$

$(5-1)i + (-2)j + (4-3)k$

$\Rightarrow 4i - 2j + k = P\phi$

$P\phi = 0$

$4i - 2j + k = 0 = r$

$f = x^2 - y^2 + 2z^2$

given find directional derivation

$\nabla \cdot f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(x^2 - y^2 + 2z^2)$

$\Rightarrow 2xi - 2yj + 4zk$

at point $(1,2,3)$

$\Rightarrow 2i - 2(2)j + 4(3)k$

$\Rightarrow 2\bar{i} - 4\bar{j} + 12\bar{k}$

$\nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$

$\Rightarrow (2i - 4j + 12k) \cdot \frac{4i - 2j + k}{\sqrt{16+4+1}}$

$\Rightarrow \frac{(2)(4) + (-4)(-2) + (12)(1)}{\sqrt{21}}$

$\Rightarrow \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$

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② (A) $\nabla r^n = n r^{n-2}\, \bar{r}$

$\nabla = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)$

$r = xi + yj + zk$

$\nabla r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) r^n$

$\boxed{\nabla r^n = \sum i \frac{\partial r^n}{\partial x}}$

$\sum i \frac{\partial}{\partial x} r^n$ $\left(\because \frac{\partial}{\partial x} r^n = n r^{n-1} \frac{\partial r}{\partial x}\right)$

$\sum i\, n\, r^{n-1} \frac{\partial r}{\partial x}$ $\left(\because r^2 = x^2+y^2+z^2 \text{ diff } 2r\frac{\partial r}{\partial x} = 2x,\ \frac{\partial r}{\partial x} = \frac{x}{r}\right)$

$\sum i \cdot n r^{n-1} \frac{x}{r}$

$\frac{n r^{n-1}}{r} \sum x\bar{i}$ $(\because r = xi + yj + zk,\ r = \sum xi)$

$\Rightarrow \frac{n \cdot r^{n-1}}{r}(r)$

$\therefore \nabla r^n = n\, r^{n-1}$

$\Rightarrow \frac{n r^{n-1}}{r} \cdot r$

$\nabla r^n = n r^{n-1}\, r^{-1}\, r$

$\nabla r^n = n\, r^{n-2}\, r$

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2 (B) $\nabla(\bar{a} \cdot \bar{b}) = (\bar{b} \cdot \nabla)\bar{a} + (\bar{a} \cdot \nabla)\bar{b}$

$+ \bar{b} \times \text{curl}\, \bar{a} + \bar{a} \times \text{curl}\, \bar{b}$

Sol: w.k.T

$\boxed{\text{curl}\, b = \nabla \times b}$

$\bar{b} \times \text{curl}\, \bar{a}$ $\left(\sum \bar{b} \times \left(i \times \frac{\partial a}{\partial x}\right)\right)$

$\Rightarrow \bar{b} \times (\nabla \times \bar{a})$

$\boxed{\because a \times (b \times c) = (a \cdot c) b - (a \cdot b) c}$

$\sum \bar{b} \times \left(i \times \frac{\partial a}{\partial x}\right)$

$= \sum\left(b \cdot \frac{\partial a}{\partial x}\right) i - \sum (b \cdot i) \frac{\partial a}{\partial x}$

$\Rightarrow \sum\left(b \cdot \frac{\partial a}{\partial x}\right) i - (b \cdot \nabla)\bar{a}$ → ①

$(\because a \cdot \nabla = a \cdot \frac{\partial}{\partial x} i + b \frac{\partial}{\partial y} j + c \cdot \frac{\partial}{\partial z} c)$

$(\because a \cdot \nabla = \sum a \cdot i \frac{\partial a}{\partial x})$

$\bar{a} \times \text{curl}\, \bar{b} = \left(\sum \bar{a} \times \left(i \times \frac{\partial b}{\partial y}\right)\right)$

$\sum\left(a \cdot \frac{\partial b}{\partial y}\right) i - \sum (a \cdot i) \frac{\partial b}{\partial y}$

$(\because \bar{a} \cdot \nabla = \sum a \cdot i \frac{\partial a}{\partial x})$

$\Rightarrow \sum\left(a \cdot \frac{\partial y}{\partial y}\right) i - (a \cdot \nabla)\bar{b}$ → ②

① + ②

$\bar{a} \times \text{curl}\, \bar{b} + \bar{b} \times \text{curl}\, \bar{a}$

$= \sum\left(\bar{a} \cdot \frac{\partial b}{\partial x}\right) i - (\bar{a} \cdot \nabla)\bar{b}$

$+ \sum\left(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}\right) i - (\bar{b} \cdot \nabla)\bar{a}$

$\bar{a} \times \text{curl}\, \bar{b} + \bar{b} \times \text{curl}\, \bar{a} + (a \cdot \nabla)\bar{b} + (b \cdot \nabla)\bar{a}$

$\Rightarrow \sum\left(\bar{a} \cdot \frac{\partial b}{\partial x}\right) i + \sum\left(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}\right) j$

$\sum \bar{a}\bar{b}\left(\frac{\partial i}{\partial x} + \frac{\partial i}{\partial x}\right)$

is grad$(a \cdot b)$

gam

2 (B) given RTP

$\nabla(a \cdot b) = (b \cdot \nabla)\bar{a} + (\bar{a} \cdot \nabla)\bar{b} + (b \times \text{curl}\, a) + (a \times \text{curl}\, b)$

consider $b \times \text{curl}\, a$

wkt $\text{curl}\, a = \nabla \times a$

$\Rightarrow b \times (\nabla \times a)$

$\boxed{\because a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c}$

$\nabla \times a = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) a$

$= \left(\frac{\partial a}{\partial x} \times i + \frac{\partial a}{\partial y} \times j + \frac{\partial a}{\partial z} \times k\right)$

$\boxed{\nabla \times a \Rightarrow \sum i \times \frac{\partial a}{\partial x}}$

$\therefore b \times (\nabla \times a)$

$\cancel{(a \cdot b)\, c - (b - a)}$

$\Rightarrow \cancel{b \times (}$

$\sum b \times \left(\sum i \times \frac{\partial a}{\partial x}\right)$

$\Rightarrow a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$

$\Rightarrow \sum\left(b \cdot \frac{\partial a}{\partial x}\right) i - \sum (b \cdot i) \frac{\partial a}{\partial x}$

$\sum a \cdot \nabla = a\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)$

$= \sum \frac{\partial a}{\partial x} i$

$\Rightarrow \sum\left(b \cdot \frac{\partial a}{\partial x}\right) i - (b \cdot \nabla)\bar{a} \rightarrow (1)$

III'ly $\bar{a} \times \text{curl}\, \bar{b}$

$\Rightarrow \bar{a} \times (\nabla \times \bar{b})$

$\Rightarrow \sum\left(a \cdot \frac{\partial b}{\partial x}\right) i - \sum (a \cdot i) \frac{\partial b}{\partial x}$

$\sum\left(a \cdot \frac{\partial b}{\partial x}\right) i - (a \cdot \nabla)\bar{b} \rightarrow (2)$

(1) + (2)

$\Rightarrow \sum\left(\bar{a} \cdot \frac{\partial b}{\partial x}\right) i - (\bar{a} \cdot \nabla)\bar{b} + \sum\left(b \cdot \frac{\partial a}{\partial x}\right) i - (b \cdot \nabla)\bar{a}$

$\Rightarrow \bar{a} \times \text{curl}\, \bar{b} + \bar{b} \times \text{curl}\, \bar{a} + (\bar{a} \cdot \nabla)\bar{b} + (b \cdot \nabla \bar{a})$

$\Rightarrow \sum\left(\bar{a} \cdot \frac{\partial b}{\partial x}\right) i - (\bar{a} \cdot \nabla)\bar{b} + \sum\left(b \cdot \frac{\partial a}{\partial x}\right) i$

$\cancel{-(b \cdot \nabla)\bar{a} + (a \cdot \nabla)\bar{b} + (b \cdot \nabla)\bar{a}}$

$\Rightarrow \sum a\left(\frac{\partial b}{\partial x}\right) i + \sum\left(b \cdot \frac{\partial a}{\partial x}\right) i$

$\Rightarrow \sum ab\left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial x}\right)$

$\Rightarrow$ grad$(a \cdot b)$

3(A). ST $\nabla(f(r)) = \frac{f'(r)}{r}\, r$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

$r^2 = x^2 + y^2 + z^2$

diff

$2r \frac{\partial r}{\partial x} = 2x$	$2r \cdot \frac{\partial r}{\partial y} = 2y$	$2r \cdot \frac{\partial r}{\partial z} = 2z$
$\frac{\partial r}{\partial x} = \frac{x}{r}$	$\frac{\partial r}{\partial y} = \frac{y}{r}$	$\frac{\partial r}{\partial z} = \frac{z}{r}$
$\rightarrow (1)$		

RTP $\nabla f(r) = \frac{f'(r)}{r}\, r$

LHS $\nabla f(r)$

$\Rightarrow \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right) F(r)$

$= i\, f'(r) \frac{\partial r}{\partial x} + j\, f'(r) \frac{\partial r}{\partial y} + k\, f'(r) \frac{\partial r}{\partial z}$

$\Rightarrow \sum \bar{i} \cdot f'(r) \cdot \frac{\partial r}{\partial x}$

$(\because$ from (1)$)$

$\sum \bar{i}\, f'(r) \cdot \frac{x}{r}$

$\Rightarrow \frac{f' r}{r} \sum \bar{i} x$

$\left[\because r = xi + yj + zk,\ \sum i x = r\right]$

$\Rightarrow \frac{f'(r)}{r} \cdot r$

$\therefore$ LHS = RHS

$\nabla F(r) = \frac{f'(r)}{r}\, (r)$

3(B) $F = xy^2 + yz^3 + zx^2$

along tangent to the curve

$x = t,\ y = t^2,\ z = t^3$ at point $(1,1,1)$

Sol:

$F = xy^2 + yz^2 + zx^2$

$P = (1,1,1)$

$x = t,\ y = t^2,\ z = t^3$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\Rightarrow i\frac{\partial(xy^2+yz^2+zx^2)}{\partial x} + \frac{\partial(xy^2+yz^2+zx^2)}{\partial y}j$$

$$+ \frac{\partial(xy^2+yz^2+zx^2)}{\partial z}k$$

$$\Rightarrow i(y^2+2xz) + (2xy+z^2)j + (2zy+x^2)k$$

at $P(1,1,1)$

$$i(1+2) + (2+1)j + (2+1)k$$

$$\nabla f = 3i + 3j + 3k$$

$$\nabla f = 3(i+j+k)$$

$r = xi + yj + zk$

$x = t$	$y = t^2$	$z = t^3$
$1 = t$	$1 = t^2$	$1 = t^3$
$t = 1$	$t = 1$	$t = 1$

$r = ti + t^2j + t^3k$

$$\frac{\partial r}{\partial t} = i + 2tj + 3t^2k$$

$$\frac{\partial r}{\partial t} = i + 2j + 3k = a$$

$$\frac{\partial r}{\partial t} = a = i + 2j + 3k$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{i+2j+3k}{\sqrt{1+4+9}}$$

$$= \frac{i+2j+3k}{\sqrt{14}}$$

$$\frac{\partial s}{\partial \phi} = \frac{a}{|a|} = (3i+3j+3k)\frac{(i+2j+3k)}{\sqrt{14}}$$

$$\Rightarrow \frac{(3)(1)+(3)(2)+(3)(3)}{\sqrt{14}}$$

$$\Rightarrow \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

1(A) ~~div~~ (ϕ) P.T

$$\mathrm{div}(\phi\cdot\bar{a}) = \mathrm{grad}(\phi)\cdot\bar{a} + \phi\,\mathrm{div}\,\bar{a}$$

$$\boxed{\mathrm{div}(F) = \nabla F}$$

$$\boxed{\mathrm{grad}\,\phi = \nabla\phi}$$

$$\mathrm{div}(\phi)a = \nabla\phi\, a \quad -(1)$$

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\boxed{\nabla\cdot\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k}$$

$$\Rightarrow \sum i\frac{\partial}{\partial x}(\phi)$$

$$\boxed{\therefore \nabla\phi a = \sum i\frac{\partial}{\partial x}(\phi a)} \quad -(2)$$

$$\cancel{\nabla\phi a} = \sum i\frac{\partial\phi}{\partial x}a + \sum i\frac{\partial a}{\partial x}\phi$$

$-(2)$

$$= (\nabla\phi a + \nabla a\phi)$$

$$\nabla\phi a = \mathrm{grad}(\phi)\cdot a + \cancel{\mathrm{div}}\,\phi(\mathrm{div}\,a)$$

$$\mathrm{div}(\phi\cdot\bar{a}) = (\mathrm{grad}\,\phi)\cdot a + \phi\,\mathrm{div}\cdot a$$

4(B) $F(x,y,z) = x^2+y^2+z^2 = 29$

$g(x,y,z) = x^2+y^2+z^2+4x-6y-8z-47=0$

at point $(4,-3,2)$

$$\nabla f = i\left(\frac{\partial f}{\partial x}\right) + j\left(\frac{\partial f}{\partial y}\right) + k\left(\frac{\partial f}{\partial z}\right)$$

$$\Rightarrow i(2x) + j(2y) + k(2z)$$

$$2xi + 2yj + 2zk$$

$P(4,-3,2)$

$$\boxed{\nabla f = 8i - 6j + 4k}$$

∂g

$$\frac{\partial g}{\partial x} = 2x+4 \quad \frac{\partial g}{\partial y} = 2y-6 \quad \frac{\partial g}{\partial z} = 2z-8$$

$$\Rightarrow \nabla g = i(2x+4) + j(2y-6) + k(2z-8)$$

$P(4,-3,2)$

$$\Rightarrow i(2(4)+4) + j(2(-3)-6) + k(2(2)-8)$$

$$\boxed{\nabla g = 12i - 12j - 4k}$$

$$\cos\theta = \frac{\nabla f\cdot\nabla g}{|\nabla f||\nabla g|}$$

$$= \frac{8(12) - 6(-12) + 4(-4)}{\sqrt{64+36+16} \cdot \sqrt{144+144+16}}$$

$$\Rightarrow \frac{96 + 72 - 8}{\sqrt{116} \cdot \sqrt{304}}$$

$$\cos\theta = \frac{152}{\sqrt{116} \cdot \sqrt{304}}$$

$$\cos^{-1}\frac{\sqrt{551}}{29} = \cos^{-1}\left(\frac{\sqrt{19 \times 29}}{29}\right.$$

$$\cos^{-1}\frac{\sqrt{19}\,\sqrt{29}}{\sqrt{29}\,\sqrt{29}} \Rightarrow \cos^{-1}\frac{\sqrt{19}}{\sqrt{29}}$$

$$\boxed{(a \times mb) = m(a \times b)}$$

$$\sum i \times \left(\frac{\partial \phi}{\partial x}\right) a + \sum i \times \left(\frac{\partial a}{\partial x}\right) \phi$$

~~$\sum$ grad~~

$$\Rightarrow \sum \phi\left(i \times \frac{\partial \phi}{\partial x}\right) + \sum \frac{\partial \phi}{\partial x}(i \times a)$$

$$\Rightarrow \phi(\nabla \times a) + \nabla\phi \times a$$

$$\therefore \cancel{\phi \cdot (\nabla \times a)} = (\nabla \phi) \times a + \phi(\nabla \times a)$$

$$\phi \cdot \nabla(\phi \times a) = (\nabla(\phi)) \times a + \phi(\nabla \times a)$$

$$\therefore \text{curl}(\phi a) = (\text{grad}\,\phi) \times a + \phi\,\text{curl}\,\bar{a}$$

5(A) $\text{curl}(\phi a) = (\text{grad}\,\phi) \times a + \phi\,\text{curl}\,\bar{a}$

$$\phi\,\text{curl}\,\bar{a} = \phi \cdot (\nabla \times a)$$

$$\text{grad}\,\phi \times a = (\nabla \cdot \phi) \times a$$

$$\text{curl}(\phi a) = (\nabla \times \phi a)$$

$$\sum i \times \frac{\partial}{\partial x}(\phi a) \quad \Big| \quad \sum i \times \frac{\partial}{\partial x}(\phi) \cdot a$$

$$\sum i \frac{\partial \phi}{\partial x} \cdot a + \sum i \cdot \frac{\partial a}{\partial x} \phi$$

5(B)

6(A) $\text{curl}(\bar{a} \times \bar{b})$

$$= a\,\text{div}\,b - b\,\text{div}\,a + (\bar{b} \cdot \nabla) a - (a \cdot \nabla) b$$

$$\text{curl}(\bar{a} \times \bar{b}) = \nabla \times (a \times b)$$

$$\sum i \times (a \times b) \frac{\partial}{\partial x} \quad \left[\therefore \nabla \times f = \sum i \times \frac{\partial f}{\partial x}\right]$$

$$\sum i \times \left(a \times \frac{\partial b}{\partial x}\right) + \sum i \times \left(b \times \frac{\partial a}{\partial x}\right)$$

$$\Rightarrow \sum i \times \left(a \times \frac{\partial b}{\partial x}\right) + \sum i \times \left(\frac{\partial a}{\partial x} \times b\right)$$

$$\boxed{a \times (b \times c) = (a \cdot b) \cdot b - (a \cdot b) \cdot c}$$

$$\Rightarrow \sum\left(i \cdot \frac{\partial b}{\partial x}\right) \cdot a - \sum (i \cdot a) \cdot \frac{\partial b}{\partial x} + \sum (i \cdot b) \frac{\partial a}{\partial x} - \sum\left(i \cdot \frac{\partial a}{\partial x}\right) b$$

$$\Rightarrow \sum\left(i \cdot \frac{\partial b}{\partial x}\right) a - (\nabla \cdot b)\bar{a} - (\nabla \cdot a) b + \sum (i \cdot b) \frac{\partial a}{\partial x}$$

$$(\nabla \cdot b) a - (a \cdot \nabla) b + (\bar{b} \cdot \nabla) a - (a \cdot \nabla) b$$

$$= a\,\text{div}\,b - b\,\text{div}\,a + (\bar{b} \cdot \nabla) a - (a \cdot \nabla) b$$

6 (B) PT div curl F = 0

$F = f_1 i + f_2 j + f_3 k$

$\text{div} F = \nabla f$

$\text{curl} f = \nabla \times f$

div . curl f

$\Rightarrow \nabla (\nabla \times f)$

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$f = f_1 i + f_2 j + f_3 k$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\Rightarrow i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\nabla(\nabla \times f) = \frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right] + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \cdot \partial y} - \frac{\partial^2 f_2}{\partial x \cdot \partial z} - \frac{\partial^2 f_1}{\partial y \cdot \partial z} + \frac{\partial^2 f_3}{\partial x \cdot \partial y} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial y \cdot \partial z}$$

$= 0$

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7 (A) ~~curl~~ PT curl grad $\phi = 0$

~~$\nabla \phi$~~ grad $\phi = \nabla \phi$

curl $\phi = \nabla \times \phi$

curl grad $\phi = \nabla \times (\nabla \phi)$

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$

$\nabla \times \nabla \phi = 0$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$\Rightarrow i\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) - j\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \cdot \partial z}\right) + k\left(\frac{\partial^2 \phi}{\partial x \cdot \partial y} - \frac{\partial^2 \phi}{\partial x \cdot \partial y}\right)$$

$i(0) - j(0) + k(0)$

$= 0$

$\therefore$ curl grad $(\phi) = 0$

PTP

(B) $\nabla^2 r^n = n(n+1) r^{n-2}$

$\nabla^2 r^n$

$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

$\nabla^2 = \frac{\partial^2}{\partial x^2} i + \frac{\partial^2}{\partial y^2} j + \frac{\partial^2}{\partial z^2} k$

$\nabla^2 = \sum i \frac{\partial^2}{\partial x^2}$

$\nabla^2 r^n = \sum i \frac{\partial^2}{\partial x^2} r^n$ $\quad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$

$\sum i \frac{\partial}{\partial x}\, n r^{n-1} \cdot \frac{\partial r}{\partial x}$ $\quad \left(r = \frac{1}{r} = r^{-1}\right)$

$\sum i \left[\frac{\partial}{\partial x}\, n r^{n-1} \cdot \frac{x}{r}\right]$ $\quad \left[\frac{\partial}{\partial x} uv = u\frac{dv}{dx} + v\frac{du}{dx}\right]$

$\Rightarrow \frac{\partial}{\partial x}\, n r^{n-2} x$

$\Rightarrow \sum i \left[r^{n-2}(1) + x(n-2) r^{n-2-1} \cdot \frac{\partial r}{\partial x}\right]$

$\Rightarrow \sum i \left[r^{n-2} + x(n-2) r^{n-3} \cdot \frac{\partial r}{\partial x}\right]$

LHS: $n \sum (r^{n-2} + (n-2) r^{n-3} \cdot x \left(\frac{x}{r}\right)$

$\Rightarrow n\left[\sum r^{n-2} + (n-2) r^{n-3} \cdot \frac{x^2}{r}\right]$

$\Rightarrow n\left[\sum r^{n-2} + (n-2) r^{n-3-1} x^2\right]$

$\Rightarrow n\left[3 r^{n-2} + (n-2) r^{n-4} \sum x^2\right]$

$\Rightarrow n\left[3 r^{n-2} + (n-2) r^{n-4} (r^2)\right]$ $\quad \left(r^2 = \sum x^2 = x^2 + y^2 + z^2\right)$

$\Rightarrow n\left[3 r^{n-2} + (n-2) r^{n-4+2}\right]$

$\Rightarrow n\left[3 r^{n-2} + (n-2) r^{n-2}\right]$

$n r^{n-2}\, [3 + n - 2]$

$n \cdot r^{n-2}\, (n+1)$

$\therefore \nabla^2 r^n = n(n+1) r^{n-2}$

= RHS.

8(a)

8(B) $\nabla \times (\nabla \times a) = \nabla(\nabla \cdot a) - \nabla^2 a$

$L.H.S = \nabla \times (\nabla \times a)$

$= \sum i \times \frac{\partial}{\partial x} (\nabla \times a)$

$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

consider

$i \times \frac{\partial}{\partial x} (\nabla \times a)$

$= \cancel{i \times \frac{\partial}{\partial x} \left(i \times \frac{\partial a}{\partial x} + j \times \frac{\partial a}{\partial y} + k \times \frac{\partial a}{\partial z} \right)}$

$\Rightarrow \bar{i} \times \frac{\partial}{\partial x} \left(i \times \frac{\partial a}{\partial x} + \frac{\partial a}{\partial x} \times 0 + j \times \frac{\partial a}{\partial y} + \frac{\partial a}{\partial y} \times 0 + k \times \frac{\partial a}{\partial z} + \frac{\partial a}{\partial z} \times 0 \right)$

$\Rightarrow \bar{i} \times \left(i \times \frac{\partial^2 a}{\partial x^2} + \bar{j} \times \frac{\partial^2 a}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 a}{\partial x \partial z} \right)$

L.H.S

$\Rightarrow \sum \bar{i} \times \left(i \times \frac{\partial^2 a}{\partial x} + \bar{j} \times \frac{\partial^2 a}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 a}{\partial x \partial z} \right)$

9(A) unit normal vector to the surface

$z = x^2 + y^2$ at $(-1,-2,5)$

Sol:

$F = x^2 + y^2 - z$

$P(-1,-2,5)$

$$\nabla F = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(x^2 + y^2 - z)$$

$\nabla F = 2xi + 2yj - k$

at $P(-1,-2,5)$

$\Rightarrow 2(-1)i + 2(-2)j - k$

$\nabla F \Rightarrow -2i - 4j - k$

$$\hat{\nabla F} = \frac{\nabla F}{|\nabla F|} = \frac{-2i - 4j - k}{\sqrt{4 + 16 + 1}}$$

$$\Rightarrow \cancel{\frac{-2i - 4j - k}{\sqrt{45}}}$$

$$\therefore \hat{\nabla F} = \frac{-2i - 4j - k}{\sqrt{21}}$$

$\therefore$ unit vector

$$\hat{\nabla F} = \frac{-2i - 4j - k}{\sqrt{21}}$$

9(B) Find angle b/w surfaces

$x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$

at point $(2,-1,2)$

Sol: $\phi_1 = x^2 + y^2 + z^2 = 9$

$\phi_2 = x^2 + y^2 - 3$

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\nabla \phi_1 = \frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 9) i + \frac{\partial}{\partial y}(x^2 + y^2 + z^2 - 9) j + \frac{\partial}{\partial z}(x^2 + y^2 + z^2 - 9) k$$

$\nabla \phi_1 = 2xi + 2yj + 2zk$

At $P(2,-1,2)$

$\Rightarrow 2(2)i + 2(-1)j + 2(2)k$

$\nabla \phi_1 \Rightarrow 4i - 2j + 4k$

$$\nabla \phi_2 = \frac{\partial}{\partial x}(x^2 + y^2 - 3) i + \frac{\partial}{\partial y}(x^2 + y^2 - 3) j + \frac{\partial}{\partial z}(x^2 + y^2 - z) k$$

$\nabla \phi_2 \Rightarrow 2xi + 2yj + k$

at $(2,-1,2)$

$\Rightarrow 2(2)i + 2(-1)j + k$

$\Rightarrow 4i - 2j + k$

$\nabla \phi_1 = 4i - 2j + 4k$

$\nabla \phi_2 = 4i - 2j + k$

$$\cos\theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}$$

$$\Rightarrow \frac{(4)(4) + (-2)(-2) + (4)(1)}{\sqrt{16 + 4 + 16} \cdot \sqrt{16 + 4}}$$

$$\Rightarrow \frac{16 + 4 + 4}{\sqrt{36} \cdot \sqrt{20}}$$

$$\cos\theta \Rightarrow \frac{24}{\sqrt{720}} =$$

$$\theta = \cos^{-1} \frac{8}{3\sqrt{20}}$$

10(A)

10(B) $F = x^2 - y^2 + 2z^2$

$P = (1, 2, 3)$

$Q = (5, 0, 4)$

$P = i + 2j + 3k$

$Q = 5i + 0j + 4k$

$PQ = (5-1)i + (0-2)j + (4-3)k$

$PQ \Rightarrow 4i - 2j + k$

$a = 4i - 2j + k$

$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{4i - 2j + k}{\sqrt{16+4+1}}$

$\Rightarrow \cancel{\frac{4i - 2j + k}{\sqrt{21}}}$

$\hat{a} \Rightarrow \frac{4i - 2j + k}{\sqrt{21}}$

$\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$

$\nabla f = 2xi - 2yj + 4kz$

$(1, 2, 3)$

$\nabla f \rightarrow 2i - 4j + 12k$

$\nabla f \cdot \frac{\bar{a}}{|\bar{a}|} = (2i - 4j + 12k) \frac{(4i - 2j + k)}{\sqrt{21}}$

$\Rightarrow \cancel{8i + 8j} + 12$

$= \frac{8 + 8 + 12}{\sqrt{21}}$

$\Rightarrow \frac{28}{\sqrt{21}}$