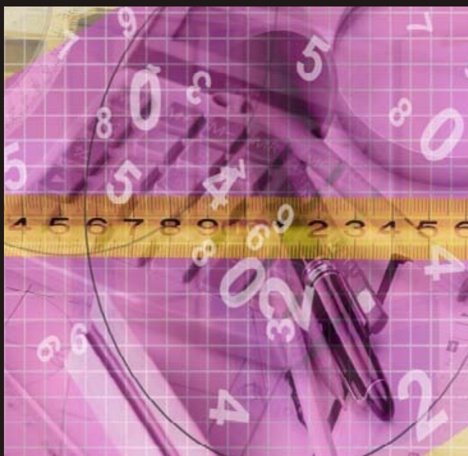
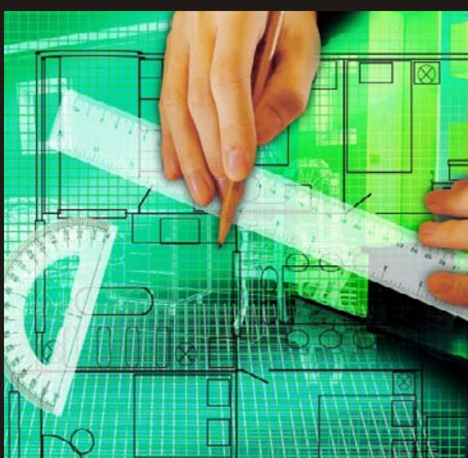


NEW AGE

SECOND EDITION

A Textbook of ENGINEERING MATHEMATICS-I

H.S. Gangwar • Prabhakar Gupta

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Preface to the Second Revised Edition

This book has been revised exhaustively according to the global demands of the students. Attention has been taken to add minor steps between two unmanageable lines where essential so that the students can understand the subject matter without mental tire.

A number of questions have been added in this edition besides theoretical portion wherever necessary in the book. Latest question papers are fully solved and added in their respective units.

Literal errors have also been rectified which have been accounted and have come to our observation. Ultimately the book is a gift to the students which is now error free and user- friendly.

Constructive suggestions, criticisms from the students and the teachers are always welcome for the improvement of this book.

AUTHORS

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Some Useful Formulae

1. $\sin ix = i \sin hx$
2. $\cos ix = \cos hx$
3. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$
4. $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$
5. $\text{Sin } h^2 x = \dfrac{1}{2} (\cos h\, 2x - 1)$
6. $\cos h^2 x = \dfrac{1}{2} (\cos h\, 2x + 1)$
7. $\int a^x dx = \dfrac{a^x}{\log a} [a \neq 1, a > 0]$
8. $\int \sin hax dx = \dfrac{1}{a} \cos hax$
9. $\int \cos hax dx = \dfrac{1}{a} \sin hax$
10. $\int \tan hax dx = \dfrac{1}{a} \log |\cos hax|$
11. $\int \dfrac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \dfrac{x}{a} = arc \sin \dfrac{x}{a}$
12. $\int \dfrac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right|$
13. $\int \dfrac{1}{x^2 + a^2} dx = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a} = arc \tan \dfrac{x}{a}$
14. $\int \sqrt{a^2 - x^2}\, dx = \dfrac{x}{2} \sqrt{a^2 - x^2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{|a|}$
15. $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$
16. $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$
17. $\int \sec ax\, dx = \dfrac{1}{a} \log |\sec ax + \tan ax|$

18. $\int \text{cosec}\, ax\, dx = \frac{1}{a} \log \left| \text{cosec}\ ax - \cot\ ax \right|$

19. $\sin\ x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} \ldots\ldots$

20. $\cos\ x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} \ldots\ldots\ldots$

21. $\tan\ x = x + \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots\ldots\ldots$

22. $\log\ (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

23. $\log\ (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots\ldots$

24. $\sin\ hx = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \ldots\ldots\ldots$

25. $\frac{d}{dx} a^x = a^x \log_e a$

26. $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

27. $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

28. $\frac{d}{dx} \text{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

29. $\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$

30. $\frac{d}{dx} \tan^{-1} \frac{x}{a} = \frac{a}{a^2+x^2}.$

Contents

UNIT I

Differential Calculus-I

1.0 INTRODUCTION

Calculus is one of the most beautiful intellectual achievements of human being. The mathematical study of change motion, growth or decay is calculus. One of the most important idea of differential calculus is derivative which measures the rate of change of a given function. Concept of derivative is very useful in engineering, science, economics, medicine and computer science.

The first order derivative of y denoted by $\frac{dy}{dx}$, second order derivative, denoted by $\frac{d^2y}{dx^2}$ third order derivative by $\frac{d^3y}{dx^3}$ and so on. Thus by differentiating a function $y = f(x)$, n times, successively, we get the nth order derivative of y denoted by $\frac{d^ny}{dx^n}$ or $D^n y$ or $y_n(x)$. Thus, the process of finding the differential co-efficient of a function again and again is called **Successive Differentiation.**

1.1 *n*th DERIVATIVE OF SOME ELEMENTARY FUNCTIONS

1. Power Function $(ax + b)^m$

Let

$$y = (ax+b)^m$$

$$y_1 = ma(ax+b)^{m-1}$$

$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$

....

....

$$y_n = m(m-1)(m-2)\ldots(m-\overline{n-1})\,a^n(ax+b)^{m-n}$$

Case I. When m is positive integer, then

$$y_n = \frac{m(m-1)\ldots(m-n+1)(m-n)\ldots3\cdot2\cdot1}{(m-n)\ldots3\cdot2\cdot1}\,a^n(ax+b)^{m-n}$$

$$\Rightarrow \quad \boxed{y_n = \frac{d^n}{dx^n}(ax+b)^m = \frac{\lfloor m}{\lfloor m-n}\,a^n(ax+b)^{m-n}}$$

Case II. When $m = n =$ +ve integer

$$y_n = \frac{\lfloor n}{\lfloor 0} a^n (ax+b)^0 = \lfloor n\, a^n \Rightarrow \boxed{\frac{d^n}{dx^n}(ax+b)^n = \lfloor n\, a^n}$$

Case III. When $m = -1$, then

$$y = (ax + b)^{-1} = \frac{1}{(ax+b)}$$

$$\therefore \quad y_n = (-1)(-2)(-3)\ \ldots\ (-n)\, a^n (ax + b)^{-1-n}$$

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\left\{\frac{1}{ax+b}\right\} = \frac{(-1)^n \lfloor n\, a^n}{(ax+b)^{n+1}}}$$

Case IV. Logarithm case: When $y = \log (ax + b)$, then

$$y_1 = \frac{a}{ax+b}$$

Differentiating $(n-1)$ times, we get

$$y_n = a^n \frac{d^{n-1}}{dx^{n-1}}(ax+b)^{-1}$$

Using case III, we obtain

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\{\log(ax+b)\} = \frac{(-1)^{n-1}\lfloor (n-1)\, a^n}{(ax+b)^n}}$$

2. Exponential Function

(*i*) Consider $y = a^{mx}$

$$y_1 = ma^{mx}.\ \log_e a$$

$$y_2 = m^2 a^{mx} (\log_e a)^2$$

....

....

$$\boxed{y_n = m^n a^{mx} (\log_e a)^n}$$

(*ii*) Consider $y = e^{mx}$

Putting $a = e$ in above $\boxed{y_n = m^n e^{mx}}$

3. Trigonometric Functions cos ($ax + b$) or sin ($ax + b$)

Let $y = \cos (ax + b)$, then

$$y_1 = -a \sin (ax + b) = a \cos \left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = -a^2 \cos (ax + b) = a^2 \cos \left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = +a^3 \sin (ax + b) = a^3 \cos \left(ax+b+\frac{3\pi}{2}\right)$$

....

....

$$y_n = \frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly,
$$y_n = \frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

4. Product Functions e^{ax} sin $(bx + c)$ or e^{ax} cos $(bx + c)$

Consider the function $y = e^{ax} \sin(bx + c)$

$$y_1 = e^{ax}\cdot b\cos(bx+c) + ae^{ax}\sin(bx+c)$$
$$= e^{ax}[b\cos(bx+c) + a\sin(bx+c)]$$

To rewrite this in the form of sin, put

$$a = r\cos\phi,\ b = r\sin\phi, \text{ we get}$$
$$y_1 = e^{ax}[r\sin\phi\cos(bx+c) + r\cos\phi\sin(bx+c)]$$
$$y_1 = re^{ax}\sin(bx+c+\phi)$$

Here, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$

Differentiating again w.r.t. x, we get

$$y_2 = rae^{ax}\sin(bx+c+\phi) + rbe^{ax}\cos(bx+c+\phi)$$

Substituting for a and b, we get

$$y_2 = re^{ax}.\ r\cos\phi\sin(bx+c+\phi) + re^{ax}\ r\sin\phi\cos(bx+c+\phi)$$
$$y_2 = r^2e^{ax}[\cos\phi\sin(bx+c+\phi) + \sin\phi\cos(bx+c+\phi)]$$
$$= r^2 e^{ax}\sin(bx+c+\phi+\phi)$$

$\therefore$ $$y_2 = r^2 e^{ax}\sin(bx+c+2\phi)$$

Similarly, $$y_3 = r^3e^{ax}\sin(bx+c+3\phi)$$

....

$$y_n = \frac{d^n}{dx^n}e^{ax}\sin(bx+c) = r^n e^{ax}\sin(bx+c+n\phi)$$

In similar way, we obtain

$$y_n = \frac{d^n}{dx^n}e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\phi)$$

Example 1. Find the nth derivative of $\frac{1}{1-5x+6x^2}$

Sol. Let $$y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$$

or $$y = \frac{2}{2x-1} - \frac{3}{3x-1} \qquad \text{(By Partial fraction)}$$

$\therefore$ $$y_n = 2\frac{d^n}{dx^n}(2x-1)^{-1} - 3\frac{d^n}{dx^n}(3x-1)^{-1}$$

$$= 2\left[\frac{(-1)^n \lfloor n \, 2^n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n \lfloor n \, 3^n}{(3x-1)^{n+1}}\right] \quad \left| \text{As } \frac{d^n}{dx^n}(ax+b)^{-1} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}} \right.$$

or $$y_n = (-1)^n \lfloor n \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right].$$

Example 2. Find the nth derivative of $e^{ax} \cos^2 x \sin x$.

Sol. Let $$y = e^{ax}\cos^2 x \sin x = e^{ax} \frac{(1+\cos 2x)}{2} \sin x$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{1}{2\times 2}e^{ax}(2\cos 2x \sin x)$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\{\sin(3x) - \sin x\}$$

or $$y = \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$$

$\therefore$ $$y_n = \frac{1}{4}\left[r^n e^{ax}\sin\,(x+n\phi)\right] + \frac{1}{4}\left[r_1{}^n e^{ax}\sin\,(3x+n\theta)\right].$$

where $r = \sqrt{a^2+1}$; $\tan\phi = 1/a$

and $r_1 = \sqrt{a^2+9}$; $\tan\theta = 3/a$.

Example 3. If $y = \tan^{-1}\dfrac{2x}{1-x^2}$, find y_n. *(U.P.T.U., 2002)*

Sol. We have $$y = \tan^{-1}\frac{2x}{1-x^2}$$

Differentiating y w.r.t. x, we get

$$y_1 = \frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{1-x^2}\right) = \frac{(1-x^2)^2}{\left(1+x^4-2x^2+4x^2\right)}\cdot\frac{2(1-x^2)+4x^2}{(1-x^2)^2}$$

$$y_1 = \frac{2(1+x^2)}{(1+x^2)^2} = \frac{2}{(1+x^2)} = \frac{2}{(x+i)(x-i)}$$

$$y_1 = \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right], \text{ (by Partial fractions)}$$

Differentiating both sides (n–1) times w.r. to 'x', we get

$$y_n = \frac{1}{i}\left[\frac{(-1)^{n-1}(\lfloor n-1)}{(x-i)^n} - \frac{(-1)^{n-1}(\lfloor n-1)}{(x+i)^n}\right]$$

$$= \frac{(-1)^{n-1}(\lfloor n-1)}{i}\left[(x-i)^{-n} - (x+i)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(\lfloor n-1)}{i}\left[r^{-n}(\cos\theta - i\sin\theta)^{-n} - r^{-n}(\cos\theta + i\sin\theta)^{-n}\right]$$

(where $x = r\cos\theta$, $1 = r\sin\theta$)

$$= \frac{(-1)^{n-1}(\lfloor n-1)r^{-n}}{i}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$$

$$y_n = 2(-1)^{n-1} \lfloor n-1 \; r^{-n} \sin n\theta, \text{ where } r = \sqrt{x^2+1}$$

$$\theta = \tan^{-1}\left(\frac{1}{x}\right).$$

Example 4. If $y = x \log \dfrac{x-1}{x+1}$. Show that

$$y_n = (-1)^{n-2} \lfloor n-2 \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right] \qquad \text{(U.P.T.U., 2002)}$$

Sol. We have $y = x \log \dfrac{x-1}{x+1} = x\,[\log(x-1) - \log(x+1)]$

Differentiating w.r. to 'x', we get

$$y_1 = \log(x-1) - \log(x+1) + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \log(x-1) - \log(x+1) + \left(1+\frac{1}{x-1}\right) + \left(-1+\frac{1}{x+1}\right)$$

or $$y_1 = \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1}$$

Differentiating $(n-1)$ times with respect to $\boldsymbol{x}$, we get

$$y_n = \frac{d^{n-1}}{dx^{n-1}}\log(x-1) - \frac{d^{n-1}}{dx^{n-1}}\log(x+1) + \frac{d^{n-1}}{dx^{n-1}}(x-1)^{-1} + \frac{d^{n-1}}{dx^{n-1}}(x+1)^{-1}$$

$$= \frac{d^{n-2}}{dx^{n-2}}\left\{\frac{d}{dx}\log(x-1)\right\} - \frac{d^{n-2}}{dx^{n-2}}\left\{\frac{d}{dx}\log(x+1)\right\} + \frac{(-1)^{n-1}\lfloor n-1}{(x-1)^n} + \frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n}$$

$$= \frac{d^{n-2}}{dx^{n-2}}\left(\frac{1}{x-1}\right) - \frac{d^{n-2}}{dx^{n-2}}\left(\frac{1}{x+1}\right) + \frac{(-1)^{n-1}\lfloor n-1}{(x-1)^n} + \frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n}$$

$$= \frac{(-1)^{n-2}\lfloor n-2}{(x-1)^{n-1}} - \frac{(-1)^{n-2}\lfloor n-2}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)\lfloor n-2}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)\lfloor n-2}{(x+1)^n}$$

$$= (-1)^{n-2}\lfloor n-2\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} - \frac{(n-1)}{(x-1)^n} - \frac{(n-1)}{(x+1)^n}\right]$$

$$= (-1)^{n-2}\lfloor n-2\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Example 5. Find $y_n(0)$ if $y = \dfrac{x^3}{x^2-1}$.

Sol. We have $$y = \frac{x^3}{x^2-1} = \frac{x^3-1+1}{x^2-1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} + \frac{1}{x^2-1}$$

or
$$y = \frac{x^2+x+1}{(x+1)} + \frac{1}{(x-1)(x+1)}$$

$$y = \frac{x^2-1+1}{x+1} + 1 + \frac{1}{(x-1)\,(x+1)}$$

$$y = x + \frac{1}{x+1} + \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

or
$$y = x + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

$\therefore$
$$y_n = 0 + \frac{1}{2}\left[\frac{(-1)^n \underline{|n}}{(x-1)^{n+1}} + \frac{(-1)^n \underline{|n}}{(x+1)^{n+1}}\right]$$

or
$$y_n = \frac{(-1)^n \underline{|n}}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

At $x = 0,\ y_n(0) = \dfrac{(-1)^n \underline{|n}}{2}\left[\dfrac{1}{(-1)^{n+1}} + \dfrac{1}{(1)^{n+1}}\right]$

When n is odd, $y_n(0) = \dfrac{(-1)^n \underline{|n}}{2}[1+1] = -\underline{|n}$

When n is even, $y_n(0) = \dfrac{(-1)^n \underline{|n}}{2}[-1+1] = 0.$

EXERCISE 1.1

1. If $y = \dfrac{x^2}{(x-1)^2(x+2)}$, find nth derivative of y. (*U.P.T.U., 2002*)

$$\left[\textbf{Ans. } y_n = \frac{(-1)^n \underline{|n+1}}{3(x-1)^{n+2}} + \frac{5(-1)^n \underline{|n}}{9(x-1)^{n+1}} + \frac{4(-1)^n \underline{|n}}{9(x+2)^{n+1}}\right]$$

2. Find the nth derivative of $\dfrac{x^2}{(x-a)(x-b)}$.
$$\left[\textbf{Ans. } \frac{(-1)^n \underline{|n}}{(a-b)}\left[\frac{a^2}{(x-a)^{n+1}} - \frac{b^2}{(x-b)^{n+1}}\right]\right]$$

3. Find the nth derivative of $\tan^{-1}\left[\dfrac{1+x}{1-x}\right]$.

[**Ans.** $(-1)^{n-1}\underline{|n-1}\sin^n\theta\sin n\theta$ where $\theta = \cot^{-1}x$]

4. If $y = \sin^3 x$, find y_n.
$$\left[\textbf{Ans. } \frac{3}{4}\sin\left(x+n\frac{\pi}{2}\right) - \frac{1}{4}.3^n.\sin\left(3x+n\frac{\pi}{2}\right)\right]$$

5. Find nth derivative of $\tan^{-1}\left(\dfrac{x}{a}\right)$.
$$\left[\textbf{Ans. } (-1)^{n-1}\underline{|n-1}\,a^{-n}\sin^n\theta\sin n\theta\right]$$

6. Find y_n, where $y = e^x.x$. $\left[\textbf{Ans. } e^x(x+n)\right]$

7. Find y_n, when $y = \dfrac{1-x}{1+x}$. $\left[\textbf{Ans. } \dfrac{2(-1)^n \lfloor n}{(x+1)^{n+1}}\right]$

8. Find nth derivative of $\log x^2$. $\left[\textbf{Ans. } (-1)^{n-1} \lfloor n-1 \cdot 2x^{-n}\right]$

9. Find y_n, $y = e^x \sin^2 x$. $\left[\textbf{Ans. } \dfrac{e^x}{2}\left[1-5^{n/2}\cos(2x+n\tan^{-1}2)\right]\right]$

10. If $y = \cos x \cdot \cos 2x \cdot \cos 3x$ find y_n.

[**Hint:** $\cos x \cdot \cos 2x \cdot \cos 3x = \dfrac{1}{4}[(\cos 6x+\cos 4x+\cos 2x+1)]$

$$\left[\textbf{Ans. } \frac{1}{4}\left[6^n\cos\left(6x+n\frac{\pi}{2}\right)+4^n\cos\left(4x+n\frac{\pi}{2}\right)+2^n\cos\left(2x+n\frac{\pi}{2}\right)\right]\right]$$

1.2 LEIBNITZ'S* THEOREM

Statement. If u and v be any two functions of x, then

$$D^n(u.v) = {}^nc_0\, D^n(u).v + {}^nc_1 D^{n-1}(u).\, D(v) + {}^nc_2\, D^{n-2}(u).D^2(v) + \ldots$$
$$+ {}^nc_r\, D^{n-r}(u).D^r(v) + \ldots + {}^nc_n\, u.\, D^n v \quad \ldots(i)$$

(*U.P.T.U., 2007*)

Proof. This theorem will be proved by Mathematical induction.

Now, $D(u.v) = D(u).v + u.D(v) = {}^1c_0\, D(u).v + {}^1c_1\, u.D(v)$ ...(*ii*)

This shows that the theorem is true for $n = 1$.

Next, let us suppose that the theorem is true for, $n = m$ from (*i*), we have

$$D^m(u.v) = {}^mc_0\, D^m(u).v + {}^mc_1\, D^{m-1}(u)\, D(v) + {}^mc_2\, D^{m-2}(u)\, D^2(v) + \ldots + {}^mc_r\, D^{m-r}(u)\, D^r(v) + \ldots + {}^mc_m\, u\, D^m(v)$$

Differentiating w.r. to x, we have

$$D^{m+1}(uv) = {}^mc_0\{D^{m+1}(u)\cdot v + D^m(u)\cdot D(v)\} + {}^mc_1\{D^m(u)D(v) + D^{m-1}(u)D^2(v)\}$$
$$+ {}^mc_2\{D^{m-1}(u)D^2(v) + D^{m-2}(u).D^3(v)\} + \ldots + {}^mc_r\{D^{m-r+1}(u)D^r v + D^{m-r}(u)D^{r+1}(v)\}$$
$$+ \ldots + {}^mc_m\{D(u)\cdot D^m(v) + uD^{m+1}(v)\}$$

But from Algebra we know that ${}^mc_r + {}^mc_{r+1} = {}^{m+1}c_{r+1}$ and ${}^mc_0 = {}^{m+1}c_0 = 1$

$$\therefore\ D^{m+1}(uv) = {}^{m+1}c_0 D^{m+1}(u)\cdot v + ({}^mc_0 + {}^mc_1)D^m(u)\cdot D(v) + ({}^mc_1 + {}^mc_2)D^{m-1}u\cdot D^2 v$$
$$+ \ldots + ({}^mc_r + {}^mc_{r+1})D^{m-r}(u)\cdot D^{r+1}(v) + \ldots + {}^{m+1}c_{m+1}u\cdot D^{m+1}(v)$$

$$\left(\text{As } {}^mc_m = {}^{m+1}c_{m+1} = 1\right)$$

---

* **Gottfried William Leibnitz (1646–1716)** was born Leipzig (Germany). He was Newton's rival in the invention of calculus. He spent his life in diplomatic service. He exhibited his calculating machine in 1673 to the Royal society. He was linguist and won fame as Sanskrit scholar. The theory of determinants is said to have originated with him in 1683. The generalization of Binomial theorem into multinomial theorem is also due to him. His works mostly appeared in the journal 'Acta eruditorum' of which he was editor-in-chief.

$$\Rightarrow D^{m+1}(uv) = {}^{m+1}c_0 D^{m+1}(u)\cdot v + {}^{m+1}c_1 D^m(u)\cdot D(v) + {}^{m+1}c_2 D^{m-1}(u)\cdot D^2(v) + \ldots$$
$$+ {}^{m+1}c_{r+1} D^{m-r}(u)\cdot D^{r+1}(v) + \ldots + {}^{m+1}c_{m+1} u\cdot D^{m+1}(v) \quad \ldots(iii)$$

Therefore, the equation (*iii*) shows that the theorem is true for $n = m + 1$ also. But from (2) that the theorem is true for $n = 1$, therefore, the theorem is true for $(n = 1 + 1)$ *i.e.*, $n = 2$, and so for $n = 2 + 1 = 3$, and so on. Hence, the theorem is true for all positive integral value of n.

Example 1. If $y^{1/m} + y^{-1/m} = 2x$, prove that
$(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$ (*U.P.T.U., 2007*)

Sol. Given $$y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$$\Rightarrow \quad y^{2/m} - 2xy^{1/m} + 1 = 0$$

or $$(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$$

$$\Rightarrow \quad z^2 - 2xz + 1 \quad (y^{1/m} = z)$$

$$\therefore \quad z = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow \quad y^{1/m} = x \pm \sqrt{x^2-1} \Rightarrow y = \left[x \pm \sqrt{x^2-1}\right]^m \quad \ldots(i)$$

Differentiating equation (*i*) w.r.t. x, we get

$$y_1 = m\left[x \pm \sqrt{x^2-1}\right]^{m-1}\left[1 \pm \frac{2x}{2\sqrt{x^2-1}}\right] = \frac{m\left[x \pm \sqrt{x^2-1}\right]^m}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad y_1 = \frac{my}{\sqrt{x^2-1}} \Rightarrow y_1\sqrt{x^2-1} = my$$

or $$y_1^2\,(x^2 - 1) = m^2y^2 \quad \ldots(ii)$$

Differentiating both sides equation (*ii*) w.r.t. x, we obtain

$$2y_1y_2(x^2 - 1) + 2xy_1^2 = 2m^2\, yy_1$$

$$\Rightarrow \quad y_2\,(x^2 - 1) + xy_1 - m^2y = 0$$

Differentiating n times by Leibnitz's theorem w.r.t. x, we get

$$D^n\,(y_2)\cdot(x^2 - 1) + {}^nc_1\, D^{n-1}y_2\cdot D^2(x^2 - 1) + {}^nc_2\, D^{n-2}\, y_2\, D^2\,(x^2 - 1) + D^n\,(y_1)x + {}^nc_1\, D^{n-1}\,(y_1)\, Dx - m^2y_n = 0$$

$$\Rightarrow \quad y_{n+2}\,(x^2 - 1) + ny_{n+1}\cdot 2x + \frac{n(n-1)}{\lfloor 2}\, y_n\cdot 2 + y_{n+1}\cdot x + ny_n - m^2y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - n + n - m^2)y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1}\, (n^2 - m^2)\, y_n = 0.$$ **Hence proved.**

Example 2. Find the nth derivative of $e^x \log x$.

Sol. Let $u = e^x$ and $v = \log x$

Then $D^n\,(u) = e^x$ and $D^n\,(v) = \dfrac{(-1)^{n-1}\lfloor n-1}{x^n}$

$$D^n(ax+b)^{-1} = \frac{(-1)^n \lfloor n}{(ax+b)^{n+1}}$$

By Leibnitz's theorem, we have

$$D^n (e^x \log x) = D^n e^x \log x + {}^nc_1\, D^{n-1} (e^x)\, D(\log x) + {}^nc_2\, D^{n-2} (e^x) D^2 (\log x) + \ldots + e^x\, D^n (\log x)$$

$$= e^x \cdot \log x + ne^x \cdot \frac{1}{x} + \frac{n(n-1)}{\lfloor 2} e^x \left(-\frac{1}{x^2}\right) + \ldots + e^x\, \frac{(-1)^{n-1} \lfloor n-1}{x^n}$$

$$\Rightarrow \quad D^n (e^x \log x) = e^x \left[\log x + \frac{n}{x} - \frac{n(n-1)}{2x^2} + \ldots + \frac{(-1)^{n-1} \lfloor n-1}{x^n}\right].$$

Example 3. Find the nth derivative of $x^2 \sin 3x$.

Sol. Let $u = \sin 3x$ and $v = x^2$

$$\therefore \quad D^n(u) = D^n (\sin 3x) = 3^n \sin \left(3x + \frac{n\pi}{2}\right)$$

$$D(u) = 2x,\ D^2 (v) = 2,\ D^3 (v) = 0$$

By Leibnitz's theorem, we have

$$D^n (x^2 \sin 3x) = D^n (\sin 3x) x^2 + {}^nc_1\, D^{n-1} (\sin 3x) \cdot D(x^2) + {}^nc_2\, D^{n-2} (\sin 3x) \cdot D^2(x^2)$$

$$= 3^n \sin\left(3x + \frac{n\pi}{2}\right) \cdot x^2 + n3^{n-1} \sin\left(3x + \frac{\overline{n-1}\pi}{2}\right) \cdot 2x + \frac{n(n-1)}{\lfloor 2} \cdot 3^{n-2} \sin\left(3x + \frac{\overline{n-2}\pi}{2}\right) \cdot 2$$

$$= 3^n x^2 \sin\left(3x + \frac{n\pi}{2}\right) + 2nx \cdot 3^{n-1} \sin\left(3x + \frac{\overline{n-1}\pi}{2}\right) + 3^{n-2} n(n-1) \cdot \sin\left(3x + \frac{\overline{n-2}\pi}{2}\right).$$

Example 4. If $y = x \log (1 + x)$, prove that

$$y_n = \frac{(-1)^{n-2} \lfloor n-2\,(x+n)}{(x+1)^n}. \qquad (U.P.T.U., 2006)$$

Sol. Let $u = \log (1 + x)$, $v = x$

$$D^n (u) = \frac{d^n}{dx^n} \log (1+x) = \frac{d^{n-1}}{dx^{n-1}} \left(\frac{d}{dx} \log (1+x)\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}} \cdot \frac{1}{x+1} = \frac{d^{n-1}}{dx^{n-1}} (x+1)^{-1}$$

$$\Rightarrow \quad D^n (u) = \frac{(-1)^{n-1} \lfloor n-1}{(x+1)^n} \quad \text{and } D(v) = 1,\ D^2(v) = 0$$

By Leibnitz's theorem, we have

$$y_n = D^n (x \log (1 + x) = D^n (\log (1 + x))\, x + {}^nc_1\, D^{n-1} (\log (1 + x))\, Dx$$

$$= x \frac{(-1)^{n-1} \lfloor n-1}{(x+1)^n} + \frac{n(-1)^{n-2} \lfloor n-2}{(x+1)^{n-1}}$$

$$\Rightarrow \quad y_n = (-1)^{n-2} \lfloor n-2 \left[\frac{-x(n-1)}{(x+1)^n} + \frac{n(x+1)}{(x+1)^n} \right]$$

$$= (-1)^{n-2} \lfloor n-2 \left[\frac{-xn+x+xn+n}{(x+1)^n} \right]$$

$$= (-1)^{n-2} \lfloor n-2 \left[\frac{x+n}{(x+1)^n} \right].$$ **Hence proved.**

Example 5. If $y = a \cos(\log x) + b \sin(\log x)$. Show that

$$x^2y_2 + xy_1 + y = 0$$

and $$x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$$ (*U.P.T.U., 2003*)

Sol. Given $y = a \cos(\log x) + b \sin(\log x)$

$$\therefore \quad y_1 = -a \sin(\log x) \left(\frac{1}{x}\right) + b \cos(\log x) \left(\frac{1}{x}\right)$$

or $$xy_1 = -a \sin(\log x) + b \cos(\log x)$$

Again differentiating w.r.t. x, we get

$$xy_2 + y_1 = -a \cos(\log x) \left(\frac{1}{x}\right) - b \sin(\log x) \left(\frac{1}{x}\right)$$

$$\Rightarrow \quad x^2y_2 + xy_1 = -\{a \cos(\log x) + b \sin(\log x)\} = -y$$

$$\Rightarrow \quad x^2y_2 + xy_1 + y = 0.$$ **Hence proved.** ...(*i*)

Differentiating (*i*) n times, by Leibnitz's theorem, we have

$$y_{n+2} \cdot x^2 + n\, y_{n+1} \cdot 2x + \frac{n(n-1)}{\lfloor 2}\, y_n \cdot 2 + y_{n+1} \cdot x + ny_n + y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - n + n + 1)\, y_n = 0$$

$$\Rightarrow \quad x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$$ **Hence proved.**

Example 6. If $y = (1-x)^{-\alpha} e^{-\alpha x}$, show that

$$(1-x)y_{n+1} - (n + \alpha x)\, y_n - n\alpha y_{n-1} = 0.$$

Sol. Given $y = (1-x)^{-\alpha} . e^{-\alpha x}$

Differentiating w.r.t. x, we get

$$y_1 = \alpha (1-x)^{-\alpha-1} e^{-\alpha x} - (1-x)^{-\alpha} e^{-\alpha x} \cdot \alpha$$

$$y_1 = (1-x)^{-\alpha} e^{-\alpha x} \cdot \alpha \left[\frac{1}{1-x} - 1\right] = y\alpha \left[\frac{x}{1-x}\right]$$

$$= \quad y_1 (1-x) = \alpha xy$$

Differentiating n times w.r.t. x, by Leibnitz's theorem, we get

$$y_{n+1}(1-x) - ny_n = \alpha y_n \cdot x + n\alpha y_{n-1}$$

$$\Rightarrow \quad (1-x)y_{n+1} - (n + \alpha x)\, y_n - n\alpha y_{n-1} = 0.$$ **Hence proved.**

Example 7. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)^m$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+m^2)\, y_n = 0$.

Sol. We have $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)^m = m \log \frac{x}{m}$

$$\Rightarrow \qquad y = b\cos\left(m\log\frac{x}{m}\right)$$

On differentiating, we have

$$y_1 = -b\sin\left(m\log\frac{x}{m}\right)\cdot\frac{m^2}{x}\cdot\frac{1}{m}$$

$$\Rightarrow \qquad xy_1 = -mb\sin\left(m\log\frac{x}{m}\right)$$

Again differentiating w.r.t. x, we get

$$xy_2 + y_1 = -mb\cos\left(m\log\frac{x}{m}\right)m\cdot\frac{1}{\frac{x}{m}}\cdot\frac{1}{m}$$

$$x(xy_2 + y_1) = -m^2b\cos\left(m\log\frac{x}{m}\right) = -m^2y$$

or $\qquad x^2y_2 + xy_1 + m^2y = 0$

Differentiating n times with respect to x, by Leibnitz's theorem, we get

$$y_{n+2}\cdot x^2 + ny_{n+1}\cdot 2x + \frac{n(n-1)}{\lfloor 2}\cdot 2y_n + xy_{n+1} + ny_n + m^2y_n = 0$$

$\Rightarrow x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - n + n + m^2)y_n = 0$

$\Rightarrow \quad x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 + m^2)\,y_n = 0.$ **Hence proved.**

Example 8. If $y = (x^2 - 1)^n$, prove that (*U.P.T.U., 2000, 2002*)

$$(x^2 - 1)y_{n+2} + 2xy_{n+1} - n\,(n+1)y_n = 0$$

Hence, if $P_n = \dfrac{d^n}{dx^n}\,(x^2-1)^n$, show that $\dfrac{d}{dx}\left\{(1-x^2)\dfrac{dP_n}{dx}\right\} + n(n+1)\,P_n = 0.$

Sol. Given $\qquad y = (x^2 - 1)^n$

Differentiating w.r. to x, we get

$$y_1 = n(x^2-1)^{n-1}.\,2x = \frac{2nx\,(x^2-1)^n}{(x^2-1)}$$

$$\Rightarrow \qquad (x^2-1)\,y_1 = 2nxy$$

Again differentiating, w.r.t. x, we obtain

$$(x^2-1)\,y_2 + 2xy_1 = 2nxy_1 + 2ny$$

Now, differentiating n times, w.r.t. x by Leibnitz's theorem

$$(x^2-1)y_{n+2} + 2nxy_{n+1} + \frac{2n(n-1)}{\lfloor 2}y_n + 2xy_{n+1} + 2ny_n = 2nxy_{n+1} + 2n^2y_n + 2ny_n$$

or $\qquad (x^2-1)y_{n+2} + 2xy_{n+1}\,(n+1-n) + (n^2 - n + 2n - 2n^2 - 2n)y_n = 0$

or $\qquad (x^2-1)\,y_{n+2} + 2xy_{n+1} - (n^2 + n)y_n = 0$

$\Rightarrow \quad (x^2-1)\,y_{n+2} + 2xy_{n+1} - n(n+1)\,y_n = 0.$ **Hence proved.** ...(*i*)

Second part: Let $y = (x^2 - 1)^n$

$$\therefore \quad P_n = \frac{d^n}{dx^n} y = y_n$$

Now

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx} y_n\right\} = \frac{d}{dx}\left\{(1-x^2) y_{n+1}\right\}$$

$$= (1 - x^2)y_{n+2} - 2xy_{n+1} = - \left[(x^2-1)y_{n+2} + 2xy_{n+1}\right]$$

$$\Rightarrow \quad \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx} P_n\right\} = - \left[n(n+1)y_n\right] \qquad \text{[Using equation } (i)]$$

or $$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n\,(n+1)y_n = 0.$$ **Hence proved.**

Example 9. Find the nth derivative of $y = x^{n-1} \log x$ at $x = \frac{1}{2}$.

Sol. Differentiating

$$y_1 = (n-1)\, x^{n-1-1} \log x + x^{n-1} \frac{1}{x}$$

or $$y_1 = \frac{(n-1)x^{n-1} \cdot \log x}{x} + \frac{x^{n-1}}{x} \Rightarrow xy_1 = (n-1)y + x^{n-1}$$

Differentiating (n–1) times by Leibnitz's theorem, we get

$$xy_n + {}^{n-1}c_1\, y_{n-1} = (n-1)y_{n-1} + \lfloor n-1 \qquad \left|\ \frac{d^{n-1}}{dx^{n-1}} x^{n-1} = (n-1)(n-2)\ldots 2.1 = \lfloor n-1\right.$$

$$\Rightarrow \quad xy_n + (n-1)y_{n-1} = (n-1)y_{n-1} + \lfloor n-1$$

$$\Rightarrow \quad xy_n = \lfloor n-1 \quad i.e.\ y_n = \frac{\lfloor n-1}{x}$$

At $$x = \frac{1}{2}$$

$$y_n\left(\frac{1}{2}\right) = 2\lfloor n-1.$$

Example 10. If $y = (1 - x^2)^{-1/2} \sin^{-1}x$, when $-1 < x < 1$ and $-\frac{\pi}{2} < \sin^{-1}x < \frac{\pi}{2}$, then show that $(1 - x^2)y_{n+1} - (2n + 1)\, xy_n - n^2\, y_{n-1} = 0$.

Sol. Given $y = (1 - x^2)^{-1/2} \sin^{-1}x$

Differentiating

$$y_1 = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)\sin^{-1}x + (1-x^2)^{-1/2}.\frac{1}{\sqrt{1-x^2}}$$

$$y_1 = \frac{x(1-x^2)^{-\frac{1}{2}}\sin^{-1}x}{(1-x^2)} + \frac{1}{(1-x^2)} = \frac{xy+1}{(1-x^2)}$$

$$\Rightarrow \quad y_1(1-x^2) = xy + 1$$

Differentiating n times w.r.t. x, by Leibnitz's theorem, we get

$$y_{n+1}(1-x^2) + ny_n(-2x) + \frac{n(n-1)}{\underline{|2}}\, y_{n-1}\cdot(-2) = xy_n + ny_{n-1}$$

$$(1-x^2)y_{n+1} - (2n+1)xy_n - (n^2 - n + n)y_{n-1} = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+1} - (2n+1)xy_n - n^2 y_{n-1} = 0. \quad \textbf{Hence proved.}$$

Example 11. If $y = x^n \log x$, then prove that

(*i*) $y_{n+1} = \dfrac{\underline{|n}}{x}$ (*ii*) $y_n = ny_{n-1} + \underline{|(n-1)}$.

Sol. (*i*) We have $y = x^n \log x$

Differentiating w.r. to x, we get

$$y_1 = nx^{n-1}\cdot \log x + \frac{x^n}{x}$$

$$\Rightarrow \quad xy_1 = nx^n . \log x + x^n$$

$$xy_1 = ny + x^n \quad \ldots(i)$$

Differentiating equation (*i*) n times, we get

$$xy_{n+1} + ny_n = ny_n + \underline{|n}$$

$$\Rightarrow \quad y_{n+1} = \frac{\underline{|n}}{x} \quad \textbf{Proved.}$$

(*ii*)

$$y_n = \frac{d^n}{dx^n}\left(x^n.\log x\right) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}x^n.\log x\right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(\frac{x^n}{x} + nx^{n-1}.\log x\right)$$

$$= n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}.\log x\right) + \frac{d^{n-1}}{dx^{n-1}}\,.\,x^{n-1}$$

$$= ny_{n-1} + \underline{|(n-1)}. \quad \textbf{Proved.}$$

As $y_n = \dfrac{d^n}{dx^n}\left(x^n \log x\right)$

$\therefore \; y_{n-1} = \dfrac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}\log x\right)$

EXERCISE 1.2

Find the *n*th derivative of the following:

1. $e^x \log x$. $\left[\textbf{Ans. } e^x\left[\log x + {}^nc_1\cdot\frac{1}{x} - {}^nc_2\cdot\frac{1}{x^2} + \underline{|2}\,{}^nc_3\cdot\frac{1}{x^3} + \ldots + (-1)^{n-1}\underline{|n-1}\,{}^nc_n x^{-n}\right]\right]$

2. $x^2 e^x$. $\left[\textbf{Ans. } e^x\left[x^2 + 2nx + n(n-1)\right]\right]$

3. $x^3 \log x$. $\left[\textbf{Ans. } \dfrac{6(-1)^n \lfloor n-4}{x^{n-3}}\right]$

4. $\dfrac{1-x}{1+x}$. $\left[\textbf{Ans. } \dfrac{2\,(-1)^n \lfloor n}{(1+x)^{n+1}}\right]$

5. $x^2 \sin 3x$.

$$\left[\textbf{Ans. } 3^n x^2 \sin\left(3x+\frac{n\pi}{2}\right)+2nx\cdot 3^{n-1}\left[\sin\{3x+\frac{1}{2}(n-1)\pi\}\right]+3^{n-2}n(n-1)\sin\left\{3x+\frac{\pi}{2}(n-2)\right\}\right]$$

6. $e^x(2x+3)^3$. $\left[\textbf{Ans. } e^x\left\{(2x+3)^2+6n(2x+3)^2+12(n-1)(2x+3)+8n(n-1)(n-2)\right\}\right]$

7. If $x = \tan y$, prove that *(U.P.T.U., 2006)*

$(1+x^2)\, y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0.$

8. If $y = e^x \sin x$, prove that $y''-2y' + 2y = 0$.

9. If $y = \sin(m \sin^{-1}x)$, prove that

$(1-x^2)y_{n+2} - (2n+1)x\cdot y_{n+1} + (m^2-n^2)y_n = 0.$ *(U.P.T.U., 2004, 2002)*

10. If $x = \cos h\left[\left(\frac{1}{m}\right)\log y\right]$, prove that $(x^2-1)y_2 + xy_1 - m^2y = 0$ and $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

11. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$.

12. If $y = e^{\tan^{-1}x}$, prove that $(1+x^2)y_{n+2} + \{2(n+1)\,x-1\}y_{n+1} + n(n+1)y_n = 0$.

13. If $\sin^{-1}y = 2\log(x+1)$, show that

$(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0.$

14. If $y = C_1\left(x+\sqrt{x^2-1}\right)^n + C_2\left(x-\sqrt{x^2-1}\right)^n$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} = 0$.

15. If $x = \cos[\log(y^{1/a})]$, then show that $(1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+a^2)\,y_n = 0$.

1.2.1 To Find $(y_n)_0$ *i.e.*, *n*th Differential Coefficient of *y*, When $x = 0$

Sometimes we may not be able to find out the *n*th derivative of a given function in a compact form for general value of x but we can find the *n*th derivative for some special value of x generally $x = 0$. The method of procedure will be clear from the following examples:

Example 1. Determine $y_n(0)$ where $y = e^{m.\cos^{-1}x}$.

Sol. We have $y = e^{m.\cos^{-1}x}$ ...(*i*)

Differentiating w.r.t. x, we get

$$y_1 = e^{m\cos^{-1}x}\, m\left(\frac{-1}{\sqrt{1-x^2}}\right) \Rightarrow \sqrt{1-x^2}\,.\,y_1 = -me^{m\cos^{-1}x}$$

or $$\sqrt{1-x^2}\,y_1 = -my \Rightarrow (1-x^2)y_1^2 = m^2y^2$$

Differentiating again

$$(1-x^2)\,2y_1\,y_2 - 2xy_1^2 = 2m^2yy_1$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = m^2y \qquad ...(ii)$$

Using Leibnitz's rule differentiating n times w.r.t. x

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\lfloor 2}y_n - xy_{n+1} - ny_n = m^2y_n$$

or $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Putting $x = 0$

$$y_{n+2}(0) - (n^2+m^2)y_n(0) = 0$$

$$\Rightarrow \quad y_{n+2}(0) = (n^2+m^2)y_n(0) \qquad ...(iii)$$

replace n by $(n-2)$

$$y_n(0) = \{(n-2)^2+m^2\}\,y_{n-2}(0)$$

replace n by $(n-4)$ in equation (iii), we get

$$y_{n-2}(0) = \{(n-4)^2+m^2\}\,y_{n-4}(0)$$

$$\therefore \quad y_n(0) = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\,y_{n-4}(0)$$

Case I. When n is odd:

$$y_n(0) = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ ....\ (1^2+m^2)y_1(0) \qquad ...(iv)$$

[The last term obtain putting $n = 1$ in eqn. (iii)]

Now we have $$y_1 = -e^{m\cos^{-1}x}\, m\cdot\frac{1}{\sqrt{1-x^2}}$$

At $x = 0$, $y_1(0) = -me^{\frac{m\pi}{2}}$ ...(v) $\quad$ As $\cos^{-1}0 = \frac{\pi}{2}$

Using (v) in (iv), we get

$$y_n(0) = -\left\{(n-2)^2+m^2\right\}\left\{(n-4)^2+m^2\right\}....\left(1^2+m^2\right)me^{\frac{m\pi}{2}}.$$

Case II. When n is even:

$$y_n(0) = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ ....\ (2^2+m^2)y_2(0) \qquad ...(vi)$$

[The last term obtain by putting $n = 2$ in (iii)]

From (ii), $$y_2(0) = m^2(y)_0$$

$$\therefore \quad y_2(0) = m^2e^{m\pi/2} \qquad ...(vii)$$

As $y = e^{m\cos^{-1}x}$

$\therefore\ y(0) = e^{m\cos^{-1}0} = e^{m\pi/2}$

From eqns. (vi) and (vii), we get

$$y_n(0) = \{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ ....\ (2^2+m^2)\,m^2e^{m\pi/2}.$$

Example 2. If $y = (\sin^{-1}x)^2$. Prove that $y_n(0) = 0$ for n odd and $y_n(0) = 2.2^2.4^2....(n-2)^2$, $n \neq 2$ for n even. *(U.P.T.U., 2005, 2008)*

Sol. We have $$y = (\sin^{-1}x)^2 \qquad ...(i)$$

On differentiating $$y_1 = 2\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}} \Rightarrow y_1\sqrt{1-x^2} = 2\sqrt{y},\ \left(\text{As } \sqrt{y} = \sin^{-1}x\right)$$

Squaring on both sides,

$$y_1^2(1-x^2) = 4y$$

Again differentiating

$$2(1-x^2)\, y_1 y_2 - 2xy_1^2 = 4y_1$$

or $$(1-x^2)\, y_2 - xy_1 = 2 \quad ...(ii)$$

Differentiating n times by Leibnitz's theorem

$$(1-x^2)\, y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\lfloor 2}\, y_n - xy_{n+1} - ny_n = 0$$

or $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$

Putting $x = 0$ in above equation

$$y_{n+2}(0) - n^2 y_n(0) = 0$$

$\Rightarrow$ $$y_{n+2}(0) = n^2 y_n(0) \quad ...(iii)$$

replace n by $(n-2)$

$$y_n(0) = (n-2)^2 y_{n-2}(0)$$

Again replace n by $(n-4)$ in (iii) and putting the value of $y_{n-2}(0)$ in above equation

$$y_n(0) = (n-2)^2 (n-4)^2 y_{n-4}(0)$$

Case I. If n is odd, then

$$y_n(0) = (n-2)^2 (n-4)^2 (n-6)^2 \ldots 1^2 \cdot y_1(0)$$

But $$y_1(0) = 2\sin^{-1}0 \cdot \frac{1}{\sqrt{1-0}} = 0$$

$\therefore$ $y_n(0) = 0.$ **Hence proved.**

Case II. If n is even, then

$$y_n(0) = (n-2)^2 (n-4)^2 \ldots 2^2 \cdot y_2(0) \quad ...(iv)$$

From (ii) $$y_2(0) = 2$$

Using this value in eqn. (iv), we get

$$y_n(0) = (n-2)^2 (n-4)^2 \ldots 2^2 \cdot 2$$

or $y_n(0) = 2.2^2.4^2 \ldots (n-2)^2$, $n \neq 2$ otherwise 0. **Proved.**

Example 3. If $y = \left[x+\sqrt{1+x^2}\right]^m$, find y_n 0).

Sol. Given $$y = \left[x+\sqrt{1+x^2}\right]^m \quad ...(i)$$

$\therefore$ $$y_1 = m\left[x+\sqrt{1+x^2}\right]^{m-1}\left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{m\left[x+\sqrt{1+x^2}\right]^m}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}}$$

or $$y_1\sqrt{1+x^2} = my$$

Squaring $$y_1^2(1+x^2) = m^2y^2 \quad ...(ii)$$

Again differentiating, $y_1^2(2x) + (1+x^2)2y_1y_2 = m^2 \cdot 2yy_1$

or $$y_2(1+x^2) + xy_1 - m^2y = 0 \quad ...(iii)$$

Differentiating n times by Leibnitz's theorem

$$(1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{2n(n-1)}{\lfloor 2}\, y_n + xy_{n+1} + ny_n - m^2y_n = 0$$

or $\quad (1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Putting $x = 0$, we get

$$y_{n+2}(0) + (n^2 - m^2)\, y_n(0) = 0$$

$$\Rightarrow \quad y_{n+2}(0) = -(n^2 - m^2)\, y_n(0) \qquad ...(iv)$$

replace n by $n - 2$

$$y_n(0) = -\{(n-2)^2 - m^2\}\, y_{n-2}(0)$$

Again replace n by $(n - 4)$ in (iv) and putting $y_{n-2}(0)$ in above equation

$$y_n(0) = (-1)^2 \{(n-2)^2 - m^2\}\{(n-4\}^2 - m^2\}\, y_{n-4}(0)$$

Case I. If n is odd

$$y_n(0) = -\{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\} \ldots \{1^2 - m^2\}\, y_1(0)$$

But $\quad y_1(0) = my(0)$

or $\quad y_1(0) = m \quad (\text{As } y(0) = 1)$

$$\Rightarrow \quad y_n(0) = \{m^2 - (n-2)^2\}\{m^2 - (n-4)^2\} \ldots (m^2 - 1^2)\cdot m.$$

Case II. If n is even

$$y_n(0) = \{m^2 - (n-2)^2\}\{m^2 - (n-4)^2\} \ldots (m^2 - 2^2)\, y_2(0)$$

$$\Rightarrow \quad y_n(0) = \{m^2 - (n-2)^2\}\{m^2 - (n-4)^2\} \ldots (m^2 - 2^2)\cdot m^2.$$

(As $y_2(0) = m^2$).

Example 4. Find the nth differential coefficient of the function on $\cos(2\cos^{-1} x)$ at the point $x = 0$.

Sol. Let $\quad y = \cos(2\cos^{-1} x) \qquad ...(i)$

On diffentiating, $\quad y_1 = -\sin(2\cos^{-1} x)\left[\dfrac{-2}{\sqrt{1-x^2}}\right]$

or $\quad y_1\sqrt{1-x^2} = 2\sin(2\cos^{-1} x)$

Squaring on both sides, we get

$$y_1^2(1 - x^2) = 4\sin^2\left(2\cos^{-1} x\right)$$

$$= 4\left\{1 - \cos^2\left(2\cos^{-1} x\right)\right\}$$

or $\quad y_1^2(1 - x^2) = 4(1 - y^2)$

Again differentiating w.r.t. x, we get

$$2y_1y_2(1 - x^2) - 2xy_1^2 = -8yy_1$$

or $\quad y_2(1 - x^2) - xy_1 + 4y = 0 \qquad ...(ii)$

Differentiating n times by Liebnitz's theorem

$$(1 - x^2)\, y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\underline{|2}}\, y_n - xy_{n+1} - ny_n + 4y_n = 0$$

$$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 - 4)\, y_n = 0$$

Putting $x = 0$ in above equation, we get

$$y_{n+2}(0) - (n^2 - 4)\, y_n(0) = 0$$

or $\quad y_{n+2}(0) = (n^2 - 4)\, y_n(0) \qquad ...(iii)$

Replace n by $n - 2$, we get

$$y_n(0) = \{(n-2)^2 - 4\}\, y_{n-2}(0)$$

Again replace n by $(n - 4)$ in (*iii*) and putting $y_{n-2}(0)$ in above then, we get

$$y_n(0) = \{(n-2)^2 - 4\}\{(n-4)^2 - 4\}\, y_{n-4}(0)$$

Case I. If n is odd

$$y_n(0) = \{(n-2)^2 - 4\}\{(n-4)^2 - 4\} \,\ldots\ldots\ldots\, (1^2 - 4)\, y_1(0)$$

But $y_1(0) = 2\sin(2\cos^{-1} 0) = 2\sin(\pi) = 0$

$\therefore$ $y_n(0) = 0.$

Case II. If n is even

$$y_n(0) = \{(n-2)^2 - 4\}\{(n-4)^2 - 4\} \,\ldots\ldots\ldots\, \{2^2 - 4\}\, y_2(0)$$

$$y_n(0) = 0$$

Hence for all values of n, even or odd,

$$y_n(0) = 0.$$

Example 5. Find the nth derivative of $y = x^2 \sin x$ at $x = 0$. (*U.P.T.U., 2008*)

Sol. We have $y = x^2 \sin x = \sin x.\, x^2$ ...(*i*)

Differentiate n times by Leibnitz's theorem, we get

$$y_n = n_{C_0} D^n(\sin x).\, x^2 + n_{C_1} D^{n-1}(\sin x)\, D(x^2) + n_{C_2} D^{n-2}(\sin x)\, D^2(x^2) + 0$$

$$= x^2 .\, \sin\left(x + \frac{n\pi}{2}\right) + 2nx .\, \sin\left(x + \frac{n-1}{2}\pi\right) + n(n-1)\sin\left(x + \frac{n-2}{2}\pi\right)$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left(x + \frac{n\pi}{2} - \frac{\pi}{2}\right) + n(n-1)\sin\left(x + \frac{n\pi}{2} - \pi\right)$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) - 2nx \cos\left(x + \frac{n\pi}{2}\right) - n(n-1)\sin\left(x + \frac{n\pi}{2}\right)$$

$$y_n = (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right)$$

Putting $x = 0$, we obtain

$$y_n(0) = (n - n^2)\sin\frac{n\pi}{2}.$$

Example 6. If $y = \sin(a \sin^{-1} x)$, prove that

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - a^2)y_n = 0$$

Also find nth derivative of y at $x = 0$. [*U.P.T.U. (C.O.), 2007*]

Sol. We have $y = \sin(a \sin^{-1} x)$

Differentiating w.r.t. x, we get

$$y_1 = \cos(a \sin^{-1} x).\, \frac{a}{\sqrt{1-x^2}} \quad \text{...(i)}$$

or $y_1\sqrt{1-x^2} = a\cos(a \sin^{-1} x)$

Squaring on both sides, we obtain

$$y_1^2(1 - x^2) = a^2\cos^2(a\sin^{-1} x) = a^2[1 - \sin^2(a \sin^{-1} x)]$$

or $y_1^2(1 - x^2) = a^2(1 - y^2)$ ...(*ii*)

Differentiating again, we get

$$2y_1\, y_2\, (1 - x^2) - 2xy_1^2 = -\,2\, a^2 yy_1$$

or $$y_2\,(1 - x^2) - xy_1 = -\,a^2 y \qquad ...(iii)$$

Differentiating n times by Leibnitz's theorem

$$(1 - x^2)y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{\lfloor 2}\, y_n - xy_{n+1} - ny_n = -\,a^2 y_n$$

or $$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - a^2)y_n = 0 \qquad ...(iv)$$

Putting $x = 0$ in relation (iv), we get

$$y_{n+2}\,(0) - (n^2 - a^2)y_n\,(0) = 0$$

or $$y_{n+2}\,(0) = (n^2 - a^2)y_n\,(0) \qquad ...(v)$$

Replace n by $(n - 2)$ in relation (v), we get

$$y_n\,(0) = \{(n-2)^2 - a^2\}y_{n-2}\,(0)$$

Again replace n by $(n - 4)$ in equation (v) and putting $y_{n-2}\,(0)$ in above relation, we get

$$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}y_{n-4}\,(0)$$

Case I. When n is odd:

$$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}\,.....\,\{1^2 - a^2\}y_1\,(0) \qquad ...(vi)$$

[The last term in (vi) obtain by putting $n = 1$ in equation (v)]

Putting $x = 0$, in equation (i), we get

$$y_1\,(0) = \cos(a \sin^{-1} 0).\, a = \cos 0\,.\,a \Rightarrow y_1\,(0) = a$$

Hence, $$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}\,.....\,\{1^2 - a^2\}.\,a$$

Case II. When n is even:

$$y_n\,(0) = \{(n-2)^2 - a^2\}\,\{(n-4)^2 - a^2\}\,.....\,\{2^2 - a^2\}y_2\,(0).$$

Putting $x = 0$ in (iii), we get

[The last term obtain by putting $n = 2$ in equation (v)]

$$y_2\,(0) = -\,a^2\, y(0) = -\,a^2\, x_0 = 0 \qquad (\text{As } y(0) = 0)$$

Hence, $$y_n\,(0) = 0.$$

EXERCISE 1.3

1. If $y = \tan^{-1} x$, find the value of $y_7\,(0)$ and $y_8\,(0)$. [**Ans.** $\lfloor 6$ and 0.]

2. If $y = e^{a \sin^{-1} x}$, prove that $(1 - x^2)\, y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)\, y_n = 0$ and hence find the value of y_n when $x = 0$.

   $\Big[$**Ans.** n is odd, $y_n\,(0) = \left\{(n-2)^2 + a^2\right\}\,\left\{(n-4)^2 + a^2\right\}\,...\,(3^2 + a^2)\,(1^2 + a^2)a$ n is even, $y_n\,(0) = \left\{(n-2)^2 + a^2\right\}\left\{(n-4)^2 + a^2\right\}......\,(4^2 + a^2)(2^2 + a^2)\,a^2.\Big]$

3. If $\log y = \tan^{-1}x$, show that $(1 + x^2)\, y_{n+2} + \{2(n + 1)x - 1\}\, y_{n+1} + n\,(n + 1)\, y_n = 0$ and hence find y_3, y_4 and y_5 at $x = 0$. [*U.P.T.U. (C.O.), 2003*]

   $\Big[$**Ans.** $y_3\,(0) = -\,1,\ y_4\,(0) = -\,1,\ y_5\,(0) = 5\Big]$

4. If $f(x) = \tan x$, then prove that

$$f^n(0) - {}^nc_2 f^{n-2}(0) + {}^nc_4 f^{n-4}(0) - \ldots\ldots = \text{Sin}\left(\frac{n\pi}{2}\right).$$

5. If $y = \sin^{-1}x$, find $y_n(0)$.

$$\left[\textbf{Ans. } n \text{ is odd, } y_n(0) = (n-2)^2 (n-4)^2 \ldots 5^2 . 3^2 . 1 \; n \text{ is even, } y_n(0) = 0.\right]$$

6. Find $y_n(0)$ when $y = \sin(m \sin h^{-1} x)$.

$$\left[\textbf{Ans. } n \text{ is odd, } y_n(0) = (-1)^{\left(\frac{n-1}{2}\right)} \left\{(n-2)^2 + m^2\right\}\left\{(n-4)^2 + m^2\right\} \ldots (1^2 + m^2)\, m \cdot n \text{ is even, } y_n = (0) = 0.\right]$$

7. If $y = \left[\log\left\{x + \sqrt{1+x^2}\right\}\right]^2$, show that

$$y_{n+2}(0) = -n^2 y_n(0) \text{ hence find } y_n(0).$$

$$\left[\textbf{Ans. } n \text{ is odd, } y_n(0), = 0 \; n \text{ is even } y_n(0) = (-1)^{\frac{n-2}{2}} (n-2)^2 (n-4)^2 \ldots 4^2\, 2^2.2.\right]$$

PARTIAL DIFFERENTIATION

Introduction

Real world can be described in mathematical terms using parametric equations and functions such as trigonometric functions which describe cyclic, repetitive activity; exponential, logrithmic and logistic functions which describe growth and decay and polynomial functions which approximate these and most other functions.

The problems in computer science, statistics, fluid dynamics, economics etc., deal with functions of two or more independent variables.

1.3 FUNCTION OF TWO VARIABLES

If $f(x, y)$ is a unique value for every x and y, then f is said to be a function of the two independent variables x and y and is denoted by

$$z = f(x, y)$$

Geometrically the function $z = f(x, y)$ represents a surface.

The graphical representation of function of two variables is shown in Figure 1.1.

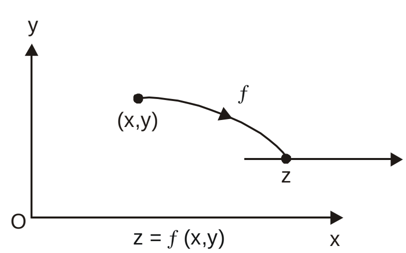


Fig. 1.1

1.4 PARTIAL DIFFERENTIAL COEFFICIENTS

The partial derivative of a function of several variables is the ordinary derivative with respect to any one of the variables whenever, all the remaining variables are held constant. The difference between partial and ordinary differentiation is that while differentiating (partially) with respect to one variable, all other variables are treated (temporarily) as constants and in ordinary differentiation no variable taken as constant,

Definition: Let $z = f(x, y)$

Keeping y constant and varying only x, the partial derivative of z w.r.t. 'x' is denoted by $\frac{\partial z}{\partial x}$ and is defined as the limit

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Partial derivative of z, w.r.t. y is denoted by $\frac{\partial z}{\partial y}$ and is defined as

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}.$$

Notation: The partial derivative $\frac{\partial z}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x}$ or f_x similarly $\frac{\partial z}{\partial y}$ is denoted by $\frac{\partial f}{\partial y}$ or f_y. The partial derivatives for higher order are calculated by successive differentiation.

Thus,
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \text{ and so on.}$$

Geometrical interpretation of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

Let $z = f(x, y)$ represents the equation of a surface in xyz-coordinate system. Suppose APB is the curve which a plane through any point P on the surface ∥ to the xz-plane, cuts. As point P moves along this curve APB, its coordinates z and x vary while y remains constant. The slope of the tangent line at P to APB represents the rate at which z-changes w.r.t. x.

Hence, $\frac{\partial z}{\partial x} = \tan\theta$ (slope of the curve APB at the point P)

and $\frac{\partial z}{\partial y} = \tan\phi$ (slope of the curve CPD at point P)

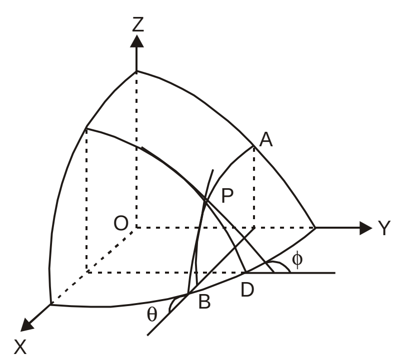


Fig. 1.2

Example 1. Verify that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ where $u(x, y) = \log_e \left(\dfrac{x^2+y^2}{xy}\right)$ (*U.P.T.U., 2007*)

Sol. We have $$u(x, y) = \log_e \left(\frac{x^2+y^2}{xy}\right)$$

$$\Rightarrow \quad u(x, y) = \log (x^2 + y^2) - \log x - \log y \quad ...(i)$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

Now differentiating partially w.r.t. y.

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{4xy}{(x^2+y^2)^2} \quad ...(A)$$

Again differentiate (*i*) partially w.r.t. y, we obtain

$$\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)} - \frac{1}{y}$$

Next, we differentiate above equation w.r.t. x.

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{(x^2+y^2)^2} \quad ...(B)$$

Thus, from (*A*) and (*B*), we find

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$ **Hence proved.**

Example 2. If $f = \tan^{-1}\left(\dfrac{y}{x}\right)$, verify that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$.

Sol. We have $$f = \tan^{-1}\left(\frac{y}{x}\right) \quad ...(i)$$

Differentiating (*i*) partially with respect to x, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \left(\frac{-y}{x^2+y^2}\right) \quad ...(ii)$$

Differentiating (*i*) partially with respect to y, we get

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2} \quad ...(iii)$$

Differentiating (*ii*) partially with respect to y, we get

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \qquad \text{...}(iv)$$

Differentiating (*iii*) partially with respect to x, we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{\left(x^2 + y^2\right)(1) - x(2x)}{\left(x^3 + y^2\right)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \qquad \text{...}(v)$$

$\therefore$ From eqns. (*iv*) and (*v*), we get $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. **Hence proved.**

Example 3. If $u(x + y) = x^2 + y^2$, prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Sol. Given $u = \frac{x^2 + y^2}{x + y}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{(x + y)(2x) - \left(x^2 + y^2\right)(1)}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

and

$$\frac{\partial u}{\partial y} = \frac{(x + y)(2y) - \left(x^2 + y^2\right)(1)}{(x + y)^2} = \frac{y^2 + 2xy - x^2}{(x + y)^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{4xy}{(x + y)^2}$$

or

$$1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 1 - \frac{4xy}{(x + y)^2} = \frac{(x - y)^2}{(x + y)^2} \qquad \text{...}(i)$$

and

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \frac{\left(x^2 + 2xy - y^2\right) - \left(y^2 + 2xy - x^2\right)}{(x + y)^2}$$

$$= \frac{2\left(x^2 - y^2\right)}{(x + y)^2} = \frac{2(x - y)}{(x + y)} \qquad \text{...}(ii)$$

$\therefore$ From (*ii*), we get

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \frac{4(x - y)^2}{(x + y)^2} = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right), \text{ from } (i).$$ **Hence proved.**

Example 4. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Sol. Given $$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \quad ...(i)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\sqrt{\left\{1-\left(\frac{x}{y}\right)^2\right\}}}\cdot\frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)$$

or $$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{(y^2-x^2)}} - \frac{yx}{x^2+y^2} \quad ...(ii)$$

and from (*i*), $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{\left\{1-\left(\frac{x}{y}\right)^2\right\}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

or $$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{(y^2-x^2)}} + \frac{xy}{x^2+y^2} \quad ...(iii)$$

Adding (*ii*) and (*iii*), we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. **Hence proved.**

Example 5. If $f(x, y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ then prove that $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$.

Sol. $$\frac{\partial f}{\partial x} = 2x\cdot\tan^{-1}\left(\frac{y}{x}\right) + x^2\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\left(-\frac{y}{x^2}\right) - y^2\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right)$$

or $$\frac{\partial f}{\partial x} = 2x\cdot\tan^{-1}\left(\frac{y}{x}\right) - \frac{yx^2}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Differentiating both sides with respect to y, we get

$$\frac{\partial^2 f}{\partial y\partial x} = 2x\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2} \quad ...(i)$$

Again $$\frac{\partial f}{\partial y} = x^2\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} - 2y\tan^{-1}\left(\frac{x}{y}\right) - y^2\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)$$

or $$\frac{\partial f}{\partial y} = \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x\left(x^2+y^2\right)}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) = x - 2y \tan^{-1}\left(\frac{x}{y}\right).$$

Differentiating both sides with respect to x, we get

$$\frac{\partial^2 f}{\partial x\, \partial y} = 1 - 2y \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \qquad ...(ii)$$

Thus, from (*i*) and (*ii*), we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$ **Hence proved.**

Example 6. If $V = (x^2 + y^2 + z^2)^{-1/2}$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Sol. Given $V = (x^2 + y^2 + z^2)^{-1/2}$. ...(*i*)

$$\therefore \quad \frac{\partial V}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}\, 2x = -x\,(x^2 + y^2 + z^2)^{-3/2}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = -\left[x\left\{-\frac{3}{2}\left(x^2+y^2+z^2\right)^{-5/2}\cdot 2x\right\}+\left(x^2+y^2+z^2\right)^{-3/2}\cdot 1\right]$$

$$= 3x^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

$$= (x^2 + y^2 + z^2)^{-5/2} [3x^2 - (x^2 + y^2 + z^2)]$$

or

$$\frac{\partial^2 V}{\partial x^2} = (x^2 + y^2 + z^2)^{-5/2} (2x^2 - y^2 - z^2) \qquad ...(ii)$$

Similarly from (*i*), we can find

$$\frac{\partial^2 V}{\partial y^2} = (x^2 + y^2 + z^2)^{-5/2} (2y^2 - x^2 - z^2) \qquad ...(iii)$$

and

$$\frac{\partial^2 V}{\partial z^2} = (x^2 + y^2 + z^2)^{-5/2} (2z^2 - x^2 - y^2) \qquad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = (x^2 + y^2 + z^2)^{-5/2} [(2x^2 - y^2 - z^2) + (2y^2 - x^2 - z^2) + (2z^2 - x^2 - y^2)]$$

$$= (x^2 + y^2 + z^2)^{-5/2} [0] = 0.$$ **Hence proved.**

Example 7. If $u = f(r)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ (*U.P.T.U., 2001, 2005*)

Sol. Given $r^2 = x^2 + y^2$ ...(*i*)

Differentiating both sides partially with respect to x, we have

$$2r\frac{\partial r}{\partial x} = 2x \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \quad ...(ii)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} \quad ...(iii)$$

Now, $u = f(r)$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}, \text{ from } (ii)$$

Again differentiating partially w.r.to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{f'(r)x}{r}\right] = \frac{r\left[f'(r).1 + xf''(r)(\partial r/\partial x)\right] - xf'(r)(\partial r/\partial x)}{r^2}$$

or $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[rf'(r) + x^2 f''(r) - \frac{x^2}{r}f'(r)\right], \text{ from } (ii).$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[rf'(r) + y^2 f''(r) - \frac{y^2}{r}f'(r)\right]$$

Adding, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2rf'(r) + (x^2+y^2)f''(r) - \frac{(x^2+y^2)}{r}f'(r)\right]$$

$$= \frac{1}{r^2}\,[2rf'(r) + r^2 f''(r) - rf'(r)], \text{ from } (i)$$

$$= \frac{1}{r}f'(r) + f''(r).$$ **Hence proved.**

Example 8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$; show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$ (*U.P.T.U., 2003*)

Sol. Given $u = \log(x^3 + y^3 + z^3 - 3xyz).$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz} \quad ...(i)$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz} \quad ...(ii)$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}. \quad ...(iii)$$

Adding eqns. (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}$$

$$\Big| \text{As } a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

or $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}. \qquad ...(iv)$$

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ from } (iv)$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2}.$$

Hence proved.

Example 9. If $u = \tan^{-1} \dfrac{xy}{\sqrt{(1+x^2+y^2)}}$, show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{\left(1+x^2+y^2\right)^{3/2}}.$$

Sol. Given $$u = \tan^{-1} \frac{xy}{\sqrt{(1+x^2+y^2)}}.$$

$\therefore$ $$\frac{\partial u}{\partial y} = \frac{1}{\left[1+\left\{x^2y^2/\left(1+x^2+y^2\right)\right\}\right]}$$

$$\times\ x\left[\frac{\sqrt{\left(1+x^2+y^2\right)}.1-y\frac{1}{2}\left(1+x^2+y^2\right)^{-1/2}2y}{\left(1+x^2+y^2\right)}\right]$$

$$= \frac{x}{1+x^2+y^2+x^2y^2}\cdot\frac{\left(1+x^2+y^2\right)-y^2}{\sqrt{\left(1+x^2+y^2\right)}}$$

or $$\frac{\partial u}{\partial y} = \frac{x}{(1+x^2)(1+y^2)} \cdot \frac{1+x^2}{\sqrt{(1+x^2+y^2)}} = \frac{x}{(1+y^2)\sqrt{(1+x^2+y^2)}}$$

Again differentiating partially w.r.to x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{x}{(1+y^2)\sqrt{(1+x^2+y^2)}}\right] = \frac{1}{(1+y^2)}\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{(1+x^2+y^2)}}\right]$$

$$= \frac{1}{(1+y^2)}\left[\frac{\sqrt{(1+x^2+y^2)}1 - x\frac{1}{2}(1+x^2+y^2)^{-1/2} 2x}{(1+x^2+y^2)}\right]$$

$$= \frac{1}{(1+y^2)} \cdot \frac{(1+x^2+y^2)-x^2}{(1+x^2+y^2)^{3/2}} = \frac{1}{(1+x^2+y^2)^{3/2}}.$$ **Hence proved.**

Example 10. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$ (*U.P.T.U., 2002*)

Sol. We have

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \quad ...(i)$$

where u is a function of x, y and z

Differentiating (*i*) partially with respect to x, we get

$$= \frac{2x}{a^2+u} - \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = 0$$

or $$\frac{\partial u}{\partial x} = \frac{2x/(a^2+u)}{\left[x^2/(a^2+u)^2 + y^2/(b^2+u)^2 + z^2/(c^2+u)^2\right]} = \frac{2x/(a^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{2y/(b^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}; \quad \frac{\partial u}{\partial z} = \frac{2z/(c^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Adding with square

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4\left[x^2/(a^2+u)^2 + y^2/(b^2+u)^2 + z^2/(c^2+u)^2\right]}{\left[\sum\left\{x^2/(a^2+u)^2\right\}\right]^2}$$

or $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{\sum\left[x^2/\left(a^2+u\right)^2\right]} \qquad ...(ii)$$

Also, $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \frac{1}{\sum\left[x^2/\left(a^2+u\right)^2\right]}\left[\frac{2x^2}{\left(a^2+u\right)}+\frac{2y^2}{\left(b^2+u\right)}+\frac{2z^2}{\left(c^2+u\right)}\right]$$

$$= \frac{2}{\sum\left[x^2/\left(a^2+u\right)^2\right]}\ [1], \text{ from } (i) \qquad ...(iii)$$

From (*ii*) and (*iii*), we have

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$ **Hence proved.**

Example 11. If $x^x\, y^y\, z^z = c$, show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x\,\partial y} = -(x \log ex)^{-1}.$$ Where z is a function of x and y.

Sol. Given $x^x.y^y.z^z = c$, where z is a function of x and y.

Taking logarithms, $x \log x + y \log y + z \log z = \log c$. ...(*i*)

Differentiating (*i*) partially with respect to x, we get

$$= \left[x\left(\frac{1}{x}\right)+(\log x)1\right]+\left[z\left(\frac{1}{z}\right)+(\log z)1\right]\frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \qquad ...(ii)$$

Similarly, from (*i*), we have $$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)} \qquad ...(iii)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\left(\frac{1+\log y}{1+\log z}\right)\right], \text{ from } (iii)$$

or $$\frac{\partial^2 z}{\partial x\partial y} = -(1+\log y)\cdot\frac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]$$

$$= -(1+\log y)\cdot\left[-(1+\log z)^{-2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]$$

or $$\frac{\partial^2 z}{\partial x\partial y} = \frac{(1+\log y)}{z(1+\log z)^2}\cdot\left\{-\left(\frac{1+\log x}{1+\log z}\right)\right\}, \text{ from } (iii)$$

$\therefore$ At $x = y = z$, we have $$\frac{\partial^2 z}{\partial x\partial y} = -\frac{(1+\log x)^2}{x\,(1+\log x)^3}$$

$$\Rightarrow \qquad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log_e e+\log x)} = (\text{As } \log_e e = 1)$$

$$= -\frac{1}{x\log(ex)} = -\{x \log (ex)\}^{-1}. \quad \textbf{Hence proved.}$$

Example 12. If $u = \log (x^3 + y^3 - x^2y - xy^2)$ then show that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Sol. We have $\qquad u = \log (x^3 + y^3 - x^2y - xy^2)$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 2xy - y^2}{(x^3 + y^3 - x^2y - xy^2)} \qquad ...(i)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - x^2 - 2xy}{(x^3 + y^3 - x^2y - xy^2)} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{(3x^2 - 2xy - y^2) + (3y^2 - x^2 - 2xy)}{(x^3 + y^3 - x^2y - xy^2)}$$

$$= \frac{2\,(x-y)^2}{(x+y)\,(x^2+y^2-2xy)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2\,(x-y)^2}{(x+y)(x-y)^2} = \frac{2}{(x+y)} \qquad ...\ (iii)$$

Now, $\quad \dfrac{\partial^2 u}{\partial x^2} + 2\,\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \cdot \frac{2}{x+y} \quad (\text{from } iii)$$

$$= 2\,\frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) + 2\,\frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)$$

$$= -\frac{2}{(x+y)^2} - \frac{2}{(x+y)^2} = -\frac{4}{(x+y)^2}. \quad \textbf{Hence proved.}$$

Example 13. If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

Sol. We have $u = e^{xyz} \therefore \dfrac{\partial u}{\partial z} = e^{xyz}.xy$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}(e^{xyz} \cdot xy) = e^{xyz}\, x^2yz + e^{xyz}.x$$

or $$\frac{\partial^2 u}{\partial y \partial z} = (x^2yz + x)\, e^{xyz}$$

Hence $$\frac{\partial^3 u}{\partial x \partial y \partial z} = (2xyz + 1)\, e^{xyz} + (x^2yz + x)\, e^{xyz} \cdot yz$$

$$= (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$ **Hence proved.**

Example 14. If $u = \log r$, where $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}.$$

Sol. Given $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2,$ ...(*i*)

Differentiating partially with respect to x, we get

$$2r\frac{\partial r}{\partial x} = 2\,(x - a) \text{ or } \frac{\partial r}{\partial x} = \left(\frac{x-a}{r}\right). \qquad ...(ii)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{(y-b)}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{(z-c)}{r}$$

Now, $u = \log r.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\left(\frac{x-a}{r}\right), \text{ from } (ii)$$

or $$\frac{\partial u}{\partial x} = \frac{x-a}{r^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right) = \frac{r^2(1)-(x-a)\,2r(\partial r/\partial x)}{r^4}$$

or $$\frac{\partial^2 u}{\partial x^2} = \frac{r^2-2\,(x-a)^2}{r^4}, \text{ from } (ii)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{r^2-2\,(y-b)^2}{r^4}; \quad \frac{\partial^2 u}{\partial z^2} = \frac{r^2-2\,(z-c)^2}{r^4}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3r^2-2\{(x-a)^2+(y-b)^2+(z-c)^2\}}{r^4}$$

$$= \frac{3r^2-2r^2}{r^4}, \text{ from } (i) = \frac{1}{r^2}.$$ **Hence proved.**

Example 15. If $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, prove that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Sol. Given $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$.

Differentiating partially with respect to x, we get

$$\frac{\partial u}{\partial x} = x^2 \frac{1}{1+(y/x)^2} \cdot \left(-\frac{y}{x^2}\right) + 2x \tan^{-1}\left(\frac{y}{x}\right) - y^2 \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y}$$

$$= -\frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2} + 2x \tan^{-1} \frac{y}{x}$$

$$= -\frac{y(x^2+y^2)}{(x^2+y^2)} + 2x \tan^{-1}\left(\frac{y}{x}\right) = -y + 2x \tan^{-1}\left(\frac{y}{x}\right)$$

Again differentiating partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{-y + 2x \tan^{-1}\left(\frac{y}{x}\right)\right\} = -1 + 2x \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= -1 + \frac{2x^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$ **Hence proved.**

Example 16. If $z = f(x - by) + \phi(x + by)$, prove that

$$b^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}.$$

Sol. Given $z = f(x - by) + \phi(x + by)$ ...(*i*)

$\therefore$ $\frac{\partial z}{\partial x} = f'(x - by) + \phi'(x + by)$

and $\frac{\partial^2 z}{\partial x^2} = f''(x - by) + \phi''(x + by)$. ...(*ii*)

Again from (*i*), $\frac{\partial z}{\partial y} = -bf'(x - by) + b\phi'(x + by)$

and $\frac{\partial^2 z}{\partial y^2} = b^2 f''(x - by) + b^2 \phi''(x + by) = b^2 \frac{\partial^2 z}{\partial x^2}$, from (*ii*). **Hence proved.**

Example 17. If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$. Prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$ (*U.P.T.U., 2006*)

Sol. $$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)}$$

$= 2.$ **Hence proved.**

EXERCISE 1.4

1. Find $\frac{\partial^3 u}{\partial x \partial y \partial z}$ if $u = e^{x^2+y^2+z^2}$. [**Ans.** $8xyzu$]

2. Find the first order derivatives of

(*i*) $u = x^{xy}$. $\left[\textbf{Ans. } \frac{\partial u}{\partial x} = x^{xy}(y \log x + y); \frac{\partial u}{\partial y} = x^{xy+1} \log x\right]$

(*ii*) $u = \log\left(x + \sqrt{x^2 - y^2}\right)$ $\left[\textbf{Ans. } \frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2 - y^2}} \frac{\partial u}{\partial y} = -y\left(x^2 - y^2\right)^{\frac{-1}{2}}\left(x + \sqrt{x^2 - y^2}\right)^{-1}\right]$

3. If $u = \sin^{-1}\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$.

4. If $u = e^x (x \cos y - y \sin y)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

5. If $u = a \log(x^2 + y^2) + b \tan^{-1}\left(\frac{y}{x}\right)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

6. If $z = \tan(y - ax) + (y + ax)^{3/2}$, prove that $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$.

7. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

8. If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$. [**Ans.** 0]

9. If $u = \log (x^2 + y^2 + z^2)$, find the value of $\dfrac{\partial^2 u}{\partial y \partial z}$. $\left[\textbf{Ans. } \dfrac{-4yz}{\left(x^2+y^2+z^2\right)^2}\right]$

10. If $u = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$, then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{u}$.

11. If $z = f(x + ay) + \phi(x - ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$.

12. If $u = \cos^{-1} \left[(x-y)/(x+y)\right]$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

13. If $u = \log (x^2 + y^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

14. If $u = x^2y + y^2z + z^2x$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x + y + z)^2$.

15. Prove that $f(x, t) = a \sin bx . \cos bt$ satisfies $\dfrac{\partial^2 f}{\partial x^2} = b^2 \dfrac{\partial^2 f}{\partial t^2}$.

16. If $u = r^m$, where $r = \sqrt{x^2+y^2+z^2}$ find $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$. [**Ans.** $m(m + 1)r^{m-2}$]

17. If $u = (x^2 + y^2 + z^2)^{-1}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 2(x^2 + y^2 + z^2)^{-2}$.

18. If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n, when $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$. $\left[\textbf{Ans. } n = -\dfrac{3}{2}\right]$

19. For $n = 2$ or -3 show that $u = r^n(3\cos^2\theta - 1)$ satisfies the differential equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0.$$

20. If $u = e^{a\theta} \cos(a \log r)$, show that $\left(\dfrac{\partial^2 u}{\partial r^2}\right) + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$.

21. If $e^{-\frac{z}{(x^2-y^2)}} = (x - y)$, show that $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x^2 - y^2$.

[**Hint:** Solve for $z = (y^2 - x^2) \log (x - y)$].

22. If $u = \dfrac{1}{r}$ and $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

23. If $u\ (x, y, z) = \cos 3x \cos 4y \sin h\ 5z$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

24. If $x^2 = au + bv$; $y^2 = au - bv$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial y}\right)_x \left(\dfrac{\partial y}{\partial v}\right)_u$.

25. If $x = r\cos\theta$, $y = r\sin\theta$, find $\left(\dfrac{\partial x}{\partial r}\right)_\theta, \left(\dfrac{\partial x}{\partial \theta}\right)_{r,} \left(\dfrac{\partial \theta}{\partial x}\right)_{y,} \left(\dfrac{\partial \theta}{\partial y}\right)_{x,} \left(\dfrac{\partial y}{\partial x}\right)_r$.

[**Ans.** $\cos\theta, -r\sin\theta, -r^{-1}\sin\theta, r^{-1}\cos\theta, -\cot\theta$.]

1.5 HOMOGENEOUS FUNCTION

A polynomial in x and y *i.e.,* a function $f\ (x, y)$ is said to be homogeneous if all its terms are of the same degree. Consider a homogeneous polynomial in x and y

$$f\ (x, y) = a_0 x^n + a_1\ x^{n-1}\ y + a_2\ x^{n-2} y^2 + \ldots.. + a_n\ y^n$$

$$= x^n \left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_n\left(\frac{y}{x}\right)^n\right]$$

or

$$\boxed{f\ (x, y) = x^n\ F\left(\frac{y}{x}\right)}$$

Hence every homogeneous function of x and y of degree n can be written in above form.

NOTE: Degree of Homogeneous function = degree of numerator – degree of denominator.

Remark 1: If $f\ (x, y) = a_0 x^n + a_1 x^{n+1} \cdot y^{-1} + a_2 x^{n+2} \cdot y^{-2} + \ldots\ldots + a_n x^{n+n}\ y^{-n}$

$$= x^n \left\{a_0 + a_1 \frac{x}{y} + a_2\left(\frac{x}{y}\right)^2 + \ldots. + a_n\left(\frac{x}{y}\right)^n\right\}$$

$$\Rightarrow \quad f\ (x, y) = x^n\ F\left(\frac{x}{y}\right);\ \text{degree} = n$$

Remark 2: If $f\ (x, y) = a_0 y^{-n} + a_1 y^{-n-1} \cdot x + \ldots\ldots + a_n y^{-n-n} \cdot x^n$

$$= y^n \left\{a_0 + a_1\left(\frac{x}{y}\right) + \ldots. + a_n\left(\frac{x}{y}\right)^n\right\}$$

$$\Rightarrow \quad f\ (x, y) = y^{-n}\ F\left(\frac{x}{y}\right);\ \text{degree} = -n$$

Another forms are also possible *i.e.,*

$$f\ (x, y) = y^n\ F\left(\frac{x}{y}\right);\ f\ (x, y) = y^n\ F(y/x)$$

1.6 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

Statement: If f is a homogeneous function of x, y of degree n then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf. \qquad \textit{(U.P.T.U., 2006)}$$

Proof. Since f is a homogeneous function

$$\therefore \qquad f(x, y) = x^n F\left(\frac{y}{x}\right) \qquad ...(i)$$

Differentiating partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x} = nx^{n-1} F\left(\frac{y}{x}\right) + x^n F'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \qquad ...(ii)$$

$$\frac{\partial f}{\partial y} = x^n F'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \qquad ...(iii)$$

Multiplying (ii) by x and (iii) by y and adding, we have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nx^n F\left(\frac{y}{x}\right) - x^{n-1} y F'\left(\frac{y}{x}\right) + x^{n-1} y F'\left(\frac{y}{x}\right)$$

$$= nx^n F\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad \boxed{x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f} \quad \text{(from } (i)).$$

In general if $f(x_1, x_2, ..., x_n)$ be a homogeneous function in $x_1, x_2, ..., x_n$ then $x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2}$

$$+ ... + x_n \frac{\partial f}{\partial x_n} = nf.$$

Corollary 1. If f is a homogeneous function of degree n, then

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1)f.$$

Proof. We have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f \qquad ...(i)$$

Differentiating (i) w.r.t. x and y respectively, we get

$$\frac{\partial f}{\partial x} + x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial x \partial y} = n \frac{\partial f}{\partial x} \qquad ...(ii)$$

and

$$x \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y} + y \frac{\partial^2 f}{\partial y^2} = n \frac{\partial f}{\partial y} \qquad ...(iii)$$

Multiplying (ii) by x and (iii) by y and adding, we have

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{xy^2} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n\left(x \frac{\partial f}{\partial x} + y \frac{\partial t}{\partial y}\right)$$

$$\Rightarrow \qquad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n^2 f - nf = n(n-1) f$$

Example 1. Verify Euler's theorem for the function

$$f(x, y) = ax^2 + 2hxy + by^2.$$

Sol. Here the given function $f(x, y)$ is homogeneous of degree $n = 2$. Hence the Euler's theorem is

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f \qquad \text{...}(i)$$

Now, we are to prove equation (i) as follows:

$$\frac{\partial f}{\partial x} = 2ax + 2hy, \quad \frac{\partial f}{\partial y} = 2hx + 2by$$

$$\therefore \qquad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= 2(ax^2 + 2hxy + by^2) = 2f$$

$$\Rightarrow \qquad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f, \text{ which proves equation } (i).$$

Example 2. Verify Euler's theorem for the function $u = x^n \sin\left(\frac{y}{x}\right)$.

Sol. Since u is homogeneous function in x and y of degree n, hence we are to prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \qquad \text{...}(i)$$

We have

$$u = x^n \sin\left(\frac{y}{x}\right)$$

$$\therefore \qquad \frac{\partial u}{\partial x} = nx^{n-1} \sin\left(\frac{y}{x}\right) + x^n \cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

or

$$x \frac{\partial u}{\partial x} = nx^n \sin\left(\frac{y}{x}\right) - x^{n-1} y \cos\left(\frac{y}{x}\right) \qquad \text{...}(ii)$$

Similarly,

$$y \frac{\partial u}{\partial y} = yx^{n-1} \cos\left(\frac{y}{x}\right) \qquad \text{...}(iii)$$

Adding (ii) and (iii), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nx^n \sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

which verifies Euler's theorem.

Example 3. If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show by Euler's theorem that

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

Sol. We have $$u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Let $$f = \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Here, f is a homogeneous function in x and y

where, $$\text{degree } n = \frac{1}{2} - \frac{1}{2} = 0$$

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0.\,f = 0$$

or $$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 0 \quad \Big| \text{ As } f = \sin u$$

or $$x\cos u \cdot \frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$ **Hence proved.**

Example 4. If $u = \log[(x^4 + y^4)/(x + y)]$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$ (*U.P.T.U., 2000*)

Sol. We have $$u = \log_e \frac{x^4+y^4}{x+y} \Rightarrow e^u = \frac{x^4+y^4}{x+y}$$

Let $$f = e^u = \frac{x^4+y^4}{x+y}$$

Here the function f is a homogeneous function in x and y of degree, $n = 4 - 1 = 3$

$\therefore$ By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 3f$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3f$$

$$\Rightarrow \qquad \left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)e^{u} = 3e^{u}$$

$$\Rightarrow \qquad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3.$$ **Hence proved.**

Example 5. Verify Euler's theorem for

$$u = \sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right). \qquad (U.P.T.U., 2006)$$

Sol. Here u is a homogeneous function of degree,

$n = 1 - 1 = 0$; hence by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$$

Now,
$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

or
$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2} \qquad ...(i)$$

and
$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right)$$

or
$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}}+\frac{xy}{(x^2+y^2)} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0,$$ hence Euler's theorem is verified.

Example 6. If $u = x\sin^{-1}\left(\frac{x}{y}\right)+y\sin^{-1}\left(\frac{y}{x}\right)$, find the value of

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}. \qquad (U.P.T.U., 2007)$$

Sol. We have

$$u = x\sin^{-1}\left(\frac{x}{y}\right)+y\sin^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \sin^{-1}\left(\frac{x}{y}\right)+\frac{x}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right)+y\cdot\frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(-\frac{y}{x^2}\right)$$

$\Rightarrow$ $$x \frac{\partial u}{\partial x} = x \sin^{-1}\left(\frac{x}{y}\right) + \frac{x^2}{\sqrt{y^2-x^2}} - \frac{y^2}{\sqrt{x^2-y^2}} \quad ...(i)$$

and $$\frac{\partial u}{\partial y} = x \cdot \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \sin^{-1}\left(\frac{y}{x}\right) + y \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(\frac{1}{x}\right)$$

$\Rightarrow$ $$y \frac{\partial u}{\partial y} = -\frac{x^2}{\sqrt{y^2-x^2}} + y \sin^{-1}\left(\frac{y}{x}\right) + \frac{y^2}{\sqrt{x^2-y^2}} \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \sin^{-1}\left(\frac{x}{y}\right) + y \sin^{-1}\left(\frac{y}{x}\right) = u \quad ...(iii)$$

Differentiating (*iii*) partially w.r.t. x and y respectively.

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} \Rightarrow x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = 0 \quad ...(iv)$$

and $$x \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y} \Rightarrow y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial^2 u}{\partial x \partial y} = 0 \quad ...(v)$$

Multiplying equation (*iv*) by x and (*v*) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Example 7. If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$ and evaluate

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}.$$

Sol. We have

$$u = \tan^{-1} \frac{x^3+y^3}{x-y} \Rightarrow \tan u = \frac{x^3+y^3}{x-y}$$

$\therefore$ $$\text{Let } f = \tan u = \frac{x^3+y^3}{x-y}$$

Since $f(x, y)$ is a homogeneous function of degree

$$n = 3 - 1 = 2$$

By Euler's theorem, we have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf \Rightarrow x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 2 \tan u$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} \cdot \sec^2 u + y \frac{\partial u}{\partial y} \cdot \sec^2 u = 2 \tan u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\tan u}{\sec^2 u} = \sin 2u. \quad \textbf{Proved.} \qquad ...(i)$$

Differentiating (*i*) partially w.r.t. *x*, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \cdot \frac{\partial u}{\partial x}$$

or $$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (2 \cos 2u - 1) \frac{\partial u}{\partial x}$$

Multiplying by *x*, we obtain

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = x (2 \cos 2u - 1) \frac{\partial u}{\partial x} \qquad ...(ii)$$

Again differentiating equation (*i*) partially w.r.t. *y*, we get

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2 \cos 2u \frac{\partial u}{\partial y}$$

or $$y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial^2 u}{\partial y \partial x} = (2 \cos 2u - 1) \frac{\partial u}{\partial y}$$

or $$y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = y (2 \cos 2u - 1) \frac{\partial u}{\partial y} \text{(multiply by } y) \qquad ...(iii)$$

Adding (*ii*) and (*iii*), we get

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= (2\cos 2u - 1) \left(x\frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \\ &= (2\cos 2u - 1) \sin 2u, \text{ (from } (i)) \\ &= (2\sin 2u \cos 2u - \sin 2u) \\ &= \sin 4u - \sin 2u \\ &= 2 \cos \left(\frac{4u+2u}{2} \right) \cdot \cos \left(\frac{4u-2u}{2} \right). \end{aligned}$$

Hence, $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2 \cos 3u \cdot \cos u.$

Example 8. If $z = x^m f\left(\dfrac{y}{x}\right) + x^n g\left(\dfrac{x}{y}\right)$, prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m + n - 1) \left(x\frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right).$$

Sol. Let $$u = x^m f\left(\frac{y}{x}\right),\ v = x^n g\left(\frac{x}{y}\right)$$

then $$z = u + v \qquad ...(i)$$

Now, *u* is homogeneous function of degree *m*. Therefore with the help of (Corollary 1, on page 36), we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = m(m-1)u \quad ...(ii)$$

Similarly for $v = x^n g\left(\frac{x}{y}\right)$, we have

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v \quad ...(iii)$$

Adding (*ii*) and (*iii*), we get

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v) + y^2 \frac{\partial}{\partial y^2}(u+v) = m(m-1)u + n(n-1)v$$

$$\Rightarrow \quad x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = m(m-1)u + n(n-1)v \text{ (As } z = u+v). \quad ...(iv)$$

Again from Euler's theorem, we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = mu \text{ and } x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv$$

Adding $x\frac{\partial}{\partial x}(u+v) + y \frac{\partial}{\partial y}(u+v) = mu + nv$

$$\Rightarrow \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = mu + nv \quad ...(v)$$

Now,
$$\begin{aligned} m(m-1)u + n(n-1)v &= (m^2 u + n^2 v) - (mu + nv) \\ &= m(m+n)u + n(m+n)v - mn(u+v) - (mu+nv) \\ &= (mu+nv)(m+n) - (mu+nv) - mnz \\ &= (mu+nv)(m+n-1) - mnz \\ &= (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz, \text{ from } (v) \end{aligned}$$

Putting this value in equation (*iv*), we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz$$

$$\Rightarrow \quad x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right).$$ **Hence proved.**

Example 9. If $u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0.$$ (*U.P.T.U., 2003*)

Sol. We have $$u = \sin^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$$

$$\Rightarrow \quad \sin u = \left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right], \text{ let } f = \sin u$$

$\therefore$ Degree of homogeneous function f, $n = 1 - 4 = -3$, from Euler's theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = -3 \sin u$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = -3 \sin u$$

or $$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)\cos u = -3 \sin u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3 \tan u = 0.$$ **Hence proved.**

Example 10. If $V = \log_e \sin\left\{\frac{\pi\left(2x^2+y^2+zx\right)^{\frac{1}{2}}}{2\left(x^2+xy+2yz+z^2\right)^{\frac{1}{3}}}\right\}$, find the value of

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}, \text{ when } x = 0, y = 1, z = 2.$$

Sol. We have $$V = \log \sin\left\{\frac{\pi\left(2x^2+y^2+zx\right)^{\frac{1}{2}}}{2\left(x^2+xy+2yz+z^2\right)^{\frac{1}{3}}}\right\}$$

$$\therefore \quad e^V = \sin\left\{\frac{\pi\left(2x^2+y^2+zx\right)^{\frac{1}{2}}}{2\left(x^2+xy+2yz+z^2\right)^{\frac{1}{3}}}\right\}$$

or $$\sin^{-1}(e^V) = \left\{\frac{\pi\left(2x^2+y^2+zx\right)^{\frac{1}{2}}}{2\left(x^2+xy+2yz+z^2\right)^{\frac{1}{3}}}\right\}$$

Let $$f = \sin^{-1}(e^v) = \left\{\frac{\pi\left(2x^2+y^2+zx\right)^{\frac{1}{2}}}{2\left(x^2+xy+2yz+z^2\right)^{\frac{1}{3}}}\right\}$$

Since f is a homogeneous function $\therefore$ $n = 1 - \frac{2}{3} = \frac{1}{3}$

By Euler's theorem, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z} = \frac{1}{3}f$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\sin^{-1}e^V)+y\frac{\partial}{\partial y}(\sin^{-1}e^V)+z\frac{\partial}{\partial z}(\sin^{-1}e^V) = \frac{1}{3}\sin^{-1}e^V$$

$$\Rightarrow \quad \left(x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}\right)\left(\frac{1\times e^V}{\sqrt{1-e^{2V}}}\right) = \frac{1}{3}\sin^{-1}e^V$$

$$\Rightarrow \quad x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} = \frac{\sqrt{1-e^{2V}}}{e^V}\times\frac{1}{3}\sin^{-1}e^V \qquad ...(i)$$

Now, $$\left(e^V\right)_{\substack{x=0\\y=1\\z=2}} = \sin\left\{\frac{\pi}{2\times 2}\right\} = \frac{1}{\sqrt{2}},\ e^{2V} = \frac{1}{2}$$

and $$\left(\sin^{-1}e^V\right)_{\substack{x=0\\y=1\\z=2}} = \frac{\pi}{4}$$

Putting all these values in equation (*i*), we get

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} = \frac{\sqrt{1-1/2}}{\frac{1}{\sqrt{2}}}\times\frac{1}{3}\times\frac{\pi}{4} = \frac{\sqrt{1/2}}{\sqrt{1/2}}\times\frac{\pi}{12}$$

$$\Rightarrow \quad x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} = \frac{\pi}{12}.$$

Example 11. If $u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 5u \text{ and } x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 20u.$$

Sol. $u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$

$$= x^5\left(\frac{y}{x}\right)^2\sin^{-1}\left(\frac{y}{x}\right) = x^5F\left(\frac{y}{x}\right) \quad \Bigg| \quad F\left(\frac{y}{x}\right)=\left(\frac{y}{x}\right)^2\sin^{-1}\frac{y}{x}$$

$\therefore$ u is a homogeneous function of degree 5 *i.e.*, $n = 5$

By Euler's theorem, we get $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 5u$. **Proved.**

Next, we know that (from Corollary 1, on page 36)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 5(5-1)u$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 20u.$$ **Hence proved.**

Example 12. If $u = x f_1\left(\frac{y}{x}\right) + f_2\left(\frac{y}{x}\right)$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Sol. Let $u_1 = x f_1\left(\frac{y}{x}\right)$ and $u_2 = x^0 f_2\left(\frac{y}{x}\right)$, then $u = u_1 + u_2$

Since u_1 is a homogeneous function of degree one. So by Corollary 1 on page 36, we get

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = 1\,(1 - 1) = 0 \quad ...(i)$$

and u_2 is also a homogeneous function of degree 0

$$\therefore \quad x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 0 \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$x^2 \frac{\partial^2}{\partial x^2}(u_1 + u_2) + 2xy \frac{\partial^2}{\partial x \partial y}(u_1 + u_2) + y^2 \frac{\partial^2}{\partial y^2}(u_1 + u_2) = 0$$

$$\Rightarrow \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$ **Hence proved.**

Example 13. If $u = \sin^{-1}(x^3 + y^3)^{2/5}$, evaluate $\frac{x^2 \partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$.

Sol. Given $u = \sin^{-1}(x^3 + y^3)^{2/5}$

$$\sin u = (x^3 + y^3)^{2/5} = x^{6/5}\left(1 + \frac{y^3}{x^3}\right)^{2/5}$$

Let $f = \sin u$

$$\therefore \quad f = x^{6/5}\left(1 + \frac{y^3}{x^3}\right)^{2/5} \quad ...(i)$$

which is homogeneous of degree $n = \frac{6}{5}$. By Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{6}{5} f$$

$$\Rightarrow \quad x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) = \frac{6}{5} \sin u$$

or

$$x \frac{\partial u}{\partial x} \cos u + y \frac{\partial u}{\partial y} \cos u = \frac{6}{5} \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{6}{5} \tan u \quad ...(ii)$$

Differentiating (*ii*) w.r. to '*x*', we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{6}{5}\sec^2 u.\frac{\partial u}{\partial x}$$

Multiplying by x

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{6}{5}\sec^2 u.\,x\frac{\partial u}{\partial x} \qquad ...(iii)$$

Differentiating (*ii*) w.r. to '*y*', we get

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{6}{5}\sec^2 u\frac{\partial u}{\partial y}$$

Multiplying by y

$$y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{6}{5}\sec^2 u.\,y\frac{\partial u}{\partial y} \qquad ...(iv)$$

Adding (*iii*) and (*iv*), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(\frac{6}{5}\sec^2 u - 1\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{6}{5}\tan u\left(\frac{6}{5}\sec^2 u - 1\right) \qquad \Bigg|\text{As } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{6}{5}\tan u$$

Example 14. If $u = 3x^4 \cot^{-1}\left(\frac{y}{x}\right) + 16y^4 \cos^{-1}\left(\frac{x}{y}\right)$ then prove that

$$x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 12u.$$

Sol. The given function is homogeneous function of degree 4. By Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u \qquad ...(i)$$

Differentiating (*i*) w.r. to x, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 4\frac{\partial u}{\partial x}$$

or $$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial y^2} = 4x\frac{\partial u}{\partial x} \qquad ...(ii)$$

Differentiating (*i*) w.r. to y, we get

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 4\frac{\partial u}{\partial y}$$

or $$xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 4y\frac{\partial u}{\partial y} \qquad ...(iii)$$

Adding (*ii*) and (*iii*), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 3\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 3\times 4u = 12u.$$ **Hence proved.**

EXERCISE 1.5

Verify Euler's theorem

1. $\sqrt{x^2-y^2}$.

2. $\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\Big/\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)$.

3. $\cos^{-1}\left(\frac{x}{y}\right)+\cot^{-1}\left(\frac{y}{x}\right)$.

4. $x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$.

5. $\frac{xy}{(x+y)}$.

6. $\cos^{-1}\left(\frac{x}{y}\right)+\cot^{-1}\left(\frac{y}{x}\right)$.

7. If $u=\log_e\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$. (*U.P.T.U., 2000*)

8. If $u=\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{9}{20}u$ [*U.P.T.U. (AG), 2005*]

9. If $z=x^4y^2\sin^{-1}\left(\frac{x}{y}\right)+\log x-\log y$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=6x^4y^2\sin^{-1}\left(\frac{x}{y}\right)$. [*U.P.T.U. (C.O.)., 2003*]

10. If $u=x^3+y^3+z^3+3xyz$; show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3u$.

11. If $u=\cos^{-1}\left[\frac{x-y}{x+y}\right]$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.

12. If $u=\log\,[(x^2+y^2)/(x+y)]$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$. (*U.P.T.U., 2008*)

13. If $u=\sin^{-1}\{(x^2+y^2)/(x+y)\}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$.

14. If $u=\sin^{-1}\{(x+y)/(\sqrt{x}+\sqrt{y})\}$. Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$.

15. If $u=\tan^{-1}[(x^2+y^2)/(x+y)]$, then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$.

16. If u is a homogeneous function of degree n, show that

(*a*) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$. (*b*) $y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial y}$.

17. If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

18. If $u = \sin^{-1}\left[\frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{6}} + y^{\frac{1}{6}}}\right]$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{144}\tan u\,[\tan^2 u - 11]$.

19. Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}\tan u$ if $u = \sin^{-1}\left(\frac{x^3 + y^3}{\sqrt{x} + \sqrt{y}}\right)$.

20. Verify Euler's theorem for $f = \frac{z}{x+y} + \frac{y}{z+x} + \frac{x}{y+z}$.

21. If $u = y^2\, e^{y/x} + x^2\tan^{-1}\left(\frac{x}{y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$.

22. If $\left(\sqrt{x} + \sqrt{y}\right)\sin^2 u = x^{\frac{1}{3}} + y^{\frac{1}{3}}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$.

23. If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$, show that $x^2\frac{\partial^2 u}{\partial x^2}$

24. If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f(x, y) = 0$.

25. If $u = \sec^{-1}\{(x^3 + y^3)/(x + y)\}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.

1.7 TOTAL DIFFERENTIAL COEFFICIENT

Let $$z = f(x, y) \qquad \text{...}(i)$$

where $x = \phi(t)$ and $y = \psi(t)$, then z can be expressed as a function of t alone by substituting the values of x and y in terms of t from the last two equations in equation (i).

And we can find the ordinary differential coefficient $\frac{dz}{dt}$, which is called total differential coefficient of z with respect to t. Since it is very difficult sometimes to express z in terms of t alone by eliminating x and y. So we are now to find $\frac{dz}{dt}$ without actually substituting the values of x and y in terms of t in $z = f(x, y)$.

Let δx, δy and δz be the increaments in x, y and z corresponding to a small increament δt in the value of t.

then $z + \delta z = f(x + \delta x, y + \delta y)$ ...(*ii*)

where $x + \delta x = \phi(t + \delta t),\ y + \delta y = \psi(t + \delta t)$

Now,
$$\frac{dz}{dt} = \lim_{\delta t \to 0} \frac{\delta z}{\delta t} = \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x,y)}{\delta t} \quad \text{(from } ii\text{)}$$

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y) + f(x+\delta x, y) - f(x,y)}{\delta t}$$

{Adding and subtracting $f(x + \delta x, y)$}

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta t} + \lim_{\delta t \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta t}$$

$$= \lim_{\delta t \to 0} \left[\frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}\right] + \lim_{\delta t \to 0} \left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x} \cdot \frac{\delta x}{\delta t}\right]$$

Also, as $\delta t \to 0$, $\delta x \to 0$, $\delta y \to 0$

$$\therefore \quad \frac{dz}{dt} = \lim_{\delta t \to 0} \left[\frac{\partial f(x,y)}{\partial y} \frac{\delta y}{\delta t}\right] + \lim_{\delta t \to 0} \left[\frac{\partial f(x,y)}{\partial x} \frac{\delta x}{\delta t}\right]$$

$$= \frac{\partial f(x,y)}{\partial y} \frac{dy}{dt} + \frac{\partial f(x,y)}{\partial x} \cdot \frac{dx}{dt}$$

$$\therefore \quad \boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}} \quad \text{...(iii) (As } z = f(x, y)\text{)}$$

In general
$$\frac{dz}{dt} = \frac{\partial z}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial z}{\partial x_2} \frac{dx_2}{dt} + \dots + \frac{\partial z}{\partial x_n} \frac{dx_n}{dt}$$

The above relation can be also written as

$$\boxed{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$, **which is called total differential of z.**

Corollary: If $z = f(x, y)$ and suppose y is the function of x, then f is a function of one independent variable x. Here y is intermediate variable. Identifying t with x in (*iii*), we get

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \frac{dx}{dx} + \frac{\partial z}{\partial y} \frac{dy}{dx} \Rightarrow \boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx}}.$$

1.7.1 Change of Variables

Let $z = f(x, y)$ where $x = \phi(s, t)$ and $y = \psi(s, t)$ then z is considered as function of s and t.

Now the derivative of z with respect s is partial but not total. Keeping t constant the equation (*iii*) modified as

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \quad \text{...(A)}$$

In a similar way, we get

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \cdot \qquad ...(B)$$

The equations (*A*) and (*B*) are known as **chain rule** for partial differentiation.

Example 1. Find the total differential coefficient of $x^2 y$ w.r.t. x when x, y are connected by $x^2 + xy + y^2 = 1.$

Sol. Let $z = x^2 y$ ...(*i*)

Then the total differential coefficient of z

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial x} \cdot \frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad ...(ii)$$

From (*i*) $\frac{\partial z}{\partial x} = 2xy, \frac{\partial z}{\partial y} = x^2$ and we have

$$x^2 + xy + y^2 = 1 \qquad ...(iii)$$

Differentiating w.r.t. x, we get

$$2x + \frac{dy}{dx} x + y + 2y \frac{dy}{dx} = 0$$

$$(2x + y) + (x + 2y) \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

Putting these values in equation (*ii*), we get

$$\frac{dz}{dx} = 2xy + x^2 \left(-\frac{2x+y}{x+2y}\right) = 2xy - \frac{x^2(2x+y)}{(x+2y)}.$$

Example 2. If $f(x, y) = 0, \phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \frac{\partial \phi}{\partial z} \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

Sol. We have $f(x, y) = 0$ ...(*i*)

$\phi(y, z) = 0$ ...(*ii*)

From (*i*)

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

From (*ii*)

$$\frac{d\phi}{dy} = \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dy} = 0 \Rightarrow \frac{dz}{dy} = -\frac{\left(\frac{\partial \phi}{\partial y}\right)}{\left(\frac{\partial \phi}{\partial z}\right)}$$

Multiplying these two results, we get

$$\frac{dz}{dx} = \frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} \times \frac{\left(\frac{\partial \phi}{\partial y}\right)}{\left(\frac{\partial \phi}{\partial z}\right)}$$

or $$\frac{\partial f}{\partial y}\,\frac{\partial \phi}{\partial z}\,\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}\cdot$$ **Hence proved.**

Example 3. If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$ *(U.P.T.U., 2005)*

Sol. Let $s = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$ and $t = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$

So $$\frac{\partial s}{\partial x} = -\frac{1}{x^2},\ \frac{\partial s}{\partial y} = \frac{1}{y^2},\ \frac{\partial t}{\partial x} = -\frac{1}{x^2},\ \frac{\partial t}{\partial z} = \frac{1}{z^2},\ \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial s}{\partial z} = 0$$

Since $u = u(s, t)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial t}\left(-\frac{1}{x^2}\right)$$

or $$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \quad ...(i)$$

Next, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\,\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\,\frac{\partial t}{\partial y}$$

or $$\frac{\partial u}{\partial y} = \frac{1}{y^2}\,\frac{\partial u}{\partial s} + 0 \Rightarrow y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \quad ...(ii)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\,\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\,\frac{\partial t}{\partial z} = 0 + \frac{1}{z^2}\,\frac{\partial u}{\partial t}$$

$$\Rightarrow \quad z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} \quad ...(iii)$$

Adding (*i*), (*ii*) and (*iii*), we get

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0.$$ **Hence proved.**

Example 4. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}$, where

$$x = \xi\cos\alpha - \eta\sin\alpha,\ y = \xi\sin\alpha + \eta\cos\alpha.$$

Sol. We have $x = \xi \cos \alpha - \eta \sin \alpha,\ y = \xi \sin \alpha + \eta \cos \alpha.$

$$\therefore \quad \frac{\partial x}{\partial \xi} = \cos \alpha,\ \frac{\partial x}{\partial \eta} = -\sin \alpha \text{ and } \frac{\partial y}{\partial \xi} = \sin \alpha,\ \frac{\partial y}{\partial \eta} = \cos \alpha$$

Now, Let $u = u(x, y)$

$$\Rightarrow \quad \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi} = \cos \alpha \frac{\partial u}{\partial x} + \sin \alpha \frac{\partial u}{\partial y} \qquad \ldots(i)$$

and

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} = -\sin \alpha \frac{\partial u}{\partial x} + \cos \alpha \frac{\partial u}{\partial y} \qquad \ldots(ii)$$

Again

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right) = \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}\right)\left(\cos\alpha \frac{\partial u}{\partial x} + \sin\alpha \frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial \xi^2} = \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + 2 \sin \alpha \cos \alpha \frac{\partial^2 u}{\partial x \partial y} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2} \qquad \ldots(iii)$$

and

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right) = \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right)\left(-\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial \eta^2} = \sin^2\alpha \frac{\partial^2 u}{\partial x^2} - 2 \sin \alpha \cos \alpha \frac{\partial^2 u}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2} \qquad \ldots(iv)$$

Adding (*iii*) and (*iv*), we get

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial x^2} \cdot (\cos^2\alpha + \sin^2\alpha) + (\cos^2\alpha + \sin^2\alpha) \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$ **Hence proved.**

Example 5. If $u = f(y - z, z - x, x - y)$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0. \qquad (\textit{U.P.T.U., 2003})$$

Sol. Let $r = y - z,\ s = z - x,\ t = x - y$

Then $u = f(r, s, t)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \times 0 + \frac{\partial f}{\partial s}(-1) + \frac{\partial f}{\partial t}(1) \qquad \left| \text{As } \frac{\partial r}{\partial x} = 0, \frac{\partial s}{\partial x} = -1, \frac{\partial t}{\partial x} = 1 \right.$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \qquad \ldots(i)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-1) \qquad \left| \text{As } \frac{\partial r}{\partial y} = 1, \frac{\partial s}{\partial y} = 0, \frac{\partial t}{\partial y} = -1 \right.$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \qquad \ldots(ii)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial f}{\partial r}(-1) + \frac{\partial f}{\partial s}(1) + \frac{\partial f}{\partial t}(0) \quad \Big| \text{ As } \frac{\partial r}{\partial z} = -1, \frac{\partial s}{\partial z} = 1, \frac{\partial t}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial z} = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \qquad ...(iii)$$

Adding equations (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ **Hence proved.**

Example 6. If $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r};\ \frac{\partial x}{r\partial\theta} = r\frac{\partial \theta}{\partial x} \text{ and find the value of } \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}.$$

Sol. We have $$x = r\cos\theta,\ y = r\sin\theta$$

$$\Rightarrow \quad r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \quad 2r\cdot\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta \qquad ...(i)$$

and we have $$x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta \qquad ...(ii)$$

From equations (*i*) and (*ii*), we get

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}.$$ **Hence proved.**

Now, $$\frac{\partial x}{\partial \theta} = -r\sin\theta \Rightarrow \frac{1}{r}\frac{\partial x}{\partial\theta} = -\sin\theta \qquad ...(iii)$$

and $$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$= -\frac{1}{r}\sin\theta \Rightarrow r\frac{\partial\theta}{\partial x} = -\sin\theta \qquad ...(iv)$$

From equations (*iii*) and (*iv*), we obtain

$$\frac{1}{r}\frac{\partial x}{\partial\theta} = r\frac{\partial\theta}{\partial x}.$$ **Hence proved.**

Since $$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2+y^2} \Rightarrow \frac{\partial^2\theta}{\partial x^2} = \frac{y\times 2x}{\left(x^2+y^2\right)^2} \Rightarrow \frac{\partial^2\theta}{\partial x^2} = \frac{2xy}{\left(x^2+y^2\right)^2} \qquad ...(v)$$

and $$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{\left(x^2+y^2\right)} \quad \therefore \frac{\partial^2\theta}{\partial y^2} = -\frac{2xy}{\left(x^2+y^2\right)^2} \qquad ...(vi)$$

Adding equations (*v*) and (*vi*), we get

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Example 7. If $\phi(x, y, z) = 0$, show that

$$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1. \qquad \text{(U.P.T.U., 2004)}$$

Sol. We have $\quad \phi(x, y, z) = 0$

Keeping y as constant, differentiate partially w.r.t. x, we get

$$\left(\frac{\partial \phi}{\partial x}\right) + \left(\frac{\partial \phi}{\partial z}\right)\cdot\left(\frac{\partial z}{\partial x}\right)_y = 0$$

$$\Rightarrow \qquad \left(\frac{\partial z}{\partial x}\right)_y = -\frac{\left(\frac{\partial \phi}{\partial x}\right)}{\left(\frac{\partial \phi}{\partial z}\right)} \qquad ...(i)$$

Next, keeping z as constant, differentiate partially w.r.t. y, we obtain

$$\frac{\partial \phi}{\partial x}\left(\frac{\partial x}{\partial y}\right)_z + \frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow \qquad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\frac{\partial \phi}{\partial y}\right)}{\left(\frac{\partial \phi}{\partial x}\right)} \qquad ...(ii)$$

Similarly,
$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\left(\frac{\partial \phi}{\partial z}\right)}{\left(\frac{\partial \phi}{\partial y}\right)} \qquad ...(iii)$$

Multiplying equations (*i*), (*ii*) and (*iii*), we get

$$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1.$$ **Hence proved.**

Example 8. If $x + y = 2e^{\theta} \cos \phi$ and $x - y = 2i\, e^{\theta} \sin \phi$, show that

$$\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial \phi^2} = 4xy \frac{\partial^2 V}{\partial x \partial y}\cdot \qquad \text{(U.P.T.U., 2001)}$$

Sol. We have
$$x + y = 2e^{\theta} \cos \phi$$
$$x - y = 2ie^{\theta} \sin \phi$$

Adding
$$2x = 2\,(e^{\theta})\,(\cos \phi + i \sin \phi)$$
$$x = e^{\theta + i\phi} \qquad ...(i)$$

and subtracting, we get $\quad y = e^{\theta - i\phi} \qquad ...(ii)$

Let
$$V = V(x, y)$$

$$\therefore \qquad \frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial V}{\partial \theta} = x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} \qquad \left| \text{As } \frac{\partial x}{\partial \theta} = e^{\theta + i\phi} = x, \frac{\partial y}{\partial \theta} = e^{\theta - i\phi} = y \right.$$

$$\therefore \quad \frac{\partial^2 V}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial \theta}\right) = \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial \theta^2} = x^2\frac{\partial^2 V}{\partial x^2} + y^2\frac{\partial^2 V}{\partial y^2} + 2xy\frac{\partial^2 V}{\partial x \partial y} + \left(x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}\right) \qquad ...(iii)$$

Now $$\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \phi} = \frac{\partial V}{\partial x}(ix) + \frac{\partial V}{\partial y}(-iy) \quad \Bigg| \text{As} \frac{\partial x}{\partial \phi} = ix = \frac{\partial y}{\partial \phi} = -iy$$

$$\Rightarrow \quad \frac{\partial V}{\partial \phi} = i\left(x\frac{\partial V}{\partial x}-y\frac{\partial V}{\partial y}\right)$$

Next $$\frac{\partial^2 V}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \phi}\right) = i\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)i\left(x\frac{\partial V}{\partial x}-y\frac{\partial V}{\partial y}\right)$$

$$= -\left(x^2\frac{\partial^2 V}{\partial x^2}+y^2\frac{\partial^2 V}{\partial y^2}-2xy\frac{\partial^2 V}{\partial x\partial y}+x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}\right)$$

$$\frac{\partial^2 V}{\partial \phi^2} = -x^2\frac{\partial^2 v}{\partial x^2}-y^2\frac{\partial^2 v}{\partial y^2}+2xy\frac{\partial^2 v}{\partial x\partial y}-x\frac{\partial v}{\partial x}-y\frac{\partial v}{\partial y} \qquad ...(iv)$$

Adding equations (*iii*) and (*iv*), we get

$$\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial \phi^2} = 4xy\frac{\partial^2 V}{\partial x \partial y}.$$ **Hence proved.**

Example 9. If $x = r\cos\theta$, $y = r\sin\theta$, $z = f(x, y)$, prove that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta \;;\; \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial z}{\partial \theta}\cos\theta.$$

Prove also that $\dfrac{\partial^2\left(r^n.\cos n\theta\right)}{\partial x\,\partial y} = -n(n-1)\,r^{n-2}\cdot\sin(n-2)\,\theta.$

Sol. Here z is a function of x and y where x and y are functions of r and θ.

$$\therefore \text{ We have} \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \qquad ...(i)$$

$$\text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} \qquad ...(ii)$$

Now, $x = r\cos\theta$, $y = r\sin\theta$, so $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

Then, $$\frac{\partial r}{\partial x} = \cos\theta,\ \frac{\partial r}{\partial y} = \sin\theta,\ \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \text{ and } \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Substituting these values in equations, (*i*) and (*ii*), we have

$$\frac{\partial z}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial z}{\partial \theta}.$$ **Hence proved.** ...(*iii*)

and $$\frac{\partial z}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial z}{\partial \theta}.$$ **Hence proved.** ...(*iv*)

Again substituting $r^n \cos n\theta$ for z in (*iv*), we get

$$\frac{\partial}{\partial y}(r^n \cos n\theta) = \sin\theta \cdot \frac{\partial}{\partial r}(r^n \cos n\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}(r^n \cos n\theta)$$

$$= \sin\theta\,(nr^{n-1}\cos n\theta) + \frac{\cos\theta}{r}(-r^n n \sin n\theta)$$

$$= nr^{n-1}(\cos n\theta \sin\theta - \cos\theta \sin n\theta)$$

or $$\frac{\partial\left(r^n \cos n\theta\right)}{\partial y} = nr^{n-1}\sin(\theta - n\theta) \qquad ...(v)$$

Now, $$\frac{\partial^2\left(r^n \cos n\theta\right)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial\left(r^n \cos n\theta\right)}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[nr^{n-1}\sin(1-n)\theta\right], \text{ from } (v) = n\frac{\partial}{\partial x}\left[r^{n-1}\sin(1-n)\theta\right]$$

$$= n\left[\cos\theta\frac{\partial}{\partial r}\left\{r^{n-1}\sin(1-n)\theta\right\} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left\{r^{n-1}\sin(1-n)\theta\right\}\right]$$

$$= n\left[\cos\theta\,(n-1)\,r^{n-2}\sin(1-n)\,\theta - \frac{\sin\theta}{r}r^{n-1}(1-n)\cos(1-n)\,\theta\right]$$

$$= -\,n\,(n-1)\,r^{n-2}\left[\sin(n-1)\theta\cos\theta - \cos(n-1)\,\theta\sin\theta\right]$$

$$= -\,n\,(n-1)\,r^{n-2}\sin(n-2)\,\theta.$$ **Hence proved.**

Example 10. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Or

Transform $\left(\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial y^2}\right) = 0$ into polars and show that $u = (Ar^n + Br^{-n})\sin n\theta$ satisfies the above equation.

Sol. We know $x = r\cos\theta$, $y = r\sin\theta$ ...(*i*)

$\therefore$ $$r^2 = x^2 + y^2 \qquad ...(ii)$$

and $$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad ...(iii)$$

From (*ii*), we get $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r}$, from (*i*)

or $$\frac{\partial r}{\partial x} = \cos\theta \qquad ...(iv)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta \qquad ...(v)$$

Also from (*iii*), $$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

or $$\frac{\partial \theta}{\partial x} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \quad ...(vi)$$

and $$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \quad ...(vii)$$

Now, we know $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}(\cos\theta) + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right), \text{ from } (iv), (vi)$$

or $$\frac{\partial}{\partial x}(u) = \cos\theta\,\frac{\partial}{\partial r}(u) - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}(u) \quad ...(viii)$$

Replacing u by $\frac{\partial u}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta.\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right),$$

$$= \cos\theta\cdot\frac{\partial}{\partial r}\left[\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right]$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \sin\theta\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}\right)\right] - \frac{\sin\theta}{r}\left[\left(-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\cdot\frac{\partial^2 u}{\partial \theta\,\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\cdot\frac{\partial u}{\partial \theta}\right)\right]$$

or $$\frac{\partial^2 u}{\partial x^2} = \cos\theta\left[\cos\theta\cdot\frac{\partial^2 u}{\partial r^2} - \sin\theta\left\{\frac{1}{r}\cdot\frac{\partial^2 u}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial u}{\partial \theta}\right\}\right]$$

$$- \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\cdot\frac{\partial^2 u}{\partial r\,\partial \theta} - \frac{1}{r}\left(\sin\theta\frac{\partial^2 u}{\partial \theta^2} + \cos\theta\cdot\frac{\partial u}{\partial \theta}\right)\right]$$

or $$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta\,\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 u}{\partial r\,\partial \theta} + \frac{\sin^2\theta}{r^2}\,\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\,\frac{\partial u}{\partial r}$$

$$+ \frac{2\cos\theta\sin\theta}{r^2}\,\frac{\partial u}{\partial \theta} \quad ...(ix)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\,\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 u}{\partial r\,\partial \theta} + \frac{\cos^2\theta}{r^2}\,\frac{\partial^2 u}{\partial \theta^2}$$

$$+ \frac{\cos^2\theta}{r}\,\frac{\partial u}{\partial r} - \frac{2\cos\theta\sin\theta}{r^2}\,\frac{\partial u}{\partial \theta} \quad ...(x)$$

Adding equations (*ix*) and (*x*), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta)\,\frac{\partial^2 u}{\partial \theta^2}$$

$$+ \frac{1}{r}(\sin^2\theta + \cos^2\theta)\,\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\,\frac{\partial u}{\partial r}.$$

Hence proved.

Or

If $u = (Ar^n + Br^{-n}) \sin n\theta$, then

$$\frac{\partial u}{\partial r} = n\,(Ar^{n-1} - Br^{-n-1}) \sin n\theta; \quad \frac{\partial u}{\partial \theta} = (Ar^n + Br^{-n})\, n \cos n\theta$$

and
$$\frac{\partial^2 u}{\partial r^2} = n\,[A(n-1)r^{n-2} + B\,(n+1)\,r^{-n-2}] \sin n\theta;$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = -\,(Ar^n + Br^{-n})\, n^2 \sin n\theta$$

$$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = n[A\,(n-1)\,r^{n-2} + B\,(n+1)\,r^{-n-2}] \sin n\theta$$

$$- \frac{1}{r^2}\,(Ar^n + Br^{-n})n^2 \sin n\,\theta + \frac{1}{r}\, n(Ar^{n-1} - Br^{-n-1}) \sin n\theta$$

$$= [A\{n^2 - n - n^2 + n\}r^{n-2} + B(n^2 + n - n^2 - n)r^{-n-2}] \sin n\theta$$

$= 0.$ **Hence proved.**

Example 11. If $x = r \cos \theta$, $y = r \sin \theta$ or $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}.$$

Sol. Since $x = r \cos \theta$ and $y = r \sin \theta$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r};\ \frac{\partial r}{\partial y} = \frac{y}{r};\ \frac{\partial^2 r}{\partial x^2} = \frac{r^2 - x^2}{r^3};\ \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3},$$

Adding
$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{(r^2 - x^2)}{r^3} + \frac{(r^2 - y^2)}{r^3} = \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{2r^2 - r^2}{r^3},\ \because x^2 + y^2 = r^2$$

or
$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \qquad \text{...}(i)$$

Also,
$$\frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\} = \frac{1}{r}\left\{\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right\} = \frac{1}{r}\left(\frac{x^2 + y^2}{r^2}\right)$$

$$= \frac{1}{r}\left(\frac{r^2}{r^2}\right),\ \because x^2 + y^2 = r^2$$

$= \dfrac{1}{r} = \dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2}$, from (i) **Hence proved.**

Example 12. If $V = f\,(2x - 3y, 3y - 4z, 4z - 2x)$, compute the value of $6V_x + 4V_y + 3V_z$.
(*U.P.T.U., 2008*)

Sol. Let $r = 2x - 3y$, $s = 3y - 4z$, $t = 4z - 2x$

$\therefore$ $V = f(r, s, t)$

$$V_x = \frac{\partial V}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial f}{\partial r}\cdot 2+\frac{\partial f}{\partial s}\cdot 0+\frac{\partial f}{\partial t}(-2) \qquad \left| \text{As } \frac{\partial r}{\partial x}=2, \frac{\partial s}{\partial x}=0, \frac{\partial t}{\partial x}=-2 \right.$$

$$\Rightarrow \quad V_x = 2f_r - 2f_t \qquad ...(i)$$

$$V_y = \frac{\partial V}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial f}{\partial r}(-3)+\frac{\partial f}{\partial s}(3)+\frac{\partial f}{\partial t}\cdot 0 \qquad \left| \text{As } \frac{\partial r}{\partial y} = -3, \frac{\partial s}{\partial y} = 3, \frac{\partial t}{\partial y} = 0 \right.$$

$$\Rightarrow \quad V_y = -3f_r + 3f_s \qquad ...(ii)$$

Similarly

$$V_z = 4f_r - 4f_s \qquad ...(iii)$$

Multiplying (*i*), (*ii*) and (*iii*) by 6, 4, 3 respectively and adding, we get

$$6V_x + 4V_y + 3V_z = 12f_r - 12f_t - 12f_r + 12f_s + 12f_r - 12f_s$$

$$\Rightarrow \quad 6V_x + 4V_y + 3V_z = 0.$$ **Hence Proved.**

Example 13. If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$. [*U.P.T.U. (C.O.), 2005*]

Sol. By total differentiation, we know that

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \qquad ...(i)$$

we have $u = x \log xy$

$$\frac{\partial u}{\partial x} = \log xy + \frac{1}{y}\cdot y = \log xy + 1 \qquad ...(ii)$$

$$\frac{\partial u}{\partial y} = \frac{x}{xy}(x) = \frac{x}{y} \qquad ...(iii)$$

Also, given that

$$x^3 + y^3 + 3xy = 1$$

Differentiating w.r.t. '*x*', we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x^2 + y) + (x + y^2)\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{(x^2+y)}{(x+y^2)} \qquad ...(iv)$$

Using (*ii*), (*iii*) and (*iv*) in (*i*), we get

$$\frac{du}{dx} = (1 + \log xy) - \frac{x}{y}\left(\frac{x^2+y}{x+y^2}\right)$$

Example 14. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and f is a function of x, y, z, show that $u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w} = x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} + 3z\frac{\partial f}{\partial z}$.

Sol. Let $f = f(x, y, z)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{\partial f}{\partial x} + (w+v)\frac{\partial f}{\partial y} + vw\frac{\partial f}{\partial z} \qquad \left| \text{As } \frac{\partial x}{\partial u} = 1, \frac{\partial y}{\partial u} = (w+v), \frac{\partial z}{\partial u} = vw \right.$$

or
$$u\frac{\partial f}{\partial u} = u\frac{\partial f}{\partial x} + u(w+v)\frac{\partial f}{\partial y} + uvw\frac{\partial f}{\partial z} \qquad ...(i)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial f}{\partial x} + (u+w)\frac{\partial f}{\partial y} + uw\frac{\partial f}{\partial z} \qquad \left| \text{As } \frac{\partial x}{\partial v} = 1, \frac{\partial y}{\partial v} = (u+w), \frac{\partial z}{\partial v} = uw \right.$$

or
$$v\frac{\partial f}{\partial v} = v\frac{\partial f}{\partial x} + v(u+w)\frac{\partial f}{\partial y} + uvw\frac{\partial f}{\partial z} \qquad ...(ii)$$

Similarly

$$w\frac{\partial f}{\partial w} = w\frac{\partial f}{\partial x} + w(u+v)\frac{\partial f}{\partial y} + uvw\frac{\partial f}{\partial z} \qquad ...(iii)$$

Adding (*i*), (*ii*) and (*iii*), we get

$$u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w} = (u+v+w)\frac{\partial f}{\partial x} + 2(vw+wu+uv)\frac{\partial f}{\partial y} + 3uvw\frac{\partial f}{\partial z}$$

or
$$u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w} = x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} + 3z\frac{\partial f}{\partial z}. \quad \textbf{Proved.}$$

Example 15. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \phi(u, v)$ show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right).$$

Sol. We have $f(x, y) = \phi(u, v)$

Differentiating partialy w.r.t. 'x'.

$$\frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x} = 2x\frac{\partial \phi}{\partial u} + 2y\frac{\partial \phi}{\partial v} \qquad \left| \text{As } \frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial x} = 2y \right.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial \phi}{\partial u} + 2y\frac{\partial \phi}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(2x\frac{\partial \phi}{\partial u} + 2y\frac{\partial \phi}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(2x\frac{\partial \phi}{\partial u} + 2y\frac{\partial \phi}{\partial v}\right)\cdot\frac{\partial v}{\partial x}$$

$$= \left(2x\frac{\partial^2\phi}{\partial u^2} + 2y\frac{\partial^2\phi}{\partial u\partial v}\right).2x + \left(2x\frac{\partial^2\phi}{\partial v\partial u} + 2y\frac{\partial^2\phi}{\partial v^2}\right).2y$$

or
$$\frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2\phi}{\partial u^2} + 8xy\frac{\partial^2\phi}{\partial u\partial v} + 4y^2\frac{\partial^2\phi}{\partial v^2} \quad ...(i)$$

Again differentiating $f(x, y)$ partially w.r. to y

$$\frac{\partial f}{\partial y} = \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} = -2y\frac{\partial\phi}{\partial u} + 2x\frac{\partial\phi}{\partial v} \quad \Bigg| \text{As } \frac{\partial u}{\partial y} = -2y, \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-2y\frac{\partial\phi}{\partial u} + 2x\frac{\partial\phi}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(-2y\frac{\partial\phi}{\partial u} + 2x\frac{\partial\phi}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(-2y\frac{\partial\phi}{\partial u} + 2x\frac{\partial\phi}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \left(-2y\frac{\partial^2\phi}{\partial u^2} + 2x\frac{\partial^2\phi}{\partial u\partial v}\right)(-2y) + \left(-2y\frac{\partial^2\phi}{\partial u\partial v} + 2x\frac{\partial^2\phi}{\partial v^2}\right)(2x)$$

or
$$\frac{\partial^2 f}{\partial y^2} = 4y^2\frac{\partial^2\phi}{\partial u^2} - 8xy\frac{\partial^2\phi}{\partial u\partial v} + 4x^2\frac{\partial^2\phi}{\partial v^2} \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\left(x^2+y^2\right)\frac{\partial^2\phi}{\partial u^2} + 4\left(x^2+y^2\right)\frac{\partial^2\phi}{\partial v^2}$$

$$= 4\left(x^2+y^2\right)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right). \quad \text{Hence Proved.}$$

Example 16. If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$

Sol. We have

$$x^2 + y^2 + z^2 - 2xyz = 1$$

or
$$x^2 - 2xyz + y^2 z^2 = 1 - y^2 - z^2 + y^2 z^2$$

$$(x - yz)^2 = (1 - y^2)(1 - z^2)$$

or
$$(x - yz) = \sqrt{(1-y^2)}\ \sqrt{(1-z^2)} \quad ...(i)$$

Again
$$y^2 - 2xyz + z^2 x^2 = 1 - x^2 - z^2 + z^2 x^2$$

or
$$(y - zx)^2 = (1 - x^2)(1 - z^2)$$

or
$$(y - zx) = \sqrt{(1-x^2)}\ \sqrt{(1-z^2)} \quad ...(ii)$$

Similarly
$$(z - xy) = \sqrt{(1-x^2)}\cdot\sqrt{(1-y^2)} \quad ...(iii)$$

Let
$$u \equiv x^2 + y^2 + z^2 - 2xyz - 1 = 0$$

By total differentiation, we get

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

or $(2x - 2yz)dx + (2y - 2zx)dy + (2z - 2xy)dz = 0$

$$\left| \begin{aligned} \text{As } \frac{\partial u}{\partial x} &= 2x - 2yz \\ \frac{\partial u}{\partial y} &= 2y - 2zx \\ \frac{\partial u}{\partial z} &= 2z - 2xy \end{aligned} \right.$$

or $(x - yz)dx + (y - zx)dy + (z - xy)dz = 0$ ...(*iv*)

Putting (*i*), (*ii*) and (*iii*) in equation (*iv*), we get

$$\sqrt{(1-y^2)}\ \sqrt{(1-z^2)}\ .dx + \sqrt{(1-x^2)}\ \sqrt{(1-z^2)}\ dy + \sqrt{(1-x^2)}\ \sqrt{(1-y^2)}\ dz = 0$$

Dividing by $\sqrt{(1-x^2)}\ \sqrt{(1-y^2)}\ \sqrt{(1-z^2)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0. \quad \text{Hence proved.}$$

EXERCISE 1.6

1. Find $\frac{dy}{dx}$ if $x^y + y^x = c$. $\left[\textbf{Ans. } \frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]\right]$

2. If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$. $\left[\textbf{Ans. } \frac{du}{dx} = (1 + \log xy) + \frac{x}{y}\left[-\frac{(x^2+y)}{(y^2+x)}\right]\right]$

3. If $u = x^2 y$, where $x^2 + xy + y^2 = 1$, find $\frac{du}{dx}$. $\left[\textbf{Ans. } \frac{du}{dx} = 2xy - x^2\left[\frac{(2x+y)}{(x+2y)}\right]\right]$

4. If V is a function of u, v where $u = x - y$ and $v = x - y$, prove that

$$x\frac{\partial^2 V}{\partial x^2} + y\frac{\partial^2 V}{\partial y^2} = (x + y)\left(\frac{\partial^2 V}{\partial u^2} + xy\frac{\partial^2 V}{\partial v^2}\right).$$

5. Transform the Laplacian equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ by change of variables from x, y to r, θ when $x = e^r \cos\theta$, $y = e^r \sin\theta$. $\left[\textbf{Ans. } e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right) = 0\right]$

6. Find $\frac{du}{dx}$, if $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$. $\left[\textbf{Ans. } \frac{du}{dx} = 1 + \log xy - \frac{x}{y}.\frac{x^2+y}{x+y^2}\right]$

7. If the curves $f(x, y) = 0$ and $\phi(x, y) = 0$ touch, show that at the point of contact

$$\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial x}.$$

8. Find $\frac{dy}{dx}$, when $(\cos x)^y = (\sin y)^x$. $\left[\textbf{Ans. } \frac{dy}{dx} = \frac{y\tan x + \log\sin y}{\log\cos x - x\cot y}\right]$

9. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial y\partial x}\right)^2.$$

10. If z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

11. If $u = \log(\tan x + \tan y + \tan z)$ prove that

$$(\sin 2x)\frac{\partial u}{\partial x} + (\sin 2y)\frac{\partial u}{\partial y} + (\sin 2z)\frac{\partial u}{\partial z} = 2.$$

12. If $u = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$ and $l^2 + m^2 + n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

13. If $u = x^2 + 2xy - y\log z$, where $x = s + t^2$, $y = s - t^2$, $z = 2t$, find

$\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}$ at (1, 2, 1). $\left[\textbf{Ans. } \frac{\partial u}{\partial s} = 8, \frac{\partial u}{\partial t} = 8t - 4\right]$

14. If $u = x^2 - y^2 + \sin yz$, where $y = e^x$ and $z = \log x$, find $\frac{du}{dx}$.

$\left[\textbf{Ans. } 2\left(x - e^x\right) + e^x\cos\left(e^x\log x\right)\left(\log x + \frac{1}{x}\right)\right]$

15. If $z = z(u, v)$, $u = x^2 - 2xy - y^2$ and $v = y$, show that $(x + y)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y} = 0$ is equivalent to $\frac{\partial z}{\partial v} = 0$.

CURVE TRACING

Introduction

It is analytical method in which we draw approximate shape of any curve with the help of symmetry, intercepts, asymptotes, tangents, multiple points, region of existence, sign of the first and second derivatives. In this section, we study tracing of standard and other curves in the cartesian, polar and parametric form.

1.8 PROCEDURE FOR TRACING CURVES IN CARTESIAN FORM

The following points should be remembered for tracing of cartesian curves:

1.8.1 Symmetry

(a) Symmetric about x-axis: If all the powers of y occurring in the equation are even then the curve is symmetrical about x-axis.

Example. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ y^2 = 4ax.$ (*U.P.T.U., 2008*)

(b) Symmetric about y-axis: If all the powers of x occurring in the equation are even then the curve is symmetrical about y-axis.

Example. $x^2 = 4ay,\ x^4 + y^4 = 4x^2y^2.$

(c) Symmetric about both x- and y-axis: If only even powers of x and y appear in equation then the curve is symmetrical about both axis.

Example. $x^2 + y^2 = a^2.$ (*U.P.T.U., 2008*)

(d) Symmetric about origin: If equation remains unchanged when x and y are replaced by $-x$ and $-y$.

Example. $x^5 + y^5 = 5a^2x^2y.$

Remark: Symmetry about both axis is also symmetry about origin but not the converse (due to odd powers).

(e) Symmetric about the line $y = x$: A curve is symmetrical about the line $y = x$, if on interchanging x and y its equation does not change.

Example. $x^3 + y^3 = 3axy.$ (*U.P.T.U., 2008*)

(f) Symmetric about $y = -x$: A curve is symmetrical about the line $y = -x$, if the equation of curve remains unchanged by putting $x = -y$ and $y = -x$ in equation.

Example. $x^3 - y^3 = 3axy.$ (*U.P.T.U., 2008*)

1.8.2 Regions

(a) Region where the curve exists: It is obtained by solving y in terms of x or vice versa. Real horizontal region is defined by values of x for which y is defined. Real vertical region is defined by values of y for which x is defined.

(b) Region where the curve does not exist: This region is also called imaginary region, in this region y becomes imaginary for values of x or vice versa.

1.8.3 Origin and Tangents at the Origin

If there is no constant term in the equation then the curve passes through the origin otherwise not.

If the curve passes through the origin, then the tangents to the curve at the origin are obtained by equating to zero the lowest degree terms.

Example. The curve $a^2y^2 = a^2x^2 - x^4$, lowest degree term $(y^2 - x^2)$ equating to zero gives $y = \pm x$ as the two tangents at the origin.

1.8.4 Intercepts

(a) Intersection point with x- and y-axis: Putting $y = 0$ in the equation we can find points where the curve meets the x–axis. Similarly, putting $x = 0$ in the equation we can find the points where the curve meets y-axis.

(b) Points of intersection: When curve is symmetric about the line $y = \pm x$, the points of intersection are obtained by putting $y = \pm x$ in given equation of curve.

(c) Tangents at other points say (h, k) can be obtained by shifting the origin to these points (h, k) by the substitution $x = x + h$, $y = y + k$ and calculating the tangents at origin in the new xy plane.

Remark. The point where $dy/dx = 0$, the tangent is parallel to x-axis. And the point where $dy/dx = \infty$, the tangent is vertical *i.e.*, parallel to y-axis.

1.8.5 Asymptotes

If there is any asymptotes then find it.

(a) Parallel to x-axis: Equate the coefficient of the highest degree term of x to zero, if it is not constant.

(b) Parallel to y-axis: Equate the coefficient of the highest degree term of y to zero, if it is not constant.

Example. $x^2y - y - x = 0$

highest power coefficient of x *i.e.*, $x^2 = y$

Thus asymptote parallel to x-axis is $y = 0$

Similarly asymptote parallel to y-axis are $x^2 - 1 = 0 \Rightarrow x = \pm 1$.

(c) Oblique asymptotes (not parallel to x-axis and y-axis): The asymptotes are given by

$y = mx + c$, where $m = \lim\limits_{x \to \infty} \left(\dfrac{y}{x}\right)$ and $c = \lim\limits_{x \to \infty} (y - mx)$.

(d) Oblique asymptotes (when curve is represented by implicit equation $f(x, y) = 0$): The asymptotes are given by $y = mx + c$ where m is solution of $\phi_n(m) = 0$ and c is the solution of

$c\phi_n'(m) + \phi_{n-1}(m) = 0$ or $c = \dfrac{-\phi_{n-1}(m)}{\phi_n'(m)}$. Here $\phi_n(m)$ and $\phi_{n-1}(m)$ are obtained by putting $x = 1$ and

$y = m$ in the collection of highest degree terms of degree n and in the collection of the next highest degree terms of degree $(n - 1)$.

1.8.6 Sign of First Derivatives *dy/dx* ($a \le x \le b$)

(a) $\dfrac{dy}{dx} > 0$, then curve is increasing in $[a, b]$.

(b) $\dfrac{dy}{dx} < 0$, then curve is decreasing in $[a, b]$.

(c) If $\dfrac{dy}{dx} = 0$, then the point is stationary point where maxima and minima can occur.

1.8.7 Sign of Second Derivative $\dfrac{d^2y}{dx^2}$ ($a \le x \le b$)

(a) $\dfrac{d^2y}{dx^2} > 0$, then curve is convex or concave upward (holds water).

(b) $\dfrac{d^2y}{dx^2} < 0$, then the curve is concave or concave downward (spills water).

1.8.8 Point of Inflexion

A point where $d^2y/dx^2 = 0$ is called an inflexion point where the curve changes the direction of concavity from downward to upward or vice versa.

Example 1. Trace the curve $y = x^3 - 3ax^2$.

Sol. 1. Symmetry: Here the equation of curve do not hold any condition of symmetry. So there is no symmetry.

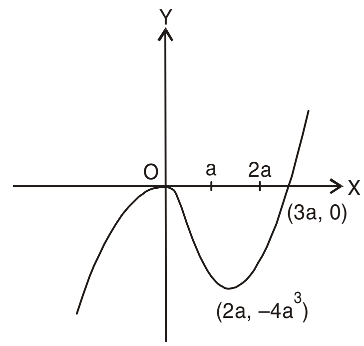


Fig. 1.3

2. Origin: Since there is no constant add in equation so the curve passes through the origin. The equation of tangent at origin is $y = 0$ *i.e.*, x-axis (lowest degree term).

3. Intercepts: Putting $y = 0$ in given equation, we get

$$x^3 - 3ax^2 = 0 \Rightarrow x = 0, 3a$$

Thus, the curve cross x-axis at (0, 0) and $(3a, 0)$.

4. There is no asymptotes.

5. $\dfrac{dy}{dx} = 3x^2 - 6ax.$

For stationary point $\dfrac{dy}{dx} = 0 = 3x^2 - 6ax = 0 \Rightarrow x = 0, 2a.$

6. $\dfrac{d^2y}{dx^2} = 6x - 6a,\ \left(\dfrac{d^2y}{dx^2}\right)_{x=0} = -6a < 0$ (concave) and $y_{\max} = 0$ and $\left(\dfrac{d^2y}{dx^2}\right)_{x=2a} = 12a - 6a = 6a > 0$ (convax) and $y_{\min.} = -4a^3$.

7. Inflexion point: $\dfrac{d^2y}{dx^2} = 0 \Rightarrow 6x - 6a = 0 \Rightarrow x = 0, a.$

8. Region: $-\infty < x < \infty$ since y is defined for all x.

9. Sign of derivative

Interval	*Sign of y*	*Quadrant*	*Sign of y′*	*Nature of curve*
$-\infty < x < 0$	$y < 0$	III	$y' > 0$	increasing
$0 < x < 2a$	$y < 0$	IV	$y' < 0$	decreasing
$2a < x < 3a$	$y < 0$	IV	$y' > 0$	increasing
$3a < x < \infty$	$y > 0$	I	$y' > 0$	increasing

Using the above calculations. We draw the graph in Figure 1.3.

Example 2. Trace the curve $y^2(a - x) = x^3,\ a > 0.$ *(U.P.T.U., 2006)*

Sol. 1. Symmetry: Since y has even power so the curve is symmetric about x-axis.

2. Origin: The curve passes through the origin.

3. Tangent at origin: The coefficient of lowest degree term is $y^2 = 0$ or $y = 0$ and $y = 0$ *i.e.*, there is a cusp at the origin.

4. Intercepts: Putting $x = 0$, then $y = 0 \Rightarrow$ origin is the only point where the curve meets the co-ordinate axes.

5. Asymptotes: Asymptotes parallel to y-axis obtained by equating to zero the highest degree term of y *i.e.*, $(x - a) = 0 \Rightarrow x = a$.

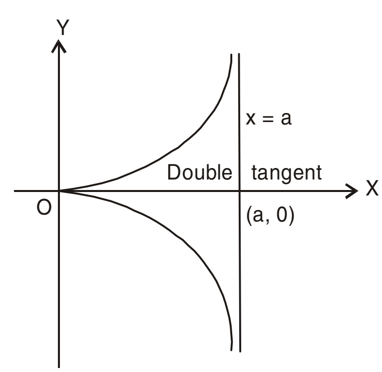


Fig. 1.4

6. **Region:** $y^2 = \dfrac{x^3}{a-x}$ this shows that the values of $x > a$, y is imaginary. So the curve does not exist $x > a$. Similarly, the curve does not exist when $x < 0$.

Here the curve only for $0 \le x < a$.

7. Sign of derivation:

$\dfrac{dy}{dx} = \pm \dfrac{x^{\frac{1}{2}}\left(\dfrac{3a}{2} - x\right)}{(a-x)\sqrt{a-x}}$, in the first quadrant for $0 \leqslant x < a$, $\dfrac{dy}{dx} > 0$, curve increasing in the first quadrant.

The shape of figure is shown in the (Fig. 1.4).

Example 3. Trace the following curve and write its asymptotes. (*U.P.T.U., 2003*)

$$x^3 + y^3 = 3axy.$$

Sol. 1. Symmetry: Interchage x and y. Then equation of curve remain unchanged.

$\therefore$ The curve is symmetric about the line $y = x$.

2. Origin: The curve passes through origin.

3. Tangent at origin: The coefficient of lowest degree term is $x = 0$ or $y = 0$

$\therefore$ $x = 0$ and $y = 0$ are tangents at origin.

4. Intercepts: Putting $y = x$, we get

$$2x^3 = 3ax^2 \Rightarrow 2x^3 - 3ax^2 = 0$$

$$\Rightarrow x^2 (2x - 3a) = 0$$

$$\Rightarrow \quad x = 0 \text{ and } x = \frac{3a}{2}$$

At $x = 0, y = 0$ and at $x = \dfrac{3a}{2}, y = \dfrac{3a}{2}$

Thus, points of intersection are (0, 0) and $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$ along the line $y = x$.

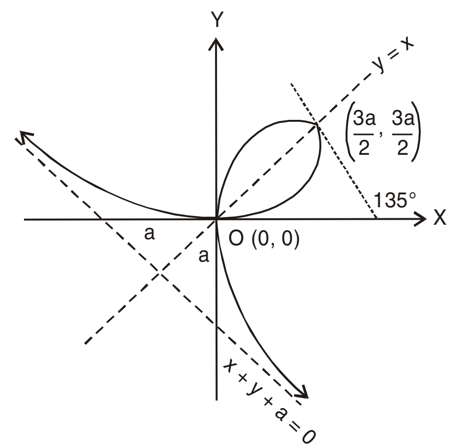


Fig. 1.5

5. Asymptotes: Since the coefficients of highest powers of x and y are constants, there are no asymptotes parallel to x-and y-axis.

Putting $x = 1$ and $y = m$ in highest degree term $(x^3 + y^3)$, we get

$$\phi_3(m) = 1 + m^3 = 0$$

$$\Rightarrow \quad m = -1 \text{ (Real solution)}$$

Again putting $x = 1$ and $y = m$

in next highest degree term ($- 3axy$), we get

$$c = \frac{(-3am)}{3m^2} = \frac{a}{m}$$

At $m = -1, c = -a.$

$$\text{Asymptotes } y = mx + c \Rightarrow y = -x-a \text{ or } y + x + a = 0.$$

6. Derivative: $\dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax} \Rightarrow \left(\dfrac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = -1$ *i.e.*, the tangent at $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$ making an angle 135° with x-axis.

7. Region: When both x and y are negative simultaneously equation of curve is not satisfied

$$(-x)^3 + (-y)^3 = 3a\,(-x)\,(-y) \Rightarrow -\,(x^3 + y^3) = 3axy$$

$\Rightarrow$ There is no part of the curve exists in 3rd quadrant

The shape of the curve is shown in Fig. 1.5.

Example 4. Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (Astroid).

Sol. We have $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \Rightarrow \left(\dfrac{x}{a}\right)^{2/3} + \left(\dfrac{y}{a}\right)^{2/3} = 1$

or
$$\left(\frac{x^2}{a^2}\right)^{1/3} + \left(\frac{y^2}{a^2}\right)^{1/3} = 1 \qquad ...(i)$$

1. Symmetry: Since there are even powers of x and y so the curve is symmetric about both axis.

2. Origin: Since there is constant term so the curve does not passes through the origin.

3. Intercept: Putting $y = 0$, we get $x = \pm a$.

$\Rightarrow$ The curve cross x-axis at $(a, 0)$ and $(-a, 0)$

Similarly, putting $x = 0$ then $y = \pm a$ *i.e.*, the curve cross y-axis at $(0, a)$ and $(0, -a)$.

4. Derivative: $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{\frac{1}{3}} \Rightarrow \left(\dfrac{dy}{dx}\right)_{(a,0)} = 0$ *i.e.*, tangent at $(a, 0)$ is along x-axis and $\left(\dfrac{dy}{dx}\right)_{(0,a)} = -\infty = \tan\left(-\dfrac{\pi}{2}\right)$ *i.e.*, tangent at $(0, a)$ is y-axis.

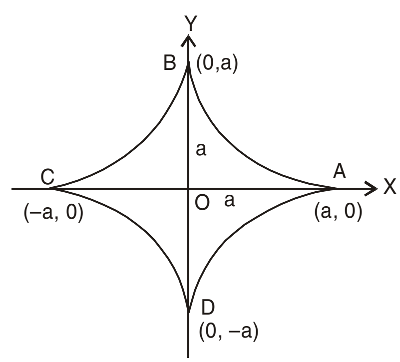


Fig. 1.6

5. Region: $\left(\dfrac{y^2}{a^2}\right)^{\frac{1}{3}} = 1 - \left(\dfrac{x^2}{a^2}\right)^{\frac{1}{3}}$

$\therefore$ if $\dfrac{x}{a} > 1$ *i.e.*, $x > a$, we have $\dfrac{y^2}{a^2} < 0$ *i.e.*, $y^2 < 0 = y$ is imaginary, so the curve does not exist for $x > a$. Similarly, the curve does not exist for $x < -a$ and $y > a$, $y < -a$.

6. Asymptotes: No asymptotes.

The shape of the figure is shown in the Fig. 1.6.

Example 5. Trace the curve $x^2y^2 = a^2\,(y^2 - x^2)$.

Sol. 1. Symmetry: The curve is symmetric about both axis.

2. Origin: The curve passes through origin.

3. Tangents: The tangents at the origin $y^2 - x^2 = 0$. *i.e.*, $y = \pm x$.

4. Intercepts: Cross the line at origin *i.e.*, (0, 0).

5. Region: $y^2 = a^2x^2/(a^2 - x^2)$ this shows that the curve does not exist for $x^2 > a^2$ *i.e.*, for $x > a$ and $x < -a$.

6. As $x \to a$, $y^2 \to \infty$ and as $x \to -a$, $y^2 \to \infty$

The shape of the curve is shown in Fig. 1.7.

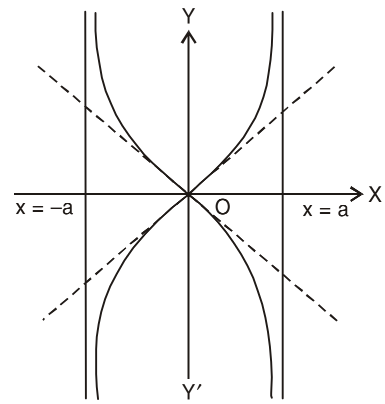


Fig. 1.7

Example 6. Trace the curve $y^2 (a + x) = x^2 (b - x)$, (Strophoid).

Sol. 1. Symmetry: There is only even power of y so the curve is symmetric about x-axis.

2. Origin: The curve passes through the origin.

3. Tangents at origin: The lowest degree term is $ay^2 - bx^2$

$$\therefore \text{ tangents are } ay^2 - ax^2 = 0 \Rightarrow y = \pm \sqrt{\frac{b}{a}}\, x.$$

4. Intercepts: Putting $y = 0$, so $x^2 (b - x) = 0$

$\Rightarrow$ $x = 0, b$

The curve meets x-axis at (0, 0) and (b, 0). y-intercept: put $x = 0$ then $y = 0$. So (0, 0) is the y-intercept.

5. Asymptotes: Asymptotes parallel to y-axis is

$$x + a = 0 \Rightarrow x = -a.$$

6. Region: $y = \pm x\sqrt{\frac{b-x}{a+x}}$, y becomes imaginary when $x > b$ and $x < -a$. Thus curve exist only in the region $-a < x < b$.

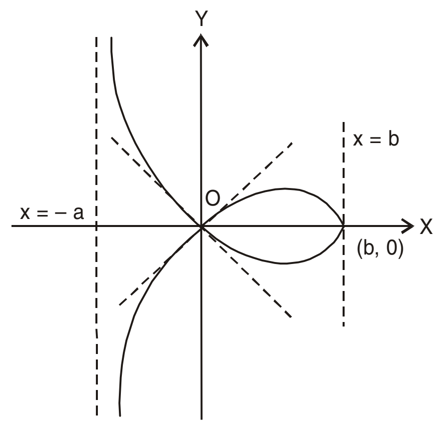


Fig. 1.8

7. Derivative: $\dfrac{dy}{dx} = \dfrac{(-2x^2 - 3ax + bx + 2ab)}{2(a+x)^{3/2}(b-x)} \Rightarrow \left(\dfrac{dy}{dx}\right)_{(b,0)} = \infty = \tan\dfrac{\pi}{2}$

Thus, the tangent at (b, 0) is parallel to y-axis.

Therefore from above calculations, we draw the curve (Fig. 1.8) of given equation.

Example 7. Trace the curve $y^2 (x^2 + y^2) + a^2 (x^2 - y^2) = 0$.

Sol. 1. Symmetry: The curve is symmetric about both axis.

2. Origin: Curve passes through origin.

3. Tangents at origin: The tangents at the origin are $x^2 - y^2 = 0$ or $y = \pm x$.

4. Intercepts: At x-axis = (0, 0)

At y-axis $\Rightarrow$ $y^4 - a^2y^2 = 0$ (put $x = 0$)

$y^2 (y^2 - a^2) = 0$

$\Rightarrow$ $y = \pm a, 0$

Thus, the curve cross y-axis at (0, a) and (0, – a) and (0, 0).

5. Tangent at new point $(0, -a)$ and $(0, a)$

Putting $y = y + a$, we get

$$(y + a)^2 \left[x^2 + (y+a)^2\right] + a^2\left[x^2 - (y+a)^2\right] = 0$$

$\therefore$ The tangent at the new points are $y = 0$, $y = 0$

i.e., parallel to x-axis.

6. Region: $x^2 = y^2 \, (a^2 - y^2)/(a^2 + y^2)$

The curve does not exist when $y^2 > a^2$

or $y > a$ and $y < -a$

$\therefore$ The curve exist in the region when $-a < y < a$.

7. No asymptotes: The shape of the curve shown in Fig. 1.9.

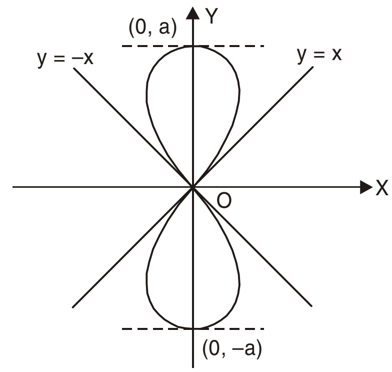


Fig. 1.9

Example 8. Trace the curve $x^2y^2 = a^2 \, (x^2 + y^2)$.

Sol. 1. Symmetry: Curve is symmetric about both axis.

2. Origin: Curve passes through the origin. Tangnt at (0, 0) are $x^2 + y^2 = 0 \Rightarrow y = \pm\, ix$, which give imaginary tangents. So (0, 0) is a conjugate point.

3. The curve does not cross the axis.

4. Asymptotes: Asymptotes are $x = \pm\, a$ and $y = \pm\, a$.

5. Region: $y^2 = a^2x^2/(x^2 - a^2)$. If $x^2 < a^2$ *i.e.*, $x < \pm\, a$ then y is imaginary *i.e.*, the curve does not exist when $x < a$ and $x < -a$.

Similarly does not exist when $y < a$ and $y < -a$

6. $y^2 \to \infty$ as $x \to a$ and $x^2 \to \infty$ as $y \to a$

$\therefore$ The shape of the curve is shown in Fig. 1.10.

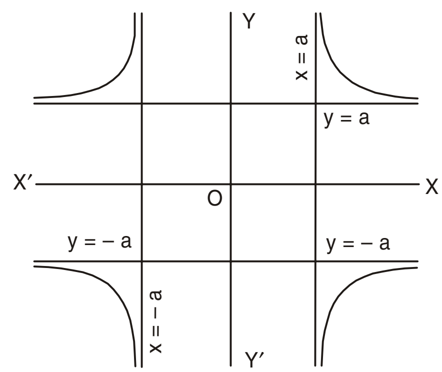


Fig. 1.10

Example 9. Trace the curve $(x^2 - a^2)\,(y^2 - b^2) = a^2b^2$.

Sol. The given curve is $(x^2 - a^2)\,(y^2 - b^2) = a^2b^2$

or $\quad x^2y^2 - b^2x^2 - a^2y^2 = 0$ or $x^2y^2 = b^2x^2 + a^2y^2$

1. Symmetry about both the axes.

2. (0, 0) satisfies the equation of the curve. Tangents at the origin are $a^2y^2 + b^2x^2 = 0$, which give imaginary tangents. So (0, 0) is a conjugate point.

3. The curve does not cross the axes.

4. Equating the coefficients of highest powers of x and y we find that $x = \pm\, a$ and $y = \pm\, b$ are the asymptotes.

5. Solving for y, we get $y^2 = \dfrac{b^2x^2}{x^2 - a^2}$.

$\therefore$ If $x^2 < a^2$ *i.e.*, x is numerically less than a, y^2 is negative *i.e.*, y is imaginary *i.e.*, the curve does not exist between the lines $x = -a$ and $x = a$.

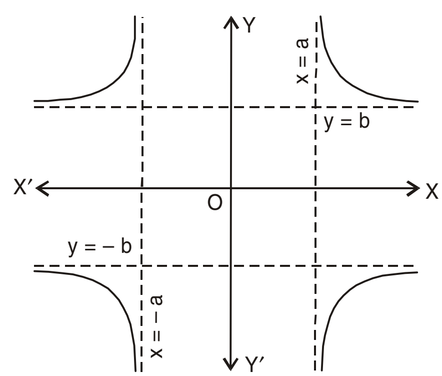


Fig. 1.11

Similarly arguing we find that the curve does not exist between the lines $y = -b$ and $y = b$.

6. $y^2 \to \infty$ as $x \to a$ and $x^2 \to \infty$ as $y \to a$.

With the above data, the shape of the curve is as shown in Fig. 1.11.

Example 10. Trace the curve $y^2 (a - x) = x^2 (a + x)$.

Sol. 1. Symmetry about x-axis.

2. Passes through (0, 0), the tangents are $y^2 = x^2$ or $y = \pm x$. Tangents being real and distinct, node is expected at the origin.

3. Curve crosses the x-axis at $(-a, 0)$ and (0, 0). Shifting the origin to $(-a, 0)$ and equating the lowest degree terms to zero, we get new y-axis as the tangent at the new origin.

4. $x = a$ is the asymptote.

5. For $x < -a$, the curve does not exist. Similarly for $x > a$, the curve does not exist.

6. As $x \to a$, $y^2 \to \infty$.

7. No point of inflexion.

$\therefore$ Shape of the curve is as shown in Fig. 1.12.

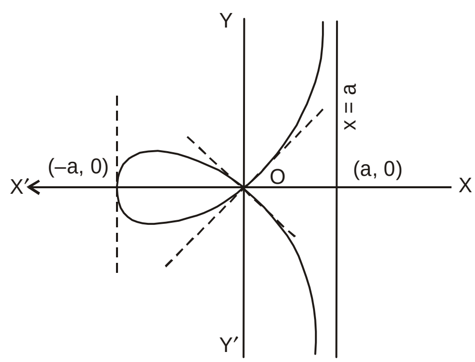
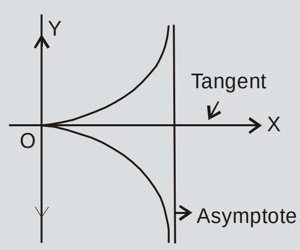


Fig. 1.12

EXERCISE 1.7

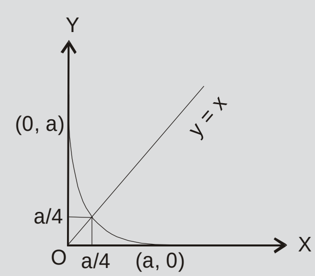
1. Trace the curve $y^2 (2a - x) = x^3$ *(U.P.T.U., 2004)*

Ans.



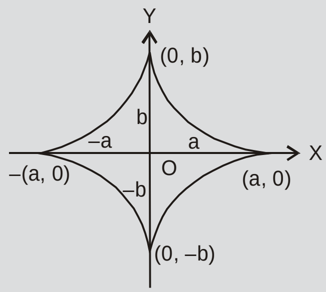
2. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ *(U.P.T.U. (C.O.), 2003)*

Ans.



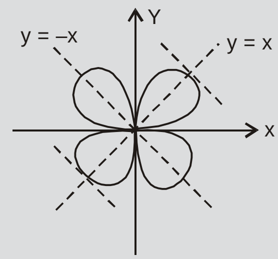
3. $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$

Ans.



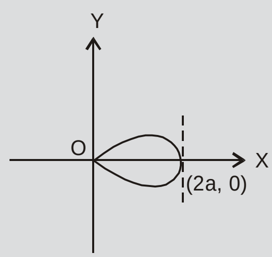
4. $x^6 + y^6 = a^2x^2y^2$

Ans.



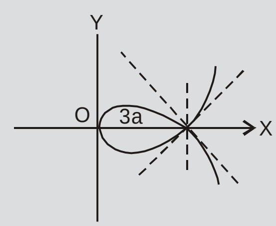
5. $a^2y^2 = x^3\ (2a - x)$

Ans.



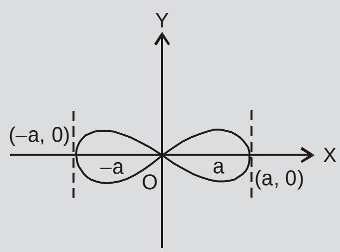
6. $9ay^2 = x(x - 3a)^2$

Ans.



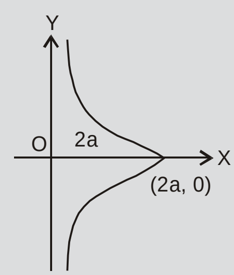
7. $a^4y^2 = a^2\ x^4 - x^6$

Ans.



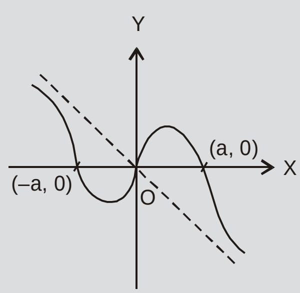
8. $xy^2 = 4a^2\ (2a - x)$

Ans.



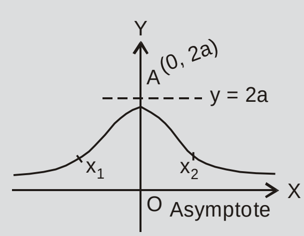
9. $y^3 = x(a^2 - x^2)$

Ans.



10. $y = 8a^3/(x^2 + 4a^2)$

Ans.



1.9 POLAR CURVES

The general form (explicit) of polar curve is $r = f(\theta)$ or $\theta = f(r)$ and the implicit form is $F(r, \theta) = 0$.

Procedure:

1. Symmetry: (*a*) If we replace θ by $-\theta$, the equation of the curve remains unchanged then there is symmetry about initial line $\theta = 0$ (usually the positive x-axis in cartesian form).

Example. $r = a\ (1 \pm \cos\ \theta)$.

(*b*) If we replace θ by $\pi - \theta$, the equation of the curve remains unchanged, then there is a symmetry about the line $\theta = \frac{\pi}{2}$ (passing through the pole and ⊥ to the initial line) which is usually the positive y-axis in cartesian).

Example. $r = a \sin 3\theta$.

(*c*) There is a symmetry about the pole (origin) if the equation of the curve remains unchanged by replacing r into $-r$.

Example. $r^2 = a \cos 2\theta$.

(*d*) Curve is symmetric about pole if $f(r, \theta) = f(r, \theta + \pi)$

Example. $r = 4 \tan\ \theta$.

(*e*) Symmetric about $\theta = \frac{\pi}{4}$ *i.e.*, $(y = x)$, if $f(r, \theta) = f\left(r, \frac{\pi}{2} - \theta\right)$

(*f*) Symmetric about $\theta = \frac{3\pi}{4}$ *i.e.*, $(y = -x)$, if $f(r, \theta) = f\left(r, \frac{3\pi}{2} - \theta\right)$

2. Pole (origin): If $r = f(\theta_1) = 0$ for some $\theta = \theta_1$ = constant then curve passes through the pole (origin) and the tangent at the pole (origin) is $\theta = \theta_1$.

Example. $r = a\ (1 + \cos\ \theta) = 0$, at $\theta = \pi$.

3. Point of intersection: Points of intersection of the curve with initial line and line $\theta = \frac{\pi}{2}$ are obtained by putting $\theta = 0$ and $\theta = \frac{\pi}{2}$.

4. Region: If r^2 is negative *i.e.*, imaginary for certain values of θ then the curve does not exist for those values of θ.

5. Asymptote: If $\lim_{\theta \to \alpha} r = \infty$ then an asymptote to the curve exists and is given by equation

$$r \sin (\theta - \alpha) = f'(\alpha)$$

where α is the solution of $\frac{1}{f(\theta)} = 0$.

6. Tangent at any point (r, θ): Tangent at this is obtained from $\tan \phi = \frac{r d\theta}{dr}$, where ϕ is the angle between radius vector and the tangent.

7. Plotting of points: Solve the equation for r and consider how r varies as θ varies from 0 to ∞ or 0 to $-\infty$. The corresponding values of r and θ give a number of points. Plot these points. This is sufficient for tracing of the curve. (Here we should observe those values of θ for which r is zero or attains a minimum or maximum value).

Example 1. Trace the curve $r^2 = a^2 \cos 2\theta$ (*U.P.T.U., 2000, 2008*)

Sol. 1. Symmetry: Since there is no change in the curve when θ replace by – θ. So the curve is symmetric about initial line.

2. Pole: Curve passes through the pole when $r^2 = a^2 \cos 2\theta = 0$

i.e., $\cos 2\theta = 0 \Rightarrow 2\theta = \pm \dfrac{\pi}{2}$ or $\theta = \pm \dfrac{\pi}{4}$

Hence, the straight lines $\theta = \pm \dfrac{\pi}{4}$ are the tangents at origin to the curve.

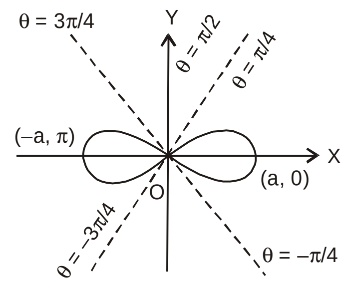


Fig. 1.13

3. Intersection: Putting θ = 0

∴ $r^2 = a^2 \Rightarrow r = \pm a$ the curve meets initial line to the points $(a, 0)$ and $(-a, \pi)$.

4. As θ varies from 0 to π, *r* varies as given below:

θ =	0	30	45	90	135	150	180
r^2 =	a^2	$a^2/2$	0	$-a^2$	0	$a^2/2$	a^2

←imaginary→

5. Region: The above data shows that curve does not exist for values of θ which lying between 45° and 135°.

Example 2. Trace the curve $r = a \sin 3\theta$ (*U.P.T.U., 2002*)

Sol. 1. Symmetry: The curve is not symmetric about the initial line.

2. Origin: Curve passes through the origin when $r = 0$

$$\Rightarrow \quad a \sin 3\theta = 0$$

$$\Rightarrow \quad 3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$$

$$\Rightarrow \quad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

are the tangents at the pole.

3. Asymptote: No asymptote since *r* is finite for any value of θ.

4. Region: Since the maximum value of sin 3θ is 1.

So, $$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{etc.}$$

or $$\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \text{etc.}$$

for which $r = a$ (maximum value).

∴ The curve exist in all quadrant about the lines $\theta = \dfrac{\pi}{6}, \dfrac{3\pi}{6}, \dfrac{5\pi}{6}$ at distance $r = a$.

The shape of the curve is given in the (Fig. 1.14).

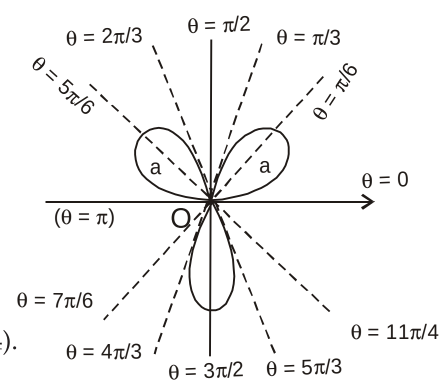


Fig. 1.14

Example 3. Trace the curve $r = ae^{\theta}$

Sol. 1. No symmetry about the initial line.

2. As $\theta \to \infty$, $r \to \infty$ and r always positive.

3. Corresponding values of θ and r are given below:

θ =	0	$\pi/2$	π	$3\pi/2$	2π
r =	a	$ae^{\pi a/2}$	$ae^{\pi a}$	$ae^{3\pi/2}$	$ae^{2\pi a}$

with the above data the shape of the curve is shown below:

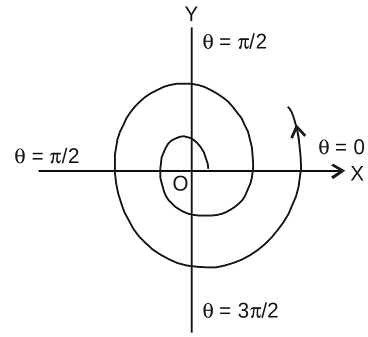


Fig. 1.15

Example 4. Trace the curve $r^2 \cos 2\theta = a^2$ *(Hyperbola)*

Sol. 1. Symmetry about pole and about the line $\theta = \frac{1}{2}\pi$.

2. Changing to cartesian the equation becomes

$$x^2 - y^2 = a^2.$$

$\therefore$ The equation of the asymptotes are $y = \pm x$ or $\theta = \pm \frac{1}{4}\pi$ are its polar asymptotes.

3. When $\theta = 0$, $r^2 = a^2$ or $r = \pm a$ *i.e.*, the points $(a, 0)$ and $(-a, 0)$ lie on the curve. (Here co-ordinates of the points are polar coordinates).

4. Solving for r we get $r^2 = a^2/\cos 2\theta$. This shows that as θ increases from 0 to $\frac{1}{4}\pi$, r increases from a to ∞.

5. For values of θ lying between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, r^2 is negative *i.e.*, r is imaginary. So the curve does not exist for $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$.

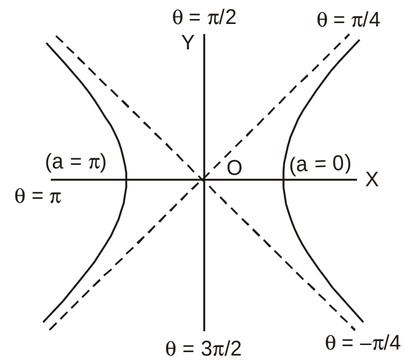


Fig. 1.16

Example 5. Trace the curve $r = a \cos 2\theta$.

Sol. 1. Symmetry about the initial line and the line $\theta = \frac{\pi}{2}$ *i.e.*, y-axis.

2. Putting $r = 0$, $\cos 2\theta = 0$ or $2\theta = \pm \frac{1}{2}\pi$ or $\theta = \pm \frac{1}{4}\pi$, *i.e.*, the straight lines $\theta = \pm \frac{1}{4}\pi$ are the tangents to the curve at the pole.

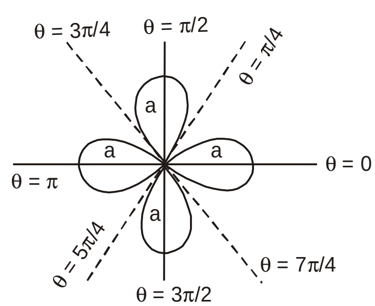


Fig. 1.17

3. Corresponding values of θ and r are given below:

$\theta = 0°$	30°	45°	60°	90°	120°	135°	150°	180°
$r = a$	$\frac{1}{2}a$	0	$-\frac{1}{2}a$	$-a$	$-\frac{1}{2}a$	0	$\frac{1}{2}a$	a

Plot these points and due to symmetry about the initial line the other portion can be traced.

Example 6. Trace the curve $r = a\,(1 - \cos\theta)$ (cardoid).

Sol. 1. Symmetry: No change in equation when we replace θ by $-\theta$. So the equation of curve is symmetric about initial line.

2. Pole (region): If the curve passes through origin then

$$r = 0 \Rightarrow a\,(1 - \cos\theta) = 0$$

$$\Rightarrow \quad \cos\theta = 1 = \cos 0 \Rightarrow \theta = 0.$$

Here, the straight line $\theta = 0$ is tangent at origin.

3. Intersection: Putting $\theta = 0$ then $r = 0$

and putting $\theta = \pi$, then $r = 2a$

$\therefore$ Intersection points on initial line = (0, 0) and $(2a, \pi)$.

4. Region: It exists in all quadrant.

5. Asymptotes: No. asymptotes.

6. As θ increases from 0 to π, r also increases from 0 to $2a$.

The corresponding values of r and θ given below:

$\theta = 0$	60°	90°	120°	180°
$r = a$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

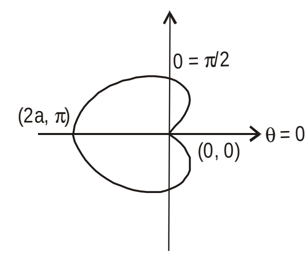


Fig. 1.18

With the above data the shape of the curve is given in (Fig. 1.18).

Example 7. Trace the curve $r = a + b\cos\theta$, when $a < b$.

Sol. 1. Symmetry: The curve is symmetric about initial line.

2. Origin: For origin $r = 0 \Rightarrow a + b\cos\theta = 0 \Rightarrow \cos\theta = -\frac{a}{b}$

3. Corresponding values of θ and r are given below:

$\theta = 0$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$r = a + b$	$a+\left(\frac{b}{\sqrt{2}}\right)$	$a+\left(\frac{b}{2}\right)$	a	$\left(a-\frac{b}{2}\right)$	$\left(a-\frac{b}{\sqrt{2}}\right)$	$(a - b)$

Let $a > \frac{b}{2}$ then r is positive for all values of θ from 0 to $\frac{2\pi}{3}$ but r is negative when $\theta = \frac{3}{4}\pi$ or π. Here, r must vanish somewhere between $\theta = \frac{2\pi}{3}$ and $\frac{3\pi}{4}$. Let $\theta = \alpha$ (lying between $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$).

For which $r = 0$ then $\theta = \alpha$ is the straight line which is tangent to the curve at the pole and for values of θ lying between α and π, r is negative and points corresponding to such values of θ will be marked in the opposite direction on these lines as r is negative for them.

Thus in this case $a < b$, the curve passes through the origin when $\theta = \alpha = \cos^{-1}\left\{-\left(\frac{a}{b}\right)\right\}$ and form two loops, one inside the other as shown in the Fig. 1.19.

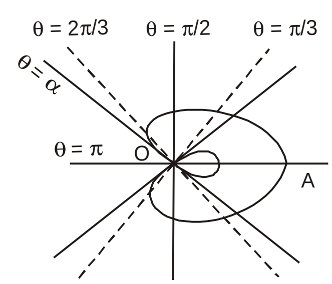


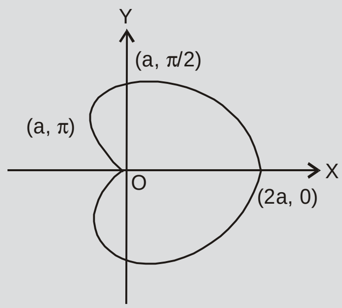
Fig. 1.19

EXERCISE 1.8

1. Trace the curve

$$r = a\,(1 + \cos\theta)$$

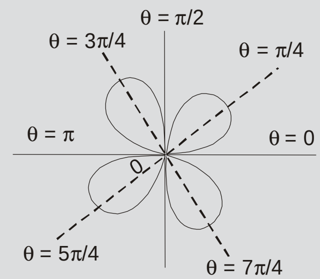
Ans.



2. Trace the curve

$$r = a\sin 2\theta$$

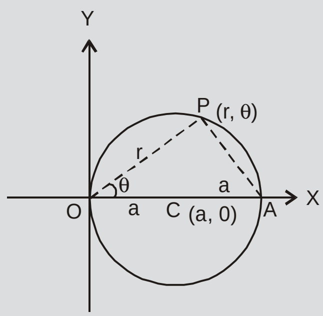
Ans.



3. Trace the curve

$$r = 2a\cos\theta$$

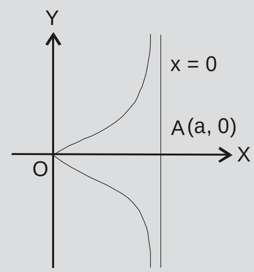
Ans.



4. Trace the curve

$$r = \frac{a\sin^2\theta}{\cos\theta}$$

Ans.

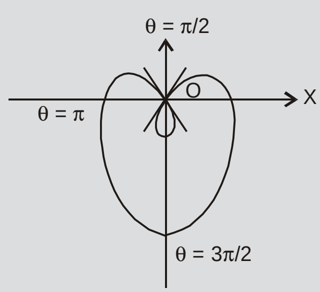


[**Hint:** Change it in cartesian coordinates, $y^2\,(x - a) = -\,x^3$]

5. Trace the curve

$$r = 2\,(1 - 2\sin\theta)$$

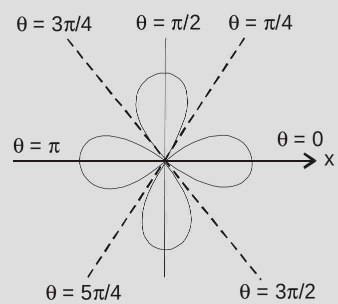
Ans.



6. Trace the curve (*U.P.T.U., 2003*)

$$r = a\cos 2\theta$$

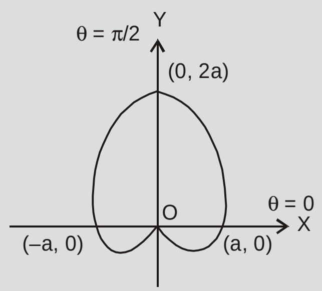
Ans.



7. Trace the curve

$$r = a\,(1 + \sin\theta)$$

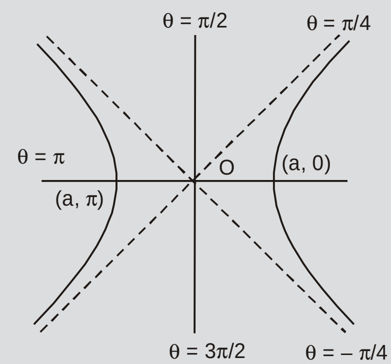
Ans.



8. Trace the curve

$$r^2 \cos 2\theta = a^2$$

Ans.



[**Hint:** Change in cartesian form $x^2 - y^2 = a^2$]

1.10 PARAMETRIC CURVES

Let $x = f_1(t)$ and $y = f_2(t)$ be the parametric equations of a curve where t is a parameter.

Method I: Eliminate the parameter 't' if possible and we shall get the cartesian equation of curve which can easily traced.

Example. $x = a \cos t$, $y = a \sin t \Rightarrow x^2 + y^2 = a^2$.

Method II: When the parameter 't' cannot be eliminated.

1. **Symmetry:** If $x = f_1(t)$ is even and $y = f_2(t)$ is odd then curve is symmetric about x-axis: Similarly, if $x = f_1(t)$ is odd and $y = f_2(t)$ is even then curve is symmetric about y-axis.
2. **Origin:** Find 't' for which $x = 0$ and $y = 0$.
3. **Intercept:** x-intercept obtained for values of t for which $y = 0$, y-intercept for values of t for which $x = 0$.
4. Determine least and greatest values of x and y.
5. **Asymptotes:** $\lim\limits_{t \to t_1} x(t) = \infty$, $\lim\limits_{t \to t_1} y(t) = \infty$, then $t = t_1$ is asymptote.
6. **Tangents:** $\dfrac{dy}{dx} = \infty$ (vertical tangent) and $\dfrac{dy}{dx} = 0$ (horizontal tangent).

Remark. If the given equations of the curves are periodic functions of t having a common period, then it is enough to trace the curve for one period.

Example 1. Trace the curve

$$x = a\,(t - \sin t),\ y = a\,(1 - \cos t) \qquad (\textit{Cycloid})$$

Sol. 1. Symmetry: Since x is odd and y is even so the curve is symmetric about y-axis.

2. Origin: Putting $x = 0$ and $y = 0$, we get

$$0 = a\,(t - \sin t) \Rightarrow t = 0$$

and

$$0 = a\,(1 - \cos t) \Rightarrow \cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi, \text{etc.}$$

3. $\dfrac{dx}{dt} = a\,(1 - \cos t)$, $\dfrac{dy}{dt} = a \sin t$ $\quad\therefore\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a\,(1-\cos t)}$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2a \sin t/2 . \cos t/2}{a(1-1+2\sin^2 t/2)} = \cot\frac{t}{2}$$

Corresponding values of x, y, $\dfrac{dy}{dx}$ for different values of t are given below:

$t =$	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π
$x =$	0	$a\left(\dfrac{\pi}{2}-1\right)$	$a\pi$	$a\left(\dfrac{3\pi}{2}+1\right)$	$2a\pi$
$y =$	0	a	$2a$	a	0
$\dfrac{dy}{dx} =$	∞	1	0	-1	$-\infty$

4. Tangents at $y = 0$ are vertical and at $y = 2a$ is horizontal. Curve is periodic for period 2π in the interval $[0, 2\pi]$. Curve repeats over intervals of $[0, 2\,a\pi]$ refer Fig. 1.20.

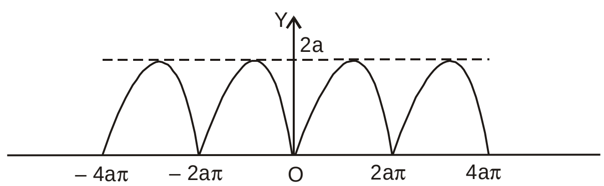


Fig. 1.20

Example 2. Trace the curve $x = a \cos^3 t$, $y = a \sin^3 t$. *(Astroid)*

Sol. 1. $dy/dt = 3a \sin^2 t \cos t$.

$$dx/dt = -3a \cos^2 t \sin t$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t}$$

or $$\frac{dy}{dx} = -\tan t.$$

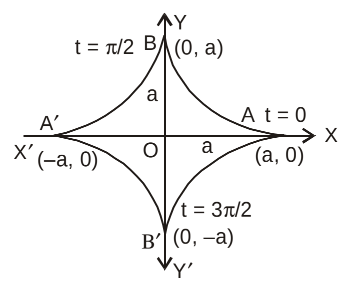


Fig. 1.21

2. Corresponding values of x, y and dy/dx for different values of t are given below:

$t =$	0	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	$\dfrac{3\pi}{2}$	π	$\dfrac{5\pi}{4}$	$\dfrac{3\pi}{2}$	$\dfrac{7\pi}{4}$	2π
$x =$	a	$\dfrac{a}{2\sqrt{2}}$	0	$\dfrac{-a}{2\sqrt{2}}$	$-a$	$\dfrac{a}{2\sqrt{2}}$	0	$\dfrac{a}{2\sqrt{2}}$	a
$y =$	0	$\dfrac{a}{2\sqrt{2}}$	a	$\dfrac{a}{2\sqrt{2}}$	0	$\dfrac{-a}{2\sqrt{2}}$	$-a$	$\dfrac{-a}{2\sqrt{2}}$	0
$\dfrac{dy}{dx} =$	0	-1	$-\infty$	1	0	$-\infty$	1	1	0

Plotting the above points and observing the inclinations of the tangents at these points the shape of the curve is as shown in Fig. 1.21.

Example 3. $x = a\,(t + \sin t),$
$y = a\,(1 - \cos t)$

Sol. 1. $dy/dt = a\,(\sin t);$
$dx/dt = a\,(1 + \cos t)$

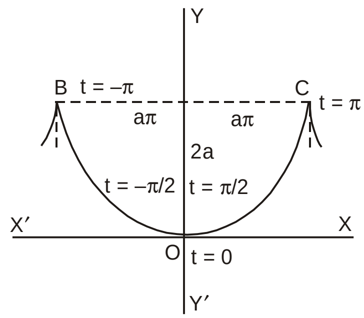


Fig. 1.22

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(\sin t)}{a(1+\cos t)}$$

$$= \frac{2a \sin \frac{1}{2}t \cos \frac{1}{2}t}{a(2\cos^2 \frac{1}{2}t)}$$

or $$dy/dx = \tan \frac{1}{2}t.$$

2. Corresponding values of x, y and dy/dx for different values of t are given below:

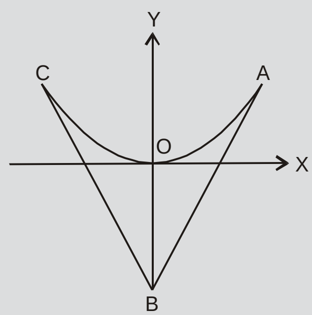
t	$= -\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π
x	$= -\pi a$	$-a\,(\frac{1}{2}\pi - 1)$	0	$a\,(\frac{1}{2}\pi + 1)$	$a\pi$
y	$= 2a$	a	0	a	$2a$
dy/dx	$= -\infty$	-1	0	1	∞

EXERCISE 1.9

Trace the following curves:

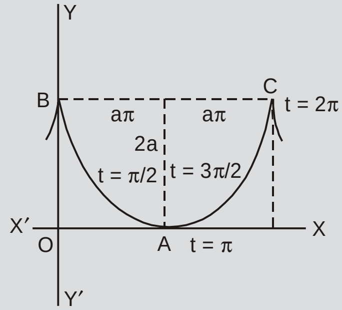
1. $x = a \sin 2t\,(1 + \cos 2t),\ y = a \cos 2t\,(1 - \cos 2t)$

Ans.



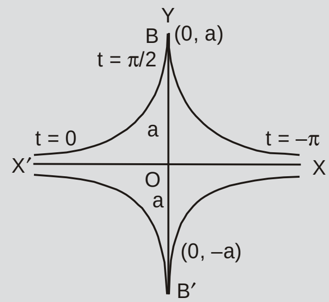
2. $x = a\,(t - \sin t),\ y = a\,(1 + \cos t)$

Ans.



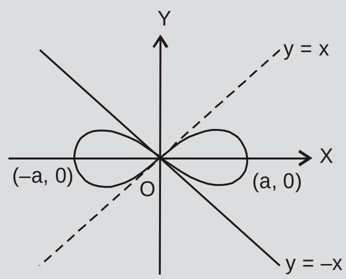
3. $x = a \cos t + \frac{1}{2} a \log \tan^2 \left(\frac{t}{2}\right)$, $y = a \sin t$ (*Tractrix*)

Ans.



4. $x = \frac{a(t+t^3)}{1+t^4}$, $y = \frac{a(t-t^3)}{1+t^4}$

Ans.



EXPANSION OF FUNCTION OF SEVERAL VARIABLES

1.11 TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Let $f(x, y)$ be a function of two independent variables x and y. If the function $f(x, y)$ and its partial derivatives up to nth order are continuous throughout the domain centred at a point (x, y). Then

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}\right]$$

$$+ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right]$$

$$+ \frac{1}{\lfloor 3}\left[h^3\frac{\partial^3 f(a,b)}{\partial x^3} + 3h^2k\frac{\partial^3 f(a,b)}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f(a,b)}{\partial x \partial y^2} + k^3\frac{\partial^3 f(a,b)}{\partial y^3}\right] + \ldots$$

Or

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a, b)$$

$$+ \frac{1}{\lfloor 3}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f(a,b) + \ldots$$

Proof. Suppose $P(x, y)$ and $Q\,(x + h, y + k)$ be two neighbouring points. Then $f\,(x + h, y + k)$, the value of f at Q can be expressed in terms of f and its derivatives at P.

Here, we treat $f(x + h, y + k)$ as a function of single variable x and keeping y as a constant. Expanded as follows using Taylor's theorem for single variable.*

$$f(x + h,\ y + k) = f(x,\ y + k) + h\ \frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{\lfloor 2}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \ldots \qquad \ldots(i)$$

Now expanding all the terms on the R.H.S. of (i) as function of y, keeping x as constant.

$$f(x + h,\ y + k) = \left[f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{\lfloor 2}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right]$$

$$+\ h\frac{\partial}{\partial x}\left[f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{\lfloor 2}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right]$$

$$+\ \frac{h^2}{\lfloor 2}\frac{\partial^2}{\partial x^2}\left[f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{\lfloor 2}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right] + \ldots$$

$$f(x + h,\ y + k) = f(x,\ y) + \left[h\frac{\partial f(x, y)}{\partial x} + k\frac{\partial f(x, y)}{\partial y}\right]$$

$$+\ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(x, y)}{\partial x^2} + 2hk\frac{\partial^2 f(x, y)}{\partial x \partial y} + k^2\frac{\partial^2 f(x, y)}{\partial y^2}\right] + \ldots$$

For any point (a, b) putting $x = a$, $y = b$ in above equation then, we get

$$f\,(a + h,\ b + k) = f\,(a,\ b) + \left[h\frac{\partial f\,(a, b)}{\partial x} + k\frac{\partial f\,(a, b)}{\partial y}\right]$$

$$+\ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f\,(a, b)}{\partial x^2} + 2hk\frac{\partial^2 f\,(a, b)}{\partial y^2} + k^2\frac{\partial^2 f\,(a, b)}{\partial y^2}\right] + \ldots$$

Or

$$f\,(a + h,\ b + k) = f\,(a,\ b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f(a, b) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f\,(a,\ b)$$

$$+\ \frac{1}{\lfloor 3}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f(a, b) + \ldots$$

Hence proved.

Alternative form:

Putting $a + h = x \Rightarrow h = x - a$

$b + k = y \Rightarrow k = y - b$

then
$$f\,(x, y) = f\,(a,\ b) + \left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y}\right] f\,(a,\ b) + \frac{1}{\lfloor 2}\left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y}\right]^2 f(a, b) + \ldots \qquad \ldots(ii)$$

---

* **Taylor's theorem for single variable**

$$f\,(x + h) = f\,(x) + h\ \frac{\partial f}{\partial x} + \frac{h^2}{\lfloor 2}\frac{\partial^2 f}{\partial x^2} + \frac{h^3}{\lfloor 3}\frac{\partial^3 f}{\partial x^3} + \ldots$$

1.11.1 Maclaurin's Series Expansion

It is a special case of Taylor's series when the expansion is about the origin (0, 0).

So, putting $a = 0$ and $b = 0$ in equation (2), we get

$$f(x, y) = f(0, 0) + \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] f(0,0) + \frac{1}{\lfloor 2}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^2 f(0,0) + \ldots$$

Example 1. Expand $e^x \cos y$ about the point $\left(1, \frac{\pi}{4}\right)$ (*U.P.T.U., 2007*)

Sol. We have $f(x, y) = e^x \cos y$ ...(*i*)

and $a = 1,\ b = \frac{\pi}{4},\ f\left(1, \frac{\pi}{4}\right) = e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}}$

$\therefore$ From (1) $\frac{\partial f}{\partial x} = e^x \cos y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial x} = e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}}$

$$\frac{\partial f}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial y} = -e \sin \frac{\pi}{4} = -\frac{e}{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y \Rightarrow \frac{\partial^2 f\left(1, \frac{\pi}{4}\right)}{\partial x^2} = \frac{e}{\sqrt{2}},\ \frac{\partial^2 f}{\partial x \partial y} = -e^x \sin y$$

$$\frac{\partial^2 f\left(1, \frac{\pi}{4}\right)}{\partial x \partial y} = -\frac{e}{\sqrt{2}},\ \frac{\partial^2 f}{\partial y^2} = -e^x \cos y = -\frac{e}{\sqrt{2}}.$$

By Taylor's theorem, we have

$$f\left(1 + h, \frac{\pi}{4} + k\right) = f\left(1, \frac{\pi}{4}\right) + \left(h\frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial x} + k\frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial y}\right) + \ldots \quad \ldots(ii)$$

Let $1 + h = x \Rightarrow h = x - 1$ and $\frac{\pi}{4} + k = y \Rightarrow k = y - \frac{\pi}{4}$, equation (2) reduce in the form

$$f((x, y) = e^x \cos y = \frac{e}{\sqrt{2}} + (x - 1) \cdot \frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \frac{1}{\lfloor 2}\left[(x-1)^2 \cdot \frac{e}{\sqrt{2}}\right.$$

$$\left. + 2(x - 1)\left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2\left(-\frac{e}{\sqrt{2}}\right)\right] + \ldots$$

$$\Rightarrow f(x, y) = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{(x-1)^2}{2} - (x-1)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2 + \ldots\right].$$

Example 2. Expand $f(x, y) = e^y \log(1 + x)$ in powers of x and y about (0, 0)

Sol. We have $f(x, y) = e^y \log(1 + x)$

Here, $a = 0$ and $b = 0$, then $f(0, 0) = e^0 \log 1 = 0$

Now, $\frac{\partial f}{\partial x} = \frac{e^y}{1+x} \Rightarrow \frac{\partial f}{\partial x}(0,0) = 1$

$$\frac{\partial f}{\partial y} = e^y \log(1 + x) \Rightarrow \frac{\partial f(0,0)}{\partial x} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{e^y}{1+x} \Rightarrow \frac{\partial^2 f(0,0)}{\partial x \partial y} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{e^y}{(1+x)^2} \Rightarrow \frac{\partial^2 f(0,0)}{\partial x^2} = -1$$

$$\frac{\partial^2 f}{\partial y^2} = e^y \log(1+x) \Rightarrow \frac{\partial^2 f(0,0)}{\partial y^2} = 0$$

Now, applying Taylor's theorem, we get

$$f(0+h, 0+k) = f(h,k) = f(0,0) + \left(h\frac{\partial f(0,0)}{\partial x} + k\frac{\partial f(0,0)}{\partial y}\right) + \frac{1}{\underline{|2}}\left(h^2 \frac{\partial^2 f(0,0)}{\partial x^2} + 2hk\frac{\partial^2 f(0,0)}{\partial x \partial y} + k^2\frac{\partial^2 f(0,0)}{\partial y^2}\right) + \ldots$$

Let $h = x$, $k = y$, then, we get

$$f(x,y) = e^y \log(1+x) = f(0,0) + \left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right) + \ldots$$

$$= 0 + (x \times 1 + y \times 0) + \frac{1}{\underline{|2}}[x^2(-1) + 2xy \times 1 + y^2 \times 0] + \ldots$$

$$\Rightarrow \quad e^y \log(1+x) = x - \frac{x^2}{2} + xy + \ldots .$$

Example 3. Find Taylor's series expansion of function $f(x, y) = e^{-x^2-y^2}$. cos xy about the point $x_0 = 0$, $y_0 = 0$ up to three terms. (*U.P.T.U., 2006*)

Sol. We have $f(x, y) = e^{-x^2-y^2} \cos xy$.

Now, we get the following terms $f(0, 0) = 1$.

$$\frac{\partial f}{\partial x} = -e^{-x^2-y^2}(2x\cos xy + y \sin xy) \Rightarrow \frac{\partial f(0,0)}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = -e^{-x^2-y^2}(2y\cos xy + x \sin xy) \Rightarrow \frac{\partial f(0,0)}{\partial y} = 0$$

Similarly, $\frac{\partial^2 f(0,0)}{\partial x \partial y} = 0, \quad \frac{\partial^2 f(0,0)}{\partial x^2} = -2, \quad \frac{\partial^2 f(0,0)}{\partial y^2} = -2$

$$\frac{\partial^3 f(0,0)}{\partial x^2 \partial y} = 0, \quad \frac{\partial^3 f(0,0)}{\partial x \partial y^2} = 0, \quad \frac{\partial^3 f(0,0)}{\partial x^3} = 0$$

$$\frac{\partial^3 f(0,0)}{\partial y^3} = 0$$

Applying Taylor's theorem

$$f(h,k) = f(0,0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(0,0) + \frac{1}{\underline{|2}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{\underline{|3}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(0,0) + \ldots$$

Putting $h = x$, $k = y$ and all values then, we get

$$f(x, y) = e^{-x^2-y^2} \cdot \cos xy = 1 + (x \times 0 + y \times 0) + \frac{1}{\lfloor 2}\left[x^2(-2) + 2xy \times 0 + y^2(-2)\right] + \frac{1}{\lfloor 3}\left[x^3 \times 0 + 3x^2y \times 0 + 3xy^2 \times 0 + y^3 \times 0\right]$$

$$\Rightarrow \quad e^{-x^2-y^2} . \cos xy = 1 - x^2 - y^2 \ .... \ .$$

Example 4. Find Taylor's expansion of $f(x, y) = \cot^{-1} xy$ in powers of $(x + 0.5)$ and $(y - 2)$ up to second degree terms. Hence compute $f(-0.4, 2.2)$ approximately.

Sol. Here $f(x, y) = \cot^{-1} xy$

$$f(-0.5, 2) = \cot^{-1}(-1) = \frac{3\pi}{4}$$

Now

$$\frac{\partial f}{\partial x} = \frac{-y}{1+x^2y^2} \Rightarrow \frac{\partial f(-0.5, 2)}{\partial x} = -1$$

$$\frac{\partial f}{\partial y} = \frac{-x}{1+x^2y^2} \Rightarrow \frac{\partial f(-0.5, 2)}{\partial y} = 1/4$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2y^2-1)}{(1+x^2y^2)^2} \Rightarrow \frac{\partial^2 f(-0.5, 2)}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy^3}{(1+x^2y^2)^2} \Rightarrow \frac{\partial^2 f(-0.5, 2)}{\partial x^2} = -2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2x^3y}{(1+x^2y^2)^2} \Rightarrow \frac{\partial^2 f(-0.5, 2)}{\partial y^2} = -\frac{1}{8}$$

Now applying Taylor's series expansion, we get

$$f(-0.5 + h, 2 + k) = f(-0.5, 2) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) \cdot f(-0.5, 2) + \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(-0.5, 2) + \ldots$$

Let

$$-0.5 + h = x \Rightarrow h = x + 0.5$$

$$2 + k = y \Rightarrow k = y - 2 \qquad | \text{ As } a = -0.5, b = 2$$

$$\therefore \quad f(x, y) = \cot^{-1} xy = \frac{3\pi}{4} + (x + 0.5)(-1) + (y - 2) \times \frac{1}{4} + \frac{1}{\lfloor 2}\left[(x+0.5)^2(-2) + 2(x+0.5)(y-2) \times 0 + (y-2)^2\left(-\frac{1}{8}\right)\right].$$

or

$$f(x, y) = \frac{3\pi}{4} - (x + 0.5) + \frac{1}{4}(y - 2) - (x + 0.5)^2 - \frac{1}{16}(y - 2)^2 + ...$$

Putting $x = -0.4$ and $y = 2.2$

$$f(-0.4, 2.2) = \frac{3\pi}{4} - (0.1) + \frac{0.2}{4} - (0.1)^2 - \frac{1}{16}(0.2)^2$$

$$= 2.29369.$$

Example 5. Calculate log $\left[(1.03)^{1/3}+(0.98)^{1/4}-1\right]$ approximately by using Taylor's expansion up to first order terms.

Sol. Let $f(x, y) = \log\left(x^{\frac{1}{3}}+y^{\frac{1}{4}}-1\right)$

$$f(1, 1) = \log 1 = 0$$

Now, $$\frac{\partial f}{\partial x} = \frac{1\times x^{-\frac{2}{3}}}{3\left(x^{\frac{1}{3}}+y^{\frac{1}{4}}-1\right)} \Rightarrow \frac{\partial f}{\partial x}(1,1) = \frac{1}{3}$$

| Taking $a = 1$, $b = 1$

$$\frac{\partial f}{\partial y} = \frac{1\times y^{-\frac{3}{4}}}{4\left(x^{\frac{1}{3}}+y^{\frac{1}{4}}-1\right)} \Rightarrow \frac{\partial f}{\partial x}(1,1) = \frac{1}{4}$$

Now, applying Taylor's theorem

$$f(1+h, 1+k) = f(1, 1) + \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(1,1)+\ldots$$

But $$f(1+h, 1+k) = \log\left[(1+h)^{\frac{1}{3}}+(1+k)^{\frac{1}{4}}-1\right]$$

$\therefore$ $$f(1+h, 1+k) = \log\left[(1+h)^{\frac{1}{3}}+(1+k)^{\frac{1}{4}}-1\right] = 0 + h\times\frac{1}{3} + k\times\frac{1}{4} \quad \ldots(i)$$

Putting $h = 0.03$ and $k = -0.02$ in equation (*i*) then, we get

$$\log\left[(1.03)^{\frac{1}{3}}+(0.98)^{\frac{1}{4}}-1\right] = 0.03\times\frac{1}{3} + (-0.02)\times\frac{1}{4}$$
$$= 0.005.$$

Example 6. Expand x^y in powers of $(x - 1)$ and $(y - 1)$ up to the third degree terms. (*U.P.T.U., 2003*)

Sol. Here $f(x, y) = x^y$ $\quad f(1, 1) = 1$

Now $$\frac{\partial f}{\partial x} = yx^{y-1} \Rightarrow \frac{\partial f(1,1)}{\partial x} = 1$$

$$\frac{\partial f}{\partial y} = x^y \log x \Rightarrow \frac{\partial f(1,1)}{\partial y} = 0$$

$$\frac{\partial^2 f}{\partial x\partial y} = x^{y-1} + yx^{y-1}\cdot\log x \Rightarrow \frac{\partial^2 f(1,1)}{\partial x\partial y} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2} \Rightarrow \frac{\partial^2 f(1,1)}{\partial x^2} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = x^y\cdot(\log x)^2 \Rightarrow \frac{\partial^2 f(1,1)}{\partial y^2} = 0$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = yx^{y-1}(\log x)^2 + 2x^{y-1} \cdot \log x \quad \Rightarrow \quad \frac{\partial^3 f(1,1)}{\partial x \partial y^2} = 0$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = (y-1)x^{y-2} + y(y-1)\, x^{y-2} \cdot \log x + yx^{y-2} \quad \Rightarrow \quad \frac{\partial^3 f(1,1)}{\partial x^2 \partial y} = 1$$

$$\frac{\partial^3 f}{\partial x^3} = y(y-1)\,(y-2)x^{y-3} \quad \Rightarrow \quad \frac{\partial^3 f(1,1)}{\partial x^3} = 0$$

$$\frac{\partial^3 f}{\partial y^3} = x^y(\log x)^3 \quad \Rightarrow \quad \frac{\partial^3 f(1,1)}{\partial y^3} = 0$$

Now applying Taylor's theorem, we get

$$f(1+h, 1+k) = f(1,1) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(1,1) + \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(1,1)$$

$$+ \frac{1}{\lfloor 3}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 \cdot f(1,1) + \ldots$$

Let $\quad 1 + h = x$ and $1 + k = y$

$\Rightarrow \quad h = x - 1$ and $k = y - 1$

$$\therefore \quad f(x,y) = f(1,1) + \left[(x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right] f + \frac{1}{\lfloor 2}\left[(x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^2 f$$

$$+ \frac{1}{\lfloor 3}\left[(x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^3 f + \ldots$$

Using all values in above equation, we get

$$f(x,y) = x^y = 1 + (x-1) + 0 + \frac{1}{\lfloor 2}\left[0 + 2\,(x-1)(y-1) + 0\right]$$

$$+ \frac{1}{\lfloor 3}\left[0 + 3\,(x-1)^2(y-1) + 0 + 0\right]$$

$$= 1 + (x-1) + (x-1)\,(y-1) + \frac{1}{2}(x-1)^2(y-1).$$

Example 7. Obtain Taylor's expansion of $\tan^{-1}\frac{y}{x}$ about (1, 1) up to and including the second degree terms. Hence compute f (1.1, 0.9). (*U.P.T.U., 2002, 2005*)

Sol. Here

$$f(x,y) = \tan^{-1}\frac{y}{x} \quad \therefore \quad f(1,1) = \frac{\pi}{4}$$

Now, $$\frac{\partial f}{\partial x} = -\frac{y}{(x^2+y^2)} \quad \Rightarrow \quad \frac{\partial f(1,1)}{\partial x} = -\frac{1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{x}{(x^2+y^2)} \Rightarrow \frac{\partial f}{\partial y}(1,1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \Rightarrow \frac{\partial^2 f}{\partial x^2}(1,1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow \frac{\partial^2 f}{\partial y^2}(1,1) = -\frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 0$$

By Taylor's theorem

$$f(1+h, 1+k) = f(1,1) + \left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]f(1,1) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^2 f(1,1)+\ldots$$

Let $1 + h = x \Rightarrow h = x - 1$

$1 + k = y \Rightarrow k = y - 1$

$$\therefore \quad f(x,y) = f(1,1)+\left[(x-1)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]f + \frac{1}{\lfloor 2}\left[(x-1)\frac{\partial}{\partial x}+(y-1)\frac{\partial}{\partial y}\right]^2 f+\ldots$$

$$= \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{\lfloor 2}\left[(x-1)^2\left(\frac{1}{2}\right)+2(x-1)(y-1) \times 0+(y-1)^2\left(-\frac{1}{2}\right)\right]$$

or

$$f(x,y) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \ldots .$$

Putting $x = 1.1$, $y = 0.9$, we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}\cdot(1.1-1) + \frac{1}{2}(0.9-1) + \frac{1}{4}(1.1-1)^2 - \frac{1}{4}(0.9-1)^2$$

$$= 0.785 - 0.05 - 0.05 + 0.0025 - 0.0025$$

$$= 0.685.$$

Example 8. Find Taylor's expansion of $\sqrt{1+x+y^2}$ in power of $(x - 1)$ and $(y - 0)$.

Sol. Here $f(x,y) = \sqrt{1+x+y^2}$ | $f(1,0) = \sqrt{2}$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{1}{2\sqrt{1+x+y^2}} \Rightarrow \frac{\partial}{\partial x}f(1,0) = 1/2\sqrt{2}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{1+x+y^2}} \Rightarrow \frac{\partial}{\partial y}f(1,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{4(1+x+y^2)^{3/2}} \Rightarrow \frac{\partial^2 f}{\partial x^2}(1,0) = -\frac{1}{4.2^{3/2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{\sqrt{1+x+y^2}} - \frac{y^2}{(1+x+y^2)^{3/2}} \Rightarrow \frac{\partial^2 f(1,0)}{\partial y^2} = \frac{1}{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{y}{2(1+x+y^2)^{3/2}} \qquad \Rightarrow \qquad \frac{\partial^2 f(1,0)}{\partial x \partial y} = 0.$$

By Taylor's theorem, we get

$$f(1+h, 0+k) = f(1,0) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(1,0) + \frac{1}{\underline{|2}}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(1,0) + \ldots$$

Let $\qquad 1 + h = x \Rightarrow h = x - 1,\ k = y$

$$\Rightarrow f(x, y) = \sqrt{2} + (x-1)\cdot\frac{1}{2\sqrt{2}} + y \times 0 + \frac{1}{\underline{|2}}(x-1)^2\left(-\frac{1}{4.2^{3/2}}\right) + (x-1)y \times 0 + y^2 \cdot \frac{1}{2\sqrt{2}} + \ldots$$

$$= \sqrt{2}\left[1 + \frac{x-1}{4} - \frac{(x-1)^2}{32} + \frac{y^2}{4} + \ldots\right].$$

EXERCISE 1.10

1. Expand $f(x, y) = x^2 + xy + y^2$ in powers of $(x - 1)$ and $(y - 2)$.

[**Ans.** $f(x, y) = 7 + 4(x-1) + 5(y-2) + (x-1)^2 + (x-1)(y-2) + (y-2)^2 + \ldots$]

2. Evaluate $\tan^{-1}\left(\frac{0.9}{1.1}\right)$. [**Ans.** 0.6904]

3. Expand $f(x, y) = \sin(xy)$ about the point $(1, \pi/2)$ up to and second degree term.

$$\left[\textbf{Ans.}\ f(x,y) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2\right] + \ldots$$

4. Obtain Taylor's expansion of $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$.

[**Ans.** $f(x, y) = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + \ldots$]

5. Expand e^{xy} in powers of $(x - 1)$ and $(y - 1)$.

$$\left[\textbf{Ans.}\ e\left\{1 + (x-1) + (y-1) + \frac{(x-1)^2}{2} + (x-1)(y-1) + \frac{(y-1)^2}{2} + \ldots\right\}\right]$$

6. Expand $\cos x \cos y$ in powers of x and y.

$$\left[\textbf{Ans.}\ f(x,y) = 1 - \frac{1}{2}(x^2 + y^2) + \frac{1}{24}(x^4 + 6x^2y^2 + y^4) + \ldots\right]$$

7. Expand $f(x, y) = e^{2x} \cos 3y$ up to second degree. [**Ans.** $1 + 2x + 2x^2 - \frac{9}{2}y^2 + \ldots$]

8. Obtain Taylor's series expansion of $(x + h)(y + k)/(x + h + y + k)$

$$\left[\textbf{Ans.}\ \frac{xy}{x+y} + \frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2} - \frac{h^2y^2}{(x+y)^3} + \frac{2hkxy}{(x+y)^3} - \frac{k^2x^2}{(x+y)^3} + \ldots\right]$$

9. Obtain Taylor's expansion of $(1 + x - y)^{-1}$ in powers of $(x - 1)$ and $(y - 1)$.

[**Ans.** $1 - x + y + x^2 - 2xy + y^2 + \ldots$]

10. Find Machaurin's expansion of $e^x \log(1 + y)$. $\left[\textbf{Ans.}\ y + xy - \frac{y^2}{2} + \frac{(x^2y - xy^2)}{2} + \frac{y^3}{3} + \ldots\right]$

OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer of the choices given below:

1. If $u = e^{xyz}$, then $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ is

(*i*) $(x^2yz + x)$ (*ii*) $(x^2yz - x)e^{xyz}$

(*iii*) $e^{xyz}.xy$ (*iv*) $(1 + 3xyz + x^2y^2z^2)e^{xyz}$

2. If $u = \sin^{-1} \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, then $\dfrac{\partial u}{\partial x}$ is equal to

(*i*) $-\dfrac{x}{y}\dfrac{\partial u}{\partial y}$ (*ii*) $\dfrac{\partial u}{\partial y}$

(*iii*) $-\dfrac{y}{x}\dfrac{\partial u}{\partial y}$ (*iv*) $xy\dfrac{\partial u}{\partial y}$

3. If $z = \log \sqrt{x^2 + y^2}$ then $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are

(*i*) $-\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}$ (*ii*) $\dfrac{x}{x^2 + y^2}, -\dfrac{y}{x^2 + y^2}$

(*iii*) $\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}$ (*iv*) $\dfrac{x}{x^2 - y^2}, \dfrac{y}{x^2 - y^2}$

4. If $u = e^{x^2 + y^2 + z^2}$, then $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ is

(*i*) $7xy$ (*ii*) $6xyzu$

(*iii*) $-8xyzu$ (*iv*) $8xyzu$

5. If $z = x^{n-1}\, y\, f(y/x)$ then $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y \partial x}$ is

(*i*) $n\dfrac{\partial z}{\partial x}$ (*ii*) nz

(*iii*) $n(n-1)z$ (*iv*) $n\dfrac{\partial z}{\partial y}$

6. If $f(x, y, z) = 0$ then $\dfrac{\partial x}{\partial y} \cdot \dfrac{\partial y}{\partial z} \cdot \dfrac{\partial z}{\partial x}$ is

(*i*) -1 (*ii*) 1

(*iii*) 0 (*iv*) 2

7. If $u = x^3 y^2 \sin^{-1}(y/x)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is:

(*i*) 0 (*ii*) $6u$

(*iii*) $-8u$ (*iv*) $5u$

8. If $u = xy\, f(y/x)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is

(*i*) yu (*ii*) xu

(*iii*) u (*iv*) $2u$

9. The degree of $u = [z^2/(x^4 + y^4)]^{1/3}$ is

(*i*) $\frac{2}{3}$ (*ii*) $\frac{1}{3}$

(*iii*) $\frac{-2}{3}$ (*iv*) 3

10. If $z = x^4 y^5$ where $x = t^2$ and $y = t^3$ then $\frac{dz}{dt}$ is

(*i*) t^{22} (*ii*) 0

(*iii*) $23t^{17}$ (*iv*) $23t^{22}$

11. If $x^2 + y^2 + z^2 = a^2$ then $\frac{\partial z}{\partial x}$ and $\frac{\partial y}{\partial x}$ at (1, –1, 2) are

(*i*) $-\frac{1}{2}, 1$ (*ii*) $-\frac{1}{2}, -1$

(*iii*) $\frac{1}{2}, -1$ (*iv*) $\frac{1}{2}, 1$

12. The equation of curve $x^3 + y^3 = 3axy$ inersects the line $y = x$ at the point

(*i*) $\left(3a, \frac{3a}{2}\right)$ (*ii*) $\left(\frac{3a}{2}, -\frac{3a}{2}\right)$

(*iii*) $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ (*iv*) $\left(-\frac{3a}{2}, -\frac{3a}{2}\right)$

13. The equation of the curve $x^2y - y - x = 0$ has maximum

(*i*) One asymptote (*ii*) Four asymptotes

(*iii*) Three asymptotes (*iv*) None of these

14. The curve $r^2 \cos 2\theta = a^2$ is symmetric about the line

(*i*) $\theta = \frac{\pi}{2}$ (*ii*) $\theta = -\frac{\pi}{2}$

(*iii*) $\theta = -\frac{3\pi}{2}$ (*iv*) $\theta = \pi$

15. The parametric form of the curve $x = a$ cos*t*, $y = a \sin t$ is symmetric about

(*i*) x-axis (*ii*) y-axis

(*iii*) both axis (*iv*) about $y = x$

B. Fill in the blanks:

1. The *n*th derivative of $y = x^{n-1} \log x$ at $x = \frac{1}{2}$ is ....

2. The *n*th derivative of $y = x \sin x$ is ....

3. If $y = e^x \sin x$, then $y'' - 2y' + 2y =$ ....

4. If $y = \sin 2x$ then $y_6\,(0) =$ ....

5. If $y = \sin hx$, then $y_{2n}(x) =$ ....

6. If $y = e^x.x$, then $y_n(x) =$ ....

7. If $y = \sin^3 x$, then $y_n(x) =$ ....

8. The nth derivative (y_n) of $y = x^2 \sin x$ at $x = 0$ is .... (*U.P.T.U., 2008*)

9. If $y = \tan^{-1} x$ and n is even, then $y_n(0) =$ ....

10. If $y = \cos(m \sin^{-1} x)$ and n is odd then $y_n(0) =$ ....

11. If $y = (\sin^{-1} x)^2$ and n is odd then $y_n(0) =$ ....

12. If $z = f(x - by) + \phi(x + by)$, then $b^2 \frac{\partial^2 z}{\partial x^2} =$ ....

13. If $u = \log(2x + 3y)$, then $\frac{\partial^2 u}{\partial x \partial y} =$ ....

14. If $u = e^{ax + by} f(ax - by)$, then $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} =$ ....

15. If $u = x^2 + y^2$, $x = s + 3t$, $y = 2s - t$, then $\frac{\partial u}{\partial s} =$ ....

16. If $f(x, y)$ be a homogeneous function of degree n, then $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} =$ ....

17. If $\log u = \frac{x^2 y^2}{x + y}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$ ....

18. If $u = (x^2 + y^2 + z^2)^{1/2}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} =$ ....

19. If $f(x, y) = 0$, $\phi(y, z) = 0$, then $\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z}\frac{dz}{dx} =$ ....

20. If $x = e^r \cos\theta$, $y = e^r \sin\theta$ then $\frac{\partial \theta}{\partial x} =$ .... and $\frac{\partial \theta}{\partial y} =$ ....

21. If $u = \frac{x^3 - x^2 y + xy^2 + y^3}{x^2 - xy - y^2}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$ ....

22. If $u = \log\frac{x^4 - y^4}{x^3 + y^3}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$ ....

23. If $\lim\limits_{t \to t_1} x(t) = \infty$, $\lim\limits_{t \to t_1} y(t) = \infty$, then t = ... is asymptote.

24. The curve $r = a(1 - \cos\theta)$ is symmetric about the ....

25. The tangent is parallel to x-axis if $\frac{dy}{dx} =$ ....

26. The curve $x^2y^2 = a^2(y^2 - x^2)$ has tangents at origin ....

27. The curve $y^2(a - x) = x^2(a + x)$ exists if ..... x .....

28. The curve $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$, cross y-axis at ..... and .....

29. If $f(x, y) = e^y \log(1 + x)$, then expansion of this function about (0, 0) up to second degree term is ....

30. $\tan^{-1}\{(0.9)(-1.2)\} =$ ....

31. If $f(x, y) = e^x \cdot \sin y$, then $\dfrac{\partial^3 f(0,0)}{\partial x^3}$ = ...

32. Expansion of e^{xy} up to first order tem is ....

33. $f(x, y) = f(1, 2) +$ ....

C. Indicate True or False for the following statements:

1. If u, v are functions of r, s are themselves functions of x, y then $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(x,y)}{\partial(r,s)}$

2. Geometrically the function $z = f(x, y)$ represents a surface in space.

3. If $f(x, y) = ax^2 + 2hxy + by^2$ then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 2f$.

4. If z is a function of two variables then dz is defined as $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$.

5. If $f(x_1, x_2, \ldots, x_n)$ be a homogeneous function then $x_1\dfrac{\partial f}{\partial x_1} + x_2\dfrac{\partial f}{\partial x_2} + \ldots + x_n\dfrac{\partial f}{\partial x_n}$ $= n(n-1)f$.

6. If $u = \dfrac{x^2+y^2}{x^2-y^2} + 4$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 4$.

7. When the function $z = f(x, y)$ differentiating (partially) with respect to one variable, other variable is treated (temporarily) as constant.

8. To satisfied Euler's theorem the function $f(x, y)$ should not be homogeneous.

9. The partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are interpreted geometrically as the slopes of the tangent lines at any point.

10. (*i*) The curve $y^2 = 4ax$ is symmetric about x-axis.

(*ii*) The curve $x^3 + y^3 = 3axy$ is symmetric about the line $y = -x$.

(*iii*) The curve $x^2 + y^2 = a^2$ is symmetric about both the axis x and y.

(*iv*) The curve $x^3 - y^3 = 3axy$ is symmetric about the line $y = x$.

11. (*i*) If there is no constant term in the equation then the curve passes through the origin otherwise not.

(*ii*) A point where $\dfrac{d^2y}{dx^2} \neq 0$ is called an inflexion point.

(*iii*) If r^2 is negative *i.e.,* imaginary for certain values of θ then the curve does not exist for those values of θ.

(*iv*) The curve $r = ae^{\theta}$ is symmetric about the line $\theta = \dfrac{\pi}{2}$.

12. (*i*) Maclaurin's series expansion is a special case of Taylor's series when the expansion is about the origin (0, 0).

(*ii*) Taylor's theorem is important tool which provide polynomial approximations of real valued functions.

(*iii*) Taylor's theorem fail to expand $f(x, y)$ in an infinite series if any of the functions $f_x(x, y)$, $f_{xx}(x, y)$, $f_{xy}(x, y)$ etc., becomes infinite or does not exist for any value of x, y in the given interval.

(*iv*) $f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} - (x-b)\frac{\partial}{\partial y}\right] f(a, b) + \ldots\ldots$.

ANSWERS TO OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer:

1. (*iv*)	**2.** (*iii*)	**3.** (*iii*)
4. (*iv*)	**5.** (*i*)	**6.** (*i*)
7. (*iv*)	**8.** (*iv*)	**9.** (*iii*)
10. (*iv*)	**11.** (*i*)	**12.** (*iii*)
13. (*iii*)	**14.** (*i*)	**15.** (*iii*)

B. Fill in the blanks:

1. $2\lfloor n-1$	**2.** $x \sin\left(x+\frac{n\pi}{2}\right) - n\cos\left(x+\frac{n\pi}{2}\right)$	
3. Zero	**4.** Zero	**5.** $\sin hx$
6. $e^x(x+n)$	**7.** Try yourself	**8.** $(n-n^2)\sin\frac{n\pi}{2}$
9. Zero	**10.** Zero	**11.** Zero
12. $\frac{\partial^2 z}{\partial y^2}$	**13.** $\frac{\partial^2 u}{\partial y \partial x}$	**14.** 2 *abu*
15. $2x + 4y$	**16.** $n(n-1)f$	**17.** $3u \log u$
18. u	**19.** $\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$	**20.** $-\frac{\sin^2\theta}{y}, \frac{\cos^2\theta}{x}$
21. u	**22.** 1	**23.** t_1
24. initial line	**25.** 0	**26.** $y = \pm x$
27. $-a < x < a$	**28.** $(0, a)$ and $(0, -a)$	**29.** $x - \frac{x^2}{2} + xy$
30. -0.823	**31.** 0	**32.** $e\{1 + (x-1) + (y-1)\}$

33. $\left[\left\{(z-1)\frac{\partial f}{\partial x} + (y \cdot 2)\frac{\partial f}{\partial y}\right\} + \frac{1}{\lfloor 2}\left\{(x-1)\frac{\partial f}{\partial x} + (y \cdot 2)\frac{\partial f}{\partial y}\right\}^2 + \ldots\ldots\right]$

C. True or False:

1. *F*	**2.** *T*	**3.** *T*	**4.** *T*
5. *F*	**6.** *F*	**7.** *T*	**8.** *F*
9. *T*			
10. (*i*) *T*	(*ii*) *F*	(*iii*) *T*	(*iv*) *F*
11. (*i*) *T*	(*ii*) *F*	(*iii*) *T*	(*iv*) *F*
12. (*i*) *T*	(*ii*) *T*	(*iii*) *T*	(*iv*) *F*

❑❑❑

UNIT II

Differential Calculus-II

2.1 JACOBIAN

The Jacobians* themselves are of great importance in solving for the reverse (inverse functions) derivatives, transformation of variables from one coordinate system to another coordinate system (cartesian to polar etc.). They are also useful in area and volume elements for surface and volume integrals.

2.1.1 Definition

If $u = u(x, y)$ and $v = v(x, y)$ where x and y are independent, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is known as the Jacobian of u, v with respect to x, y and is denoted by

$$\frac{\partial(u, v)}{\partial(x, y)} \text{ or } J(u, v)$$

Similarly, the Jacobian of three functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ is defined as

$$J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

* Carl Gustav Jacob Jacobi (1804–1851), German mathematician.

2.1.2 Properties of Jacobians

1. If $u = u\,(x, y)$ and $v = v\,(x, y)$, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1 \text{ or } JJ' = 1 \qquad (U.P.T.U., 2005)$$

where, $$J = \frac{\partial(u,v)}{\partial(x,y)} \text{ and } J' = \frac{\partial(x,y)}{\partial(u,v)}$$

Proof: Since
$$u = u\,(x, y) \qquad ...(i)$$
$$v = v\,(x, y) \qquad ...(ii)$$

Differentiating partially equations (*i*) and (*ii*) *w.r.t.* u and v, we get

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial u} \qquad ...(iii)$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\times\frac{\partial y}{\partial v} \qquad ...(iv)$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\times\frac{\partial y}{\partial u} \qquad ...(v)$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\times\frac{\partial y}{\partial v} \qquad ...(vi)$$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(By interchanging rows and columns in II determinant)

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}.\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix} \quad \text{(multiplying row-wise)}$$

Putting equations (*iii*), (*iv*), (*v*) and (*vi*) in above, we get

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

or $$\boxed{JJ' = 1.}$$ **Hence proved.**

2. Chain rule: If u, v, are function of r, s and r, s are themselves functions of x, y *i.e.*, $u = u\,(r, s)$, $v = v\,(r, s)$ and $r = r\,(x, y)$, $s = s\,(x, y)$

then $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}$$

Proof: Here $u = u\,(r, s)$, $v = v\,(r, s)$

and $r = r\,(x, y)$, $s = s\,(x, y)$

Differentiating u, v partially w.r.t. x and y

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} \qquad ...(i)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \qquad ...(ii)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} \qquad ...(iii)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \qquad ...(iv)$$

Now,
$$\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix}$$

|By interchanging the rows and columns in second determinant

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix} \quad \text{| multiplying row-wise}$$

Using equations (*i*), (*ii*), (*iii*) and (*iv*) in above, we get

$$\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}.$$

Or

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}.$$ **Hence proved.**

Example 1. Find $\frac{\partial(u,v)}{\partial(x,y)}$, when $u = 3x + 5y$, $v = 4x - 3y$.

Sol. We have
$$u = 3x + 5y$$
$$v = 4x - 3y$$

$\therefore$
$$\frac{\partial u}{\partial x} = 3,\ \frac{\partial u}{\partial y} = 5,\ \frac{\partial v}{\partial x} = 4 \text{ and } \frac{\partial v}{\partial y} = -3$$

Thus,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix} = -9 - 20 = -29.$$

Example 2. Calculate the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ of the following:

$$u = x + 2y + z,\ v = x + 2y + 3z,\ w = 2x + 3y + 5z. \qquad (U.P.T.U., 2007)$$

Sol. We have
$$u = x + 2y + z$$
$$v = x + 2y + 3z$$
$$w = 2x + 3y + 5z$$

$$\therefore \quad \frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 2,\ \frac{\partial u}{\partial z} = 1,\ \frac{\partial v}{\partial x} = 1,\ \frac{\partial v}{\partial y} = 2,\ \frac{\partial v}{\partial z} = 3,$$

$$\frac{\partial w}{\partial x} = 2,\ \frac{\partial w}{\partial y} = 3 \text{ and } \frac{\partial w}{\partial z} = 5.$$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 1\,(10 - 9) - 2\,(5 - 6) + 1\,(3 - 4) = 2.$$

Example 3. Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$, $w = \frac{4xy}{z}$.

Sol. Given
$$u = \frac{2yz}{x},\ v = \frac{3zx}{y},\ w = \frac{4xy}{z}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{-2yz}{x^2},\ \frac{\partial u}{\partial y} = \frac{2z}{x},\ \frac{\partial u}{\partial z} = \frac{2y}{x},\ \frac{\partial v}{\partial x} = \frac{3z}{y},\ \frac{\partial v}{\partial y} = -\frac{3zx}{y^2},\ \frac{\partial v}{\partial z} = \frac{3x}{y},$$

$$\frac{\partial w}{\partial x} = \frac{4y}{z},\ \frac{\partial w}{\partial y} = \frac{4x}{z} \text{ and } \frac{\partial w}{\partial z} = -\frac{4xy}{z^2}$$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & \frac{-3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & \frac{-4xy}{z^2} \end{vmatrix}$$

$$= -\frac{2yz}{x^2}\left[\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz}\right] - \frac{2z}{x}\left[\frac{-12xyz}{yz^2} - \frac{12xy}{yz}\right] + \frac{2y}{x}\left[\frac{12xz}{yz} + \frac{12xyz}{zy^2}\right]$$

$$\Rightarrow \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 + 48 + 48 = 96.$$

But, we have $\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$ (Property 1)

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{96}.$$

Example 4. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ find $J(x, y, z)$. (*U.P.T.U., 2002*)

Sol. Here we calculate $J(u, v, w)$ as follows:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y - 2z) - zx(2x - 2z) + xy(2x - 2y)$$
$$= 2[y^2z - yz^2 - zx^2 + z^2x + xy(x - y)]$$
$$= 2[-z(x^2 - y^2) + z^2(x - y) + xy(x - y)]$$
$$= 2(x - y)[-zx - zy + z^2 + xy] \qquad \Big| \text{ As } x^2 - y^2 = (x - y)(x + y)$$
$$= 2(x - y)[z(z - x) - y(z - x)]$$
$$= 2(x - y)(z - y)(z - x)$$
$$= -2(x - y)(y - z)(z - x)$$

But $\quad J(x, y, z) \cdot J(u, v, w) = 1$

$\therefore \quad J(x, y, z) =$

Example 5. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r \sin\theta \cos\phi$,

$v = r \sin\theta \sin\phi$, $w = r\cos\theta$, calculate $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

Sol. Here x, y, z are functions of u, v, w and u, v, w are functions of r, θ, ϕ so we apply IInd property.

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(r, \theta, \phi)} \qquad \ldots(i)$$

Consider
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\left[\sqrt{\frac{w}{v}\frac{v}{u}\frac{u}{w}} + \sqrt{\frac{v}{w}\frac{w}{u}\frac{u}{v}}\right] = \frac{1}{8}\left[\sqrt{1} + \sqrt{1}\right] = \frac{2}{8} = \frac{1}{4}$$

$$\Rightarrow \qquad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \qquad \text{...(ii)}$$

Next $$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r^2\sin\theta\cos\theta\cos\phi) + r^2\sin\theta\sin\phi\,(\sin^2\theta\sin\phi + \cos^2\theta\sin\phi)$$

$$= r^2\sin\theta\cos^2\phi\,(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$

$$\Rightarrow \qquad \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = r^2\sin\theta\cos^2\phi + r^2\sin\theta\sin^2\phi = r^2\sin\theta \qquad \text{...(iii)}$$

Using (*ii*) and (*iii*) in equation (*i*), we get

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{1}{4} \times r^2\sin\theta = \frac{r^2\sin\theta}{4}.$$

Example 6. If $u = x\,(1 - r^2)^{-1/2}$, $v = y\,(1 - r^2)^{-1/2}$, $w = z\,(1 - r^2)^{-1/2}$

where $r = \sqrt{x^2+y^2+z^2}$, then find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol. Since $r^2 = x^2 + y^2 + z^2$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Differentiating partially $u = x\,(1 - r^2)^{-1/2}$ w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (1-r^2)^{\frac{-1}{2}} + x\left(\frac{-1}{2}\right)(-2r)\,(1-r^2)^{\frac{-3}{2}} \cdot \frac{\partial r}{\partial x}$$

$$= (1-r^2)^{\frac{-1}{2}} + rx\,(1-r^2)^{\frac{-3}{2}} \cdot \frac{x}{r} = \frac{1}{\sqrt{1-r^2}} + \frac{x^2}{(1-r^2)^{\frac{3}{2}}}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = \frac{1-r^2+x^2}{(1-r^2)^{\frac{3}{2}}}$$

Differentiating partially u w.r.t. y, we get

$$\frac{\partial u}{\partial y} = x\left(\frac{-1}{2}\right)(1-r^2)^{\frac{-3}{2}} \cdot (-2r)\,\frac{\partial r}{\partial y} = \frac{xr}{(1-r^2)^{\frac{3}{2}}} \cdot \frac{y}{r}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial y} = \frac{xy}{(1-r^2)^{\frac{3}{2}}}$$

and $$\frac{\partial u}{\partial z} = \frac{xz}{(1-r^2)^{\frac{3}{2}}}$$

Similarly, $$\frac{\partial v}{\partial x} = \frac{yx}{\left(1-r^2\right)^{\frac{3}{2}}},\quad \frac{\partial v}{\partial y} = \frac{1-r^2+y^2}{\left(1-r^2\right)^{\frac{3}{2}}},\quad \frac{\partial v}{\partial z} = \frac{yz}{\left(1-r^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial w}{\partial x} = \frac{zx}{\left(1-r^2\right)^{\frac{3}{2}}},\quad \frac{\partial w}{\partial y} = \frac{zy}{\left(1-r^2\right)^{\frac{3}{2}}},\quad \frac{\partial w}{\partial z} = \frac{1-r^2+z^2}{\left(1-r^2\right)^{\frac{3}{2}}}.$$

Thus, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{1-r^2+x^2}{\left(1-r^2\right)^{\frac{3}{2}}} & \frac{xy}{\left(1-r^2\right)^{\frac{3}{2}}} & \frac{xz}{\left(1-r^2\right)^{\frac{3}{2}}} \\ \frac{yx}{\left(1-r^2\right)^{\frac{3}{2}}} & \frac{1-r^2+y^2}{\left(1-r^2\right)^{\frac{3}{2}}} & \frac{yz}{\left(1-r^2\right)^{\frac{3}{2}}} \\ \frac{zx}{\left(1-r^2\right)^{\frac{3}{2}}} & \frac{zy}{\left(1-r^2\right)^{\frac{3}{2}}} & \frac{1-r^2+z^2}{\left(1-r^2\right)^{\frac{3}{2}}} \end{vmatrix}$$

$$= \frac{1}{\left(1-r^2\right)^{\frac{9}{2}}} \begin{vmatrix} 1-r^2+x^2 & xy & xz \\ yx & 1-r^2+y^2 & yz \\ zx & zy & 1-r^2+z^2 \end{vmatrix}$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1-r^2+x^2)\{(1-r^2+y^2)(1-r^2+z^2)-y^2z^2\}$$
$$-xy\{xy(1-r^2+z^2)-xyz^2\}+xz\{xy^2z-zx(1-r^2+y^2)\}]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1-r^2+x^2)(1-r^2+y^2)(1-r^2+z^2)$$
$$-(1-r^2)(y^2z^2+x^2y^2+x^2z^2)-x^2y^2z^2]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1-r^2)^3+(1-r^2)^2(x^2+y^2+z^2)]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} [(1-r^2)^3+(1-r^2)^2 r^2]$$

$$= \left(1-r^2\right)^{\frac{-9}{2}} . (1-r^2)^2[1-r^2+r^2] = \left(1-r^2\right)^{\frac{-5}{2}}.$$

Example 7. Verify the chain rule for Jacobians if $x = u$, $y = u \tan v$, $z = w$. (*U.P.T.U., 2008*)

Sol. We have $$x = u \Rightarrow \frac{\partial x}{\partial u} = 1,\ \frac{\partial x}{\partial v} = \frac{\partial x}{\partial w} = 0$$

$$y = u \tan v \Rightarrow \frac{\partial y}{\partial u} = \tan v,\ \frac{\partial y}{\partial v} = u \sec^2 v,\ \frac{\partial y}{\partial w} = 0$$

$$z = w \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial v} = 0,\ \frac{\partial z}{\partial w} = 1$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \sec^2 v \quad ...(i)$$

Solving for u, v, w in terms of x, y, z, we have

$$u = x$$

$$v = \tan^{-1}\frac{y}{u} = \tan^{-1}\frac{y}{x}$$

$$w = z$$

$\therefore \quad \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0, \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}, \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}, \frac{\partial v}{\partial z} = 0, \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0$ and $\frac{\partial w}{\partial z} = 1$

$$J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{x}{x^2+y^2} = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)} = \frac{1}{u\sec^2 v} \quad \text{...}(ii)$$

Hence from (*i*) and (*ii*), we get

$$J.J' = u\sec^2 v.\frac{1}{u\sec^2 v} = 1.$$

2.1.3 Jacobian of Implicit Functions

If the variables u, v and x, y be connected by the equations

$$f_1(u, v, x, y) = 0 \quad \text{...}(i)$$

$$f_2(u, v, x, y) = 0 \quad \text{...}(ii)$$

i.e., u, v are implicit functions of x, y.

Differentiating partially (*i*) and (*ii*) w.r.t. x and y, we get

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{...}(iii)$$

$$\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial u}{\partial y} = 0 \quad \text{...}(iv)$$

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{...}(v)$$

$$\frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \text{...}(vi)$$

Now, $$\frac{\partial(f_1, f_2)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y} \end{vmatrix}$$

Using (*iii*), (*iv*), (*v*) and (*vi*) in above, we get

$$= \begin{vmatrix} -\dfrac{\partial f_1}{\partial x} & -\dfrac{\partial f_1}{\partial y} \\ -\dfrac{\partial f_2}{\partial x} & -\dfrac{\partial f_2}{\partial y} \end{vmatrix} = (-1)^2 \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix}$$

Thus, $$\frac{\partial(f_1, f_2)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1,f_2)}{\partial(x,y)}$$

$$\Rightarrow \qquad \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}$$

Similarly for three variables u, v, w

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

and so on.

Example 8. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)}. \qquad \textit{(U.P.T.U., 2006)}$$

Sol. Let $f_1 \equiv u^3 + v^3 - x - y = 0$

$f_2 \equiv u^2 + v^2 - x^3 - y^3 = 0$

Now, $$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} = 3(y^2 - x^2)$$

and $$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6\,uv\,(u-v)$$

Thus, $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} = \frac{3(y^2-x^2)}{6uv(u-v)} = \frac{(y^2-x^2)}{2uv(u-v)}.$$ Hence Proved.

Example 9. If u, v, w are the roots of the equation in k, $\dfrac{x}{a+k} + \dfrac{y}{b+k} + \dfrac{z}{c+k} = 1$, prove that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = -\dfrac{(u-v)(v-w)(w-u)}{(a-b)(b-c)(c-a)}$.

Sol. We have $\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1$

or $\quad x(b+k)(c+k) + y(a+k)(c+k) + z(a+k)(c+k) = (a+k)(b+k)(c+k)$

or $\quad k^3 + k^2(a+b+c-x-y-z) + k\{bc+ca+ab-(b+c)x-(c+a)$
$y-(a+b)z\} + (abc-bcx-cay-abz) = 0$

Since its roots are given to be u, v, w, so we have

$$u+v+w = -(a+b+c-x-y-z)$$
$$uv+vw+wu = bc+ca+ab-(b+c)x-(c+a)y-(a+b)z$$
$$uvw = -(abc-bcx-cay-abz)$$

Let
$$f_1 \equiv u+v+w+a+b+c-x-y-z = 0$$
$$f_2 \equiv uv+vw+wu-bc-ca-ab+(b+c)x+(c+a)y+(a+b)z = 0$$
$$f_3 \equiv uvw+abc-bcx-cay-abz = 0$$

Now,
$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ (b+c) & (c+a) & (a+b) \\ -bc & -ca & -ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix} = (a-b)(a-c)(b-c)$$

and
$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ (v+w) & (w+u) & (u+v) \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \quad (C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1)$$

$$= (u-v)(u-w)(v-w)$$

Thus,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{(a-b)(a-c)(b-c)}{(u-v)(u-w)(v-w)}$$

$\therefore$
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{(u-v)(v-w)(u-w)}{(a-b)(b-c)(a-c)}.$$ **Hence proved.** | As $JJ' = 1$.

Example 10. If $u = 2axy$, $v = a(x^2 - y^2)$ where $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4a^2r^3.$$

Sol. We have $u = 2axy,\ v = a\,(x^2 - y^2)$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2ay & 2ax \\ 2ax & -2ay \end{vmatrix} = -4a^2(x^2 + y^2)$$

or $$\frac{\partial(u,v)}{\partial(x,y)} = -4a^2r^2 \qquad \Big|\ \text{As } x^2 + y^2 = r^2$$

and $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = (-4a^2r^2).r = -4a^2r^3.$$ **Hence proved.**

Example 11. If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

Sol. Let
$$f_1 \equiv u^3 + v^3 + w^3 - x - y - z = 0$$
$$f_2 \equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0$$
$$f_3 \equiv u + v + w - x^2 - y^2 - z^2 = 0$$

Now, $$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -3x^2 & 3(x^2-y^2) & 3(x^2-z^2) \\ -2x & 2(x-y) & 2(x-z) \end{vmatrix} \qquad \Big|\, C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$$

$$= -6[(x^2 - y^2)(x - z) - (x^2 - z^2)(x - y)]$$
$$= -6(x - y)(x - z)[(x + y) - (x + z)]$$

$$\Rightarrow \quad \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = 6(x - y)(y - z)(z - x)$$

and $$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3u^2 & 3(v^2-u^2) & 3(w^2-u^2) \\ 2u & 2(v-u) & 2(w-u) \\ 1 & 0 & 0 \end{vmatrix} \quad \Big| C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$$

Expand it with respect to third row, we get

$$= 6[(v^2 - u^2)(w - u) - (w^2 - u^2)(v - u)]$$
$$= 6(v - u)(w - u)[(v + u) - (w + u)]$$

$$\Rightarrow \quad \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = -6(u - v)(v - w)(w - u)$$

Hence
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = +\frac{6(x-y)(y-z)(z-x)}{6(u-v)(v-w)(w-u)}$$

$$= \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$ **Hence proved.**

Example 12. u, v, w are the roots of the equation

$$(x - a)^3 + (x - b)^3 + (x - c)^3 = 0, \text{ find } \frac{\partial(u, v, w)}{\partial(a, b, c)}.$$

Sol. We have $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$

$x^3 - a^3 - 3xa(x - a) + x^3 - b^3 - 3xb(x - b) + x^3 - c^3 - 3xc(x - c) = 0$

or $3x^3 - 3x^2(a + b + c) + 3x(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3) = 0$

Since u, v, w are the roots of this equation, we have

$$u + v + w = a + b + c$$
$$uv + vw + wu = a^2 + b^2 + c^2$$
$$uvw = \frac{a^3 + b^3 + c^3}{3}$$

As $\alpha + \beta + \gamma = -b/a$, $\alpha\beta + \beta\gamma + \gamma\alpha = c/a$, $\alpha\beta\gamma = -d/a$

Let
$$f_1 \equiv u + v + w - a - b - c = 0$$
$$f_2 \equiv uv + vw + wu - a^2 - b^2 - c^2 = 0$$
$$f_3 = uvw - \frac{a^3 + b^3 + c^3}{3}$$

Now
$$\frac{\partial(f_1, f_2, f_3)}{\partial(a, b, c)} = \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -2a & 2(a-b) & 2(a-c) \\ -a^2 & (a^2-b^2) & (a^2-c^2) \end{vmatrix}, \begin{pmatrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{pmatrix}$$

$$= -2\{(a - b)(a^2 - c^2) - (a - c)(a^2 - b^2)\} = -2(a - b)(b - c)(c - a)$$

and
$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & v+u \\ vw & wu & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}, \begin{pmatrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{pmatrix}$$

$$= (u - v)\, v(u - w) - (u - w)\, w(u - v)$$
$$= -(u - v)(v - w)(w - u)$$

Thus $$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(a,b,c)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{-2(a-b)(b-c)(c-a)}{-(u-v)(v-w)(w-u)}$$

$$= -2\,\frac{(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}.$$

2.1.4 Functional Dependence

Let $u = f_1(x, y)$, $v = f_2(x, y)$ be two functions. Suppose u and v are connected by the relation $f(u, v) = 0$, where f is differentiable. Then u and v are called functionally dependent on one another (*i.e.*, one function say u is a function of the second function v) if the $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are not all zero simultaneously.

Necessary and sufficient condition for functional dependence (Jacobian for functional dependence functions):

Let u and v are functionally dependent then

$$f(u, v) = 0 \qquad ...(i)$$

Differentiate partially equation (*i*) w.r.t. x and y, we get

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\,\frac{\partial v}{\partial x} = 0 \qquad ...(ii)$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \qquad ...(iii)$$

There must be a non-trivial solution for $\frac{\partial f}{\partial u} \neq 0$, $\frac{\partial f}{\partial v} \neq 0$ to this system exists.

Thus, $$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$ | For non-trivial solution $|A| = 0$

or $$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$ | Changing all rows in columns

or $$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

Hence, two functions u and v are **"functionally dependent"** if their Jacobian is equal to zero.

Note: The functions u and v are said to be "functionally independent" if their Jacobian is not equal to zero *i.e.,* $J(u, v) \neq 0$

Similarly for three functionally dependent functions say u, v and w.

$$J(u, v, w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

Example 13. Show that the functions $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Also find the relation between them.

Sol. Here $u = x + y - z,\ v = x - y + z$ and $w = x^2 + y^2 + z^2 - 2yz$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2y-2z & 0 \end{vmatrix} \quad (C_3 \to C_3 + C_2)$$

$= 0$. Hence u, v, w are not independent.

Again
$$u + v = x + y - z + x - y + z = 2x$$
$$u - v = x + y - z - x + y - z = 2(y - z)$$
$$\therefore \quad (u + v)^2 + (u - v)^2 = 4x^2 + 4(y - z)^2$$
$$= 4(x^2 + y^2 + z^2 - 2yz) = 4w$$
$$\Rightarrow \quad (u + v)^2 + (u - v)^2 = 4w$$
or
$$2(u^2 + v^2) = 4w \text{ or } u^2 + v^2 = 2w.$$

Example 14. Find Jacobian of $u = \sin^{-1} x + \sin^{-1} y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$. Also find relation between u and v.

Sol. We have $u = \sin^{-1} x + \sin^{-1} y,\ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Now,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= -\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1 - 1 + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} = 0.$$ Hence u and v are dependent.

Next,
$$u = \sin^{-1} x + \sin^{-1} y \Rightarrow u = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$$

$$\left|\text{As } \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left\{A\sqrt{1-B^2} + B\sqrt{1-A^2}\right\}\right.$$

$$\Rightarrow \quad \sin u = x\sqrt{1-y^2} + y\sqrt{1-x^2} = v$$
or
$$v = \sin u.$$

Example 15. Show that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent unless $\frac{a}{A} = \frac{h}{H} = \frac{b}{B}$.

Sol. Let
$$u = ax^2 + 2hxy + by^2$$
$$v = Ax^2 + 2Hxy + By^2$$

If u and v are not independent, then $\frac{\partial(u,v)}{\partial(x,y)} = 0$

or
$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2ax+2hy & 2hx+2by \\ 2Ax+2Hy & 2Hx+2By \end{vmatrix} = 0$$

$\Rightarrow$ $(ax + hy)(Hx + By) - (hx + by)(Ax + Hy) = 0$

$\Rightarrow$ $(aH - hA)x^2 + (aB - bA)xy + (hB - bH)y^2 = 0$

But variable x and y are independent so the coefficients of x^2 and y^2 must separately vanish and therefore, we have

$$aH - hA = 0 \text{ and } hB - bH = 0 \text{ i.e., } \frac{a}{A} = \frac{h}{H} \text{ and } \frac{h}{H} = \frac{b}{B}$$

i.e,
$$\frac{a}{A} = \frac{h}{H} = \frac{b}{B}.$$ **Hence proved.**

Example 16. If $u = x^2 e^{-y} \cos hz$, $v = x^2 e^{-y} \sin hz$ and $w = 3x^4 e^{-2y}$ then prove that u, v, w are functionally dependent. Hence establish the relation between them.

Sol. We have $u = x^2 e^{-y} \cos hz$, $v = x^2 e^{-y} \sin hz$, $w = 3x^4 e^{-2y}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} 2xe^{-y}\cos hz & -x^2e^{-y}\cos hz & x^2e^{-y}\sin hz \\ 2xe^{-y}\sin hz & -x^2e^{-y}\sin hz & x^2e^{-y}\cos hz \\ 12x^3e^{-2y} & -6x^4e^{-2y} & 0 \end{vmatrix}$$

$$= 2x e^{-y} \cos hz\{0 + 6x^6 e^{-3y} \cos hz\} + x^2 e^{-y} \cos hz\{0 - 12x^5 e^{-3y} \cos hz\}$$
$$+ x^2 e^{-y} \sin hz\{-12x^5 e^{-3y} \sin hz + 12x^5 e^{-3y} \sin hz\}$$
$$= 12x^7 e^{-4y} \cos h^2 z - 12x^7 e^{-4y} \cos h^2 z = 0$$

Thus u, v and w are functionally dependent.

Next, $3u^2 - 3v^2 = 3(x^4 e^{-2y} \cos h^2 z - x^4 e^{-2y} \sin h^2 z) = 3x^4 e^{-2y} (\cos h^2 z - \sin h^2 z)$
$$= 3x^4 e^{-2y}$$

$\Rightarrow$ $3u^2 - 3v^2 = w.$

EXERCISE 2.1

1. If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$. $\left[\textbf{Ans. } \frac{1}{r}\right]$

2. If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$ show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4. *(U.P.T.U., 2004(CO), 2002)*

3. If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta. \qquad \text{(U.P.T.U., 2000)}$$

4. If $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$. (*U.P.T.U., 2003*) $\left[\textbf{Ans. } u^2v\right]$

5. If $u = \dfrac{y^2}{2x}$, $v = \dfrac{(x^2+y^2)}{2x}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. $\left[\textbf{Ans. } -\dfrac{y}{2x}\right]$

6. If $x = a \cos h\,\xi \cos\eta$, $y = a \sin h\,\xi \sin\eta$, show that

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{2}a^2(\cos h\, 2\xi - \cos 2\eta).$$

7. If $u^3 + v + w = x + y^2 + z^2$, $u + v^3 + w = x^2 + y + z^2$, $u + v + w^3 = x^2 + y^2 + z$, then evaluate

$\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$. $\left[\textbf{Ans. } \dfrac{(1-4xy-4yz-4zx+6xyz)}{27u^2v^2w^2+2-3(u^2+v^2+w^2)}\right]$

8. If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

(*U.P.T.U., 2001*) $\left[\textbf{Ans. } \dfrac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}\right]$

9. If $u = x_1 + x_2 + x_3 + x_4$, $uv = x_2 + x_3 + x_4$, $uvw = x_3 + x_4$ and $uvwt = x_4$, show that

$$\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u,v,w,t)} = u^3v^2w.$$

10. Calculate $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ and $J' = \dfrac{\partial(x,y)}{\partial(u,v)}$. Verify that $JJ' = 1$ given

(*i*) $u = x + \dfrac{y^2}{x}$, $v = \dfrac{y^2}{x}$. $\left[\textbf{Ans. } J = \dfrac{2y}{x}, J' = \dfrac{x}{2y}\right]$

(*ii*) $x = e^u \cos v$, $y = e^u \sin v$. $\left[\textbf{Ans. } J = e^{2u}, J' = e^{-2u}.\right]$

11. Show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$ given $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

12. If $X = u^2v$, $Y = uv^2$ and $u = x^2 - y^2$, $v = xy$, find $\dfrac{\partial(X,Y)}{\partial(x,y)}$. $\left[\textbf{Ans. } 6x^2y^2(x^2+y^2)(x^2-y^2)^2\right]$

13. Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$, if $u = x^2$, $v = \sin y$, $w = e^{-3z}$. $\left[\textbf{Ans. } -6e^{-3z}x\cos y\right]$

14. Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$, if $u = 3x + 2y - z$, $v = x - y + z$, $w = x + 2y - z$. $\left[\textbf{Ans. } -2\right]$

15. Find $J(u, v, w)$ if $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$. $\left[\textbf{Ans. } (x-y)(y-z)(z-x)\right]$

16. Prove that u, v, w are dependent and find relation between them if $u = xe^y \sin z$, $v = xe^y \cos z$, $w = x^2e^{2y}$. $\left[\textbf{Ans.}\text{ dependent}, u^2 + v^2 = w\right]$

17. $u = \dfrac{3x^2}{2(y+z)}$, $v = \dfrac{2(y+z)}{3(x-y)^2}$, $w = \dfrac{x-y}{x}$. $\left[\textbf{Ans.}\text{ dependent}, uvw^2 = 1\right]$

18. If $X = x + y + z + u$, $Y = x + y - z - u$, $Z = xy - zu$ and $U = x^2 + y^2 - z^2 - u^2$, then show that $J = \dfrac{\partial(X, Y, Z, U)}{\partial(x, y, z, u)} = 0$ and hence find a relation between X, Y, Z and U.

$\left[\textbf{Ans.}\ XY = U + 2Z\right]$

19. If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$, then prove that u, v, w are not independent and also find the relation between them. $\left[\textbf{Ans.}\ uv + vw + wu + 1 = 0\right]$

20. If $u = x + 2y + z$, $v = x - 2y + 3z$, $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w. $\left[\textbf{Ans.}\ 4w = u^2 - v^2\right]$

21. If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related? If yes find the relationship. [**Ans.** yes, $u = \tan v$]

22. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

23. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{x^2-y^2}{u^2+v^2}$.

24. Find Jacobian of u, v, w w.r.t. x, y, z when $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$, $w = \dfrac{xy}{z}$. [**Ans.** 4]

2.2 APPROXIMATION OF ERRORS

Let $u = f(x, y)$ then the total differential of u, denoted by du, is given by

$$du = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \qquad ...(i)$$

If δx and δy are increments in x and y respectively then the total increment δu in u is given by

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

or

$$f(x + \delta x, y + \delta y) = f(x, y) + \delta u \qquad ...(ii)$$

But $\delta u \approx du$, $\delta x \approx dx$ and $\delta y \approx dy$

$\therefore$ From (i)

$$\delta u \approx \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y \qquad ...(iii)$$

Using (*iii*) in (*ii*), we get the approximate formula

$$\boxed{f(x+\delta x, y+\delta y) \approx f(x, y) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y} \quad ...(iv)$$

Thus, the approximate value of the function can be obtained by equation (*iv*). Hence

$$\delta x \text{ or } dx = \text{Absolute error}$$

$$\frac{\delta x}{x} \text{ or } \frac{dx}{x} = \text{Proportional or Relative error}$$

and $$100 \times \frac{dx}{x} \text{ or } 100 \times \frac{dx}{x} = \text{Percentage error in } x.$$

Example 1. If $f(x, y) = x^2 y^{\frac{1}{10}}$, compute the value of f when $x = 1.99$ and $y = 3.01$. (*U.P.T.U., 2007*)

Sol. We have $f(x, y) = x^2 y^{\frac{1}{10}}$

$$\therefore \quad \frac{\partial f}{\partial x} = 2xy^{\frac{1}{10}}, \quad \frac{\partial f}{\partial y} = \frac{1}{10}x^2 y^{-\frac{9}{10}}$$

Let $x = 2, \delta x = -0.01$

$y = 1, \delta y = 2.01$

As $x+\delta x = 2+(-0.01) = 1.99$
$y+\delta y = 1+(2.01) = 3.01$

Now, $f(x+\delta x, y+\delta y) = f(x, y) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$

$$\Rightarrow f\{2+(-0.01), 1+2.01\} = f(2, 1) + 2\times 2(1)^{\frac{1}{10}} \times (-0.01) + \frac{1}{10}(2)^2 . (1)^{-\frac{9}{10}} \times (2.01)$$

$$\Rightarrow \quad f(1.99, 3.01) \approx 2^2 \times 1^{\frac{1}{10}} + (-0.04) + 0.804$$

$$\approx 4 - 0.04 + 0.804 = 4.764.$$

Example 2. The diameter and height of a right circular cylinder are measured to be 5 and 8 cm. respectively. If each of these dimensions may be in error by ± 0.1 cm, find the relative percentage error in volume of the cylinder.

Sol. Let diameter of cylinder = x cm.

height of cylinder = y cm.

then $$V = \frac{\pi x^2 y}{4} \quad \left(\text{radius} = \frac{x}{2}\right)$$

$$\therefore \quad \frac{\partial V}{\partial x} = \frac{\pi xy}{2}, \quad \frac{\partial V}{\partial y} = \frac{\pi x^2}{4}$$

$$\Rightarrow \quad dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$\Rightarrow \qquad dV = \frac{1}{2}\pi\, xy.\, dx + \frac{1}{4}\, \pi x^2.\, dy$$

or
$$\frac{dV}{V} = \frac{\frac{1}{2}\pi xy.dx}{\frac{\pi x^2 y}{4}} + \frac{\frac{1}{4}\pi x^2 dy}{\frac{\pi x^2 y}{4}} = 2.\ \frac{dx}{x} + \frac{dy}{y}$$

or
$$100 \times \frac{dV}{V} = 2\left(100\times\frac{dx}{x}\right) + 100 \times \frac{dy}{y}$$

Given $x = 5$ cm., $y = 8$ cm. and error $dx = dy = \pm 0.1$. So $100 \times \frac{dV}{V} = \pm 100\left(2\times\frac{0.1}{5}+\frac{0.1}{8}\right)$ $= \pm 5.25$.

Thus, the percentage error in volume = ± 5.25.

Example 3. A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m, find the percentage change in the volume of the balloon. [*U.P.T.U., 2005 (Comp.), 2002*]

Sol. Let radius $= r = 1.5$ m, $\delta r = 0.01$ m

height $= h = 4$ m, $\delta h = 0.05$ m

$$\text{volume } (V) = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

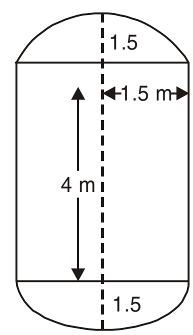


Fig. 2.1

$$\therefore \qquad \frac{\partial V}{\partial r} = 2\pi rh + 4\pi r^2, \quad \frac{\partial V}{\partial h} = \pi r^2$$

$$dV = \frac{\partial V}{\partial r}\, dr + \frac{\partial V}{\partial h}\, dh = (2\pi rh + 4\pi r^2)\, dr + \pi r^2 dh$$

or
$$\frac{dV}{V} = \frac{2\pi r(h+2r)}{\pi r^2\left(h+\frac{4r}{3}\right)}\, dr + \frac{\pi r^2}{\pi r^2\left(h+\frac{4r}{3}\right)}\, dh$$

$$= \frac{3\times 2(h+2r)}{r(3h+4r)}\, dr + \frac{3}{(3h+4r)}\, dh = \frac{3}{r(3h+4r)}\, [2(h + 2r)\, dr + r dh]$$

$$= \frac{3}{1.5(12+6)}\, [2(4 + 3)(0.01) + 1.5\, (0.05)] \qquad \left|\begin{array}{l}\delta r = dr\\ \delta h = dh\end{array}\right.$$

$$= \frac{1}{9}\, [0.14 + 0.075] = \frac{0.215}{9}$$

$$\Rightarrow \qquad 100 \times \frac{dV}{V} = 100 \times \frac{0.215}{9} = 2.389\%$$

Thus, change in the volume = 2.389%.

Example 4. Calculate the percentage increase in the pressure p corresponding to a reduction of $\frac{1}{2}$% in the volume V, if the p and V are related by $pV^{1.4} = C$, where C is a constant.

Sol. We have $$pV^{1.4} = C \qquad ...(i)$$

Taking log of equation (i)

$$\log p + 1.4 \log V = \log C$$

Differentiating

$$\frac{1}{p} \cdot dp + \frac{1.4}{V} \cdot dV = 0 \Rightarrow \frac{dp}{p} = -1.4 \frac{dV}{V}$$

or $$100 \times \frac{dp}{p} = -1.4 \left(100 \times \frac{dV}{V}\right)$$

$$= -1.4 \times \left(\frac{-1}{2}\right) \qquad \left| \frac{dV}{V} = \frac{1}{2}\% = \frac{1}{2 \times 100} \right.$$

$$= 0.7$$

Hence increase in the pressure $p = 0.7\%$.

Example 5. In estimating the cost of a pile of bricks measured as 6′ × 50′ × 4′, the tape is stretched 1% beyond the standard length. If the count is 12 bricks to one ft^3, and bricks cost Rs. 100 per 1000, find the approximate error in the cost. (*U.P.T.U., 2004*)

Sol. Let $$\text{length } (l) = x \qquad ...(i)$$

$$\text{breadth } (b) = y$$

$$\text{height } (h) = z$$

$$\therefore \quad V = lbh = xyz$$

or $$\log V = \log x + \log y + \log z$$

On differentiating $$\frac{1}{V} dV = \frac{1}{x} dx + \frac{1}{y} y + \frac{1}{z} dz$$

or $$100 \times \frac{dV}{V} = 100 \times \frac{dx}{x} + 100 \times \frac{dy}{y} + 100 \times \frac{dz}{z}$$

$$\Rightarrow \quad 100 \times \frac{dV}{V} = 1 + 1 + 1 = 3$$

$$dV = \frac{3V}{100} = \frac{3\,(6 \times 50 \times 4)}{100} = 36 \text{ cube fit}$$

$$\therefore \quad \text{Number of bricks in } dV = 36 \times 12 = 432$$

Hence, $$\text{the error in cost} = 432 \times \frac{.100}{1000} = \text{Rs. } 43.20.$$

Example 6. The angles of a triangle are calculated from the sides a, b, c of small changes δa, δb, δc are made in the sides, show that approximately $\delta A = \frac{a}{2\Delta}$ $[\delta a - \delta b \cos C - \delta c. \cos B]$ where Δ is the area of the triangle and A, B, C are the angles opposite to a, b, c respectively. Verify that $\delta A + \delta B + \delta C = 0$. (*U.P.T.U., 2001*)

Sol. From trigonometry, we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \Rightarrow b^2 + c^2 - a^2 = 2bc \cos A$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A$$

On differentiating, we get

$$a \cdot da = b \cdot db + c \cdot dc - db \cdot c \cos A - b \cdot dc \cos A + bc \sin A \cdot dA$$

$$\Rightarrow \quad bc \sin A \cdot dA = a \cdot da - b \cdot db - c \cdot dc + db \cdot c \cos A + b \cdot dc \cos A$$

$$\Rightarrow \quad 2\Delta \cdot dA = a \cdot da - (b - c \cos A) \cdot db - (c - b \cos A)\, dc$$

$$\Rightarrow \quad 2\,\Delta \cdot dA = a\, da - (a \cos C + c \cos A - c \cos A)\, db - (a \cos B + b \cos A - b \cos A)\, dc$$

$\left|\begin{array}{l} \text{As } \Delta = \frac{1}{2} bc \sin A \\ a = b\cos C + c\cos B \end{array}\right.$

or

$$2\,\Delta \cdot dA = a\, da - db.a \cos C - dc.a \cos B$$

$$\Rightarrow \quad \delta A = \frac{a}{2\Delta} [\delta a - \delta b. \cos C - \delta c. \cos B]$$

$\left|\begin{array}{c} \text{As } \delta A \approx dA \\ \delta a \approx da,\ \delta b \approx db \\ \delta c \approx dc \end{array}\right.$

Hence proved.

Similarly,

$$\delta B = \frac{b}{2\Delta} [\delta b - \delta c. \cos A - \delta a. \cos C]$$

and

$$\delta C = \frac{c}{2\Delta} [\delta c - \delta a. \cos B - \delta b. \cos A]$$

Adding δA, δB and δC, we get

$$\delta A + \delta B + \delta C = \frac{1}{2\Delta} [(a - b \cos C - c \cos B)\, \delta a + (b - a \cos C - c \cos A)\, \delta b + (c - a \cos B - b \cos A)\, \delta c]$$

$$= \frac{1}{2\Delta} [(a - a)\, \delta a + (b - b)\, \delta b + (c - c)\, \delta c]$$

$$\Rightarrow \quad \delta A + \delta B + \delta C = 0.$$ **Verified.**

Example 7. Show that the relative error in c due to a given error in θ is minimum when $\theta = 45°$ if $c = k \tan \theta$.

Sol. We have $c = k \tan\theta$ ...(*i*)

On differentiating, we get

$$dc = k \sec^2 \theta\, d\theta \qquad ...(ii)$$

From (*i*) and (*ii*), we get $\dfrac{dc}{c} = \dfrac{\sec^2 \theta d\theta}{\tan \theta} = \dfrac{2d\theta}{\sin 2\theta}$

Thus, $\dfrac{dc}{c}$ will be minimum if $\sin 2\theta$ is maximum

i.e., $\sin 2\theta = 1 = \sin 90$ | Since $\sin \theta$ lies between -1 and 1

$\Rightarrow \quad 2\theta = 90 \Rightarrow \theta = 45°$. **Hence proved.**

Example 8. Find approximate value of

$$\left[(0.98)^2 + (2.01)^2 + (1.94)^2\right]^{1/2}.$$

Sol. Suppose $f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{1/2}$

$$\therefore \quad \frac{\partial f}{\partial x} = x\left(x^2 + y^2 + z^2\right)^{-1/2}, \frac{\partial f}{\partial y} = y\left(x^2 + y^2 + z^2\right)^{-1/2},$$

$$\frac{\partial f}{\partial z} = z\left(x^2 + y^2 + z^2\right)^{-1/2}$$

Now

$$f(x + \delta x, y + \delta y, z + \delta z) = f(x, y, z) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$$

Let $x = 1,\ y = 2,\ z = 2,\ \delta x = -.02,\ \delta y = .01,\ \delta z = -.06.$

$$\Rightarrow \quad f(0.98, 2.01, 1.94) = \left(1^2 + 2^2 + 2^2\right)^{1/2} + (9)^{-1/2} \times (-0.02) + 2\,(9)^{-1/2}\,(.01) + 2\,(9)^{-1/2}\,(-0.06)$$

$$= 3 - \frac{1}{3}(0.02 - 0.02 + 0.12)$$

$$= 3 - 0.04 = 2.96.$$

Example 9. Prove that the relative error of a quotient does not exceed the sum of the relative errors of dividend and the divisor.

Sol. Let

$$x = \text{dividend}$$
$$y = \text{divisor}$$
$$z = \text{quotient}$$

Then $\dfrac{x}{y} = z$

Taking log on both sides

$$\log x - \log y = \log z$$

Differentiating $\dfrac{dx}{x} - \dfrac{dy}{y} = \dfrac{dz}{z}$

$\Rightarrow$ the relative error in quotient is equal to difference of the relative errors of dividend and divisor.

Hence $\dfrac{dz}{z}$ (relative error) in quotient does not exceed the sum of relative errors of dividend and the divisor

i.e., $\dfrac{dz}{z} < \dfrac{dx}{x} + \dfrac{dy}{y}$. **Hence proved.**

Example 10. The work that must be done to propel a ship of displacement D for a distance 'S in time 't' proportional to $\dfrac{S^2 D^{\frac{2}{3}}}{t^2}$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time diminished by 1% and the distance diminished by 2%.

Sol. Let the work $= W$

then $$W \propto \frac{S^2 D^{\frac{2}{3}}}{t^2}$$

$$\Rightarrow \qquad W = K\cdot\frac{S^2 D^{-2/3}}{t^2}, \text{ where } K \text{ is proportional constant.}$$

Taking log on both sides

$$\log W = \log K + 2\log S + \frac{2}{3}\log D - 2\log t$$

On differentiating $\quad \dfrac{dW}{W} = 2\dfrac{dS}{S} + \dfrac{2}{3}\dfrac{dD}{D} - 2\dfrac{dt}{t}$

$$\Rightarrow \qquad 100\times\frac{dW}{W} = 2\left(100\times\frac{dS}{S}\right) + \frac{2}{3}\left(100\times\frac{dD}{D}\right) - 2\left(100\times\frac{dt}{t}\right)$$

$$= 2(-2) + \frac{2}{3}(1) - 2(-1) = \frac{-4}{3}$$

$\therefore$ Approximate increase of work $= \dfrac{-4}{3}\%$.

Example 11. The height h and semi-vertical angle α, of a cone are measured from there A, the total area of the cone, including the base, is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in the area. Show further that, if $\alpha = \dfrac{\pi}{6}$, an error of + 1 per cent in h will be approximately compensated by an error of –19.8 minutes in α.

Sol. Total area of the cone

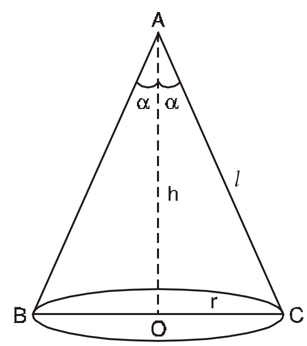


Fig. 2.2

$$A = \pi r^2 + \pi rl$$

or $\quad A = \pi h^2\tan^2\alpha + \pi(h\tan\alpha)(h\sec\alpha)$

$$= \pi h^2(\tan^2\alpha + \tan\alpha\sec\alpha)$$

$$\left|\begin{array}{l}\dfrac{r}{h} = \tan\alpha \Rightarrow r = h\tan\alpha \\ \dfrac{l}{h} = \sec\alpha \Rightarrow l = h\sec\alpha\end{array}\right.$$

Differentiating $\quad \delta A = 2\pi h\,\delta h(\tan^2\alpha + \tan\alpha\sec\alpha)$

$$+\, \pi h^2(2\tan\alpha\sec^2\alpha.\,\delta\alpha + \sec^3\alpha\,\delta\alpha + \tan^2\alpha\cdot\sec\alpha\cdot\delta\alpha)$$

or $\quad \delta A = 2\pi h\tan\alpha.\delta h(\tan\alpha + \sec\alpha) + \pi h^2(\sec^2\alpha + 2.\tan\alpha\sec\alpha$

$$+ \tan^2\alpha)\sec\alpha\,.\,\delta\alpha$$

$$= 2\pi h\tan\alpha.\,\delta h(\tan\alpha + \sec\alpha) + \pi h^2(\sec\alpha + \tan\alpha)^2\sec\alpha.\delta\alpha$$

$$\delta A = \pi h^2(\tan\alpha + \sec\alpha)\left[2\tan\alpha.\frac{\delta h}{h} + (\tan\alpha + \sec\alpha)\sec\alpha.\delta\alpha\right].$$

Now, putting $\quad \alpha = \dfrac{\pi}{6}, \dfrac{\delta h}{h}\times 100 = 1$ and $\delta A = 0$ in above.

$$0 = \pi h^2\left(\tan\frac{\pi}{6} + \sec\frac{\pi}{6}\right)\left[2\tan\frac{\pi}{6}\left(100\times\frac{\delta h}{h}\right).\frac{1}{100} + \left(\tan\frac{\pi}{6} + \sec\frac{\pi}{6}\right).\sec\frac{\pi}{6}.\delta\alpha\right]$$

$$\Rightarrow 2\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{100} + \left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)\cdot\frac{2}{\sqrt{3}}\cdot\delta\alpha = 0$$

$$\Rightarrow \frac{1}{100\sqrt{3}} + \left(\frac{3}{3}\right) \delta\alpha = 0 \Rightarrow \delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radian.}$$

or
$$\delta\alpha = -\frac{1}{100\sqrt{3}} \times \frac{180}{\pi} \text{ degree}$$
$$= -\frac{9}{5\pi\sqrt{3}} \times 60 \text{ minutes } (1° = 60 \text{ minutes})$$
$$= -\frac{9\times 60}{5\times 3.14\times 1.732} = -19.858 \text{ minutes.}$$ **Hence proved.**

Example 12. If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant, prove that $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$ where da, db, dc are small increaments in the sides a, b, c respectively.

Sol. Let R be the circum radius.

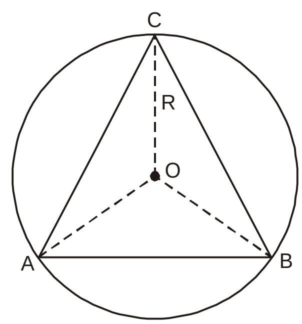


Fig. 2.3

We know that $R = \frac{a}{2\sin A}$

$$\therefore \quad \frac{\partial R}{\partial a} = \frac{1}{2\sin A}, \quad \frac{\partial R}{\partial A} = -\frac{a\times\cos A}{2\sin^2 A}$$

$$\Rightarrow \quad dR = \frac{\partial R}{\partial a} da + \frac{\partial R}{\partial A} dA$$
$$= \frac{1}{2\sin A} da - \frac{a\cos A}{2\sin^2 A} . dA$$

or
$$0 = \frac{1}{2\sin A}\left\{da - \frac{a\cos A}{2\sin A}.dA\right\} \quad |\text{As } R = \text{constant}$$

$$\Rightarrow \quad da - \frac{a\cos A}{\sin A} dA = 0 \Rightarrow \frac{da}{\cos A} = \frac{a}{\sin A} . dA$$

$$\Rightarrow \quad \frac{da}{\cos A} = 2R\, dA$$

Similarly, $\frac{db}{\cos B} = 2R\, dB$

$$\frac{dc}{\cos C} = 2R\, dC$$

Adding these equations, we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC) \quad ...(i)$$

$$\because \quad A + B + C = \pi$$

$$\therefore \quad dA + dB + dC = 0 \quad ...(ii)$$

From (*i*) and (*ii*), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$ **Hence proved.**

Example 13. Considering the area of a circular ring as an increment of area of a circle, find approximately the area of the ring whose inner and outer radii are 3 cm. and 3.02 cm. respectively.

Sol. Let area of a circle $= \pi r^2$

or $A = \pi r^2$

$\Rightarrow \quad dA = 2\pi r\ dr \qquad \ldots(i)$

Now area of circular ring δA is the difference between A_1 of outer ring and A_2 of inner sphere

choose $r = 3$ cm. and $\delta r = 0.02$, then

$A_1 - A_2 = A(r + \delta r) - A(r) = \delta A \approx dA$

$\Rightarrow \quad A_1 - A_2 = dA = 2\pi r.dr$ (from i)

$\Rightarrow \quad A_1 - A_2 = 2\pi \times 3 \times .02$

$= .12\ \pi\ \text{cm}^2.$

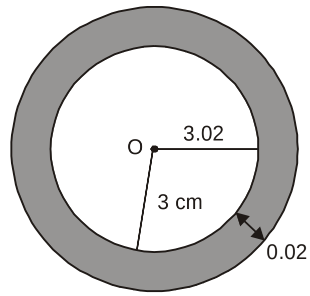


Fig. 2.4

Example 14. Calculate $(2.98)^3$

Sol. Let $f(x, y) = y^x$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = y^x.\log y, \quad \frac{\partial f}{\partial y} = xy^{x-1}$$

$$\text{Now,} \quad f(x + \delta x, y + \delta y) = f(x, y) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

Let $x = 3,\ \delta x = -0.02,\ y = 3,\ \delta y = 0$

$\Rightarrow \quad f(2.98, 3) = f(3, 3) + 3^3.\log 3 \times (-0.02) + 0$

$= 3^3 - 3^3 \times (.4771213) \times 0.02 \qquad |\log 3 = .4771213$

$= 27(1 - 0.00954) = 26.74.$

EXERCISE 2.2

1. The period T of a simple pendulum is $T = 2\pi\sqrt{\dfrac{l}{g}}$. Find the maximum error in T due to possible errors up to 1% in l and 2.5 % in g. (*U.P.T.U., 2003*) [**Ans.** 1.75%]
2. In the estimating the number of bricks in a pile which is measured to be (5 m × 10 m × 5 m), count of bricks is taken as 100 bricks per m^3. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is Rs. 2,000 per thousand bricks. (*U.P.T.U., 2000*) [**Ans.** Rs. 3000]
3. Find the approximately value of $f(0.999)$ where $f(x) = 2x^4 + 7x^3 - 8x^2 + 3x + 1$. [**Ans.** 4.984]

$$\left[\text{Hint: } f(0.999) = f(x + \delta x) = f(x) + \frac{\partial f}{\partial x}.\delta x = f(1) + f'(1)(-0.001)\right]$$

4. If the kinetic energy T is given by $T = \frac{1}{2} mv_e^2$ find approximate change in T as the mass m changes from 49 to 49.5 and the velocity v changes from 1600 to 1590.

[**Ans.** 144000 units]

5. Find the approximate value of $(1.04)^{3.01}$. [**Ans.** 1.12]

6. If Δ be the area of a triangle, prove that the error in Δ resulting from a small error in c is given by

$$\delta \Delta = \frac{\Delta}{4}\left[\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right]\delta c$$

7. Considering the volume of a spherical shell as an increment of volume of a sphere, calculate approximately the volume of a spherical shell whose inner diameter is 8 inches and whose thickness is $\frac{1}{6}$ inch. [**Ans.** 4π cubic inches]

8. A diameter and altitude of a can in the form of right circular cylinder are measured as 4 cm. and 6 cm. respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface. [**Ans.** 5.0336 cm^3, 3.146 cm^2]

9. Find the percentage error in calculating the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when error of + 1 % is made in measuring the major and minor axis. [**Ans.** 2%]

10. The quantity Q of water flowing over a v-notch is given by the formula $Q = cH^{\frac{5}{2}}$ where H is the head of water and c is a constant. Find the error in Q if the error in H is 1.5%.

[**Ans**. 3.75%]

11. Find the percentage error in calculated value of volume of a right circular cone whose altitude is same as the base radius and is measured as 5 cm. with a possible error of 0.02 cm. [**Ans**. 1.2%]

12. Find possible percentage error in computing the parallel resistance r of three resistance r_1, r_2, r_3 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r^2} + \frac{1}{r_3}$ if r_1, r_2, r_3 are each in error by plus 1.2%.

[**Ans**. 12%]

13. The diameter and the height of a right circular cylinder are measured as 4 cm. and 6 cm. respectively, with a possible error of 0.1 cm. Find approximately the maximum possible error in the computed value of the volume and surface area. [**Ans**. 1.6 π cu. cm, π sq. cm]

14. Find $\left[(3.82)^2 + 2(2.1)^3\right]^{\frac{1}{5}}$. [**Ans.** = 2.012]

15. Show that the acceleration due to gravity is reduced nearly by 1% at an altitude equal to 0.5 % of earth's radius. Given that an external point x kilometers from the earth's centre, such an acceleration is given by $g\left(\frac{r}{x}\right)^2$, where r is the radius of the earth.

16. Calculate the error in R if $RI = E$ and possible errors in E and I are 20% and 10% respectively. [**Ans**. 10%]

17. In the manufacture of closed cylindrical boxes with specified sides a, b, c ($a \neq b \neq c$) small changes of A %, B %, C % occurred in a, b, c respectively from box to box from the specified dimension. However, the volume and surface area of all boxes were according to specification, show that

$$\frac{A}{a(b-c)} = \frac{B}{b(c-a)} = \frac{C}{c(a-b)}$$

18. Find $(83.7)^{\frac{1}{4}}$. [**Ans.** 3.025]

19. Find y (1.997) where $y(x) = x^4 - 2x^3 + 9x + 7$. [**Ans.** 24.949]

20. The time T of a complete oscillation of a simple pendulum of length l is governed by $T = 2\pi\sqrt{\frac{l}{g}}$ where g is a constant.

(*a*) Find approximate error in the calculated value of T corresponding to an error of 2% in the value of L. (*U.P.T.U., 2008*) [**Ans.** 1%]

(*b*) By what percentage should the length be changed in order to correct a loss of 2 minutes per day? [**Ans.** – 0.278%]

2.3 EXTREMA OF FUNCTION OF SEVERAL VARIABLES

Introduction

In some practical and theoretical problems, it is required to find the largest and smallest values of a function of two variables where the variables are connected by some given relation or condition known as a constraint. For example, if we plot the function $z = f(x, y)$ to look like a mountain range, then the mountain tops or the high points are called local maxima of $f(x, y)$ and valley bottoms or the low points are called local minima of $f(x, y)$. The highest mountain and lowest valley in the entire range are said to be absolute maximum and absolute minimum. The graphical representation is as follows.

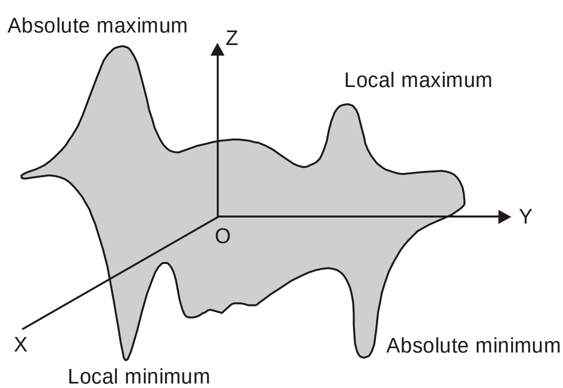


Fig. 2.5

Definition

Let $f(x, y)$ be a function of two independent variables x, y such that it is continuous and finite for all values of x and y in the neighbourhood of their values a and b (say) respectively.

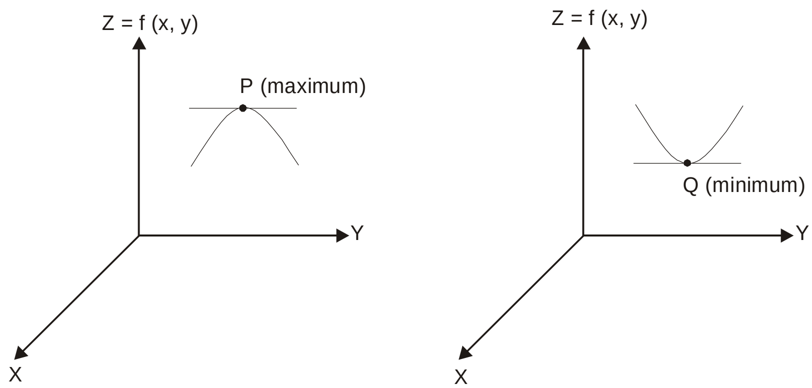
Maximum value: $f(a, b)$ is called maximum value of $f(x, y)$ if $f(a, b) > f(a + h, b + k)$. For small positive or negative values of h and k *i.e.*, $f(a, b)$ is greater than the value of function $f(x, y,)$ at all points in some small *nbd* of (a, b).

Minimum value: $f(a, b)$ is called minimum value of $f(x, y)$ if $f(a, b) < f(a + h, b + k)$.

Note: $f(a + h, b + k) - f(a, b)$ = positive, for Minimum value.
$f(a + h, b + k) - f(a, b)$ = negative, for Maximum value.

Extremum: The maximum or minimum value of the function $f(x, y)$ at any point $x = a$ and $y = b$ is called the extremum value and the point is called "extremum point".

Geometrical representation of maxima and minima: The function $f(x, y)$ represents a surface. The maximum is a point on the surface (hill top). The minimum is a point on the surface (bottom) from which the surface ascends (climbs up) in every direction.



(*a*) (*b*)

Fig. 2.6

Saddle point: It is a point where function is neither maximum nor minimum. At such point f is maximum in one direction while minimum in another direction.

Example: $z = xy$, hyperbolic paraboloid has a saddle point at the origin.

Remark 1 : If $f(x, y) \le f(a, b)$ where (x, y) is a neighbourhood of (a, b). The number $f(a, b)$ is called local maximum value of $f(x, y)$.

Remark 2 : If $f(x, y) \ge f(a, b)$ where (x, y) is a neighbourhood of $f(a, b)$. The number $f(a, b)$ is called local minimum value of $f(x, y)$.

2.3.1 Condition for the Existence of Maxima and Minima (Extrema)

By Taylor's theorem

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,\, b)} + \frac{1}{\lfloor 2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{(a,\, b)} + \ldots \quad \ldots(i)$$

Neglecting higher order terms of h^2, hk, k^2, etc. Since h, k are small, the above expansion reduce to

$$f(a + h, b + k) = f(a, b) + h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}$$

$$\Rightarrow f(a + h, b + k) - f(a, b) = h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y} \quad \ldots(ii)$$

The necessary condition for a maximum or minimum value (L.H.S. of eqn (*ii*) negative or positive) is

$$h\,\frac{\partial f(a,b)}{\partial x} + k\,\frac{\partial f(a,b)}{\partial y} = 0$$

$$\Rightarrow \qquad \frac{\partial f(a,b)}{\partial x} = 0,\ \frac{\partial f(a,b)}{\partial y} = 0 \quad \big| h \text{ and } k \text{ can take both } +\text{ve and } -\text{ve value} \qquad ...(iii)$$

The conditions (*iii*) are necessary conditions for a maxmium or a minimum value of $f(x, y)$.

Note: The conditions given by (*iii*) are not sufficient for existence of a maximum or a minimum value of $f(x, y)$.

2.3.2 Lagrange's Conditions for Maximum or Minimum (Extrema)

Using the conditions (*iii*) in equation (*i*) (2.3.1) and neglecting the higher order term h^3, k^3, $h^2 k$ etc. we get

$$f(a+h, b+k) - f(a, b) = \frac{1}{\underline{|2}}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right]_{(a,\,b)}$$

Putting $\quad \dfrac{\partial^2 f}{\partial x^2} = r,\ \dfrac{\partial^2 f}{\partial x \partial y} = s,\ \dfrac{\partial^2 f}{\partial y^2} = t$, then

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}\,[h^2 r + 2hks + k^2 t]$$

$$= \frac{1}{2}\left[\frac{h^2 r^2 + 2hkrs + k^2 tr}{r}\right]$$

$$\Rightarrow f(a+h, b+k) - f(a, b) = \frac{1}{2}\left[\frac{(hr+ks)^2 + k^2\left(rt - s^2\right)}{r}\right] \qquad ...(iv)$$

If $rt - s^2 > 0$ then the numerator in R.H.S. of (*iv*) is positive. Here sign of L.H.S. = sign of r.

Thus, if $rt - s^2 > 0$ and $r < 0$, then $f(a+h, b+k) - f(a, b) < 0$

if $rt - s^2 > 0$ and $r > 0$, then $f(a+h, b+k) - f(a, b) > 0$.

Therefore, **the Lagrange's conditions for maximum or minimum are:** (*U.P.T.U., 2008*)

1. If $rt - s^2 > 0$ and $r < 0$, then $f(x, y)$ has maximum value at (a, b).
2. If $rt - s^2 > 0$ and $r > 0$, then $f(x, y)$ has minimum value at (a, b).
3. If $rt - s^2 < 0$, then $f(x, y)$ has neither a maximum nor minimum *i.e.*, (a, b) is saddle point.
4. If $rt - s^2 = 0$, then case fail and here again investigate more for the nature of function.

2.3.3 Method of Finding Maxima or Minima

1. Solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$, for the values of x and y. Let $x = a$, $y = b$.

   The point $P(a, b)$ is called critical or stationary point.

2. Find r, s and t at $x = a$, $y = b$.
3. Now check the following conditions:
   (*i*) If $rt - s^2 > 0$ and $r < 0$, $f(x, y)$ has maximum at $x = a$, $y = b$.
   (*ii*) If $rt - s^2 > 0$ and $r > 0$, $f(x, y)$ has minimum at $x = a$, $y = b$.
   (*iii*) If $rt - s^2 < 0$, $f(x, y)$ has neither maximum nor minimum.
   (*iv*) If $rt - s^2 = 0$, case fail.

Example 1. Find the maximum and minimum of $u = x^3 + y^3 - 63(x + y) + 12xy$.

Sol. $$\frac{\partial u}{\partial x} = 3x^2 - 63 + 12y; \quad \frac{\partial u}{\partial y} = 3y^2 - 63 + 12x$$

$$r = \frac{\partial^2 u}{\partial x^2} = 6x, \; s = \frac{\partial^2 u}{\partial x \partial y} = 12, \; t = \frac{\partial^2 u}{\partial y^2} = 6y$$

Now for maximum and minimum value

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad 3x^2 + 12y - 63 = 0 \Rightarrow x^2 + 4y - 21 = 0 \quad \ldots(i)$$

and $$3y^2 + 12x - 63 = 0 \Rightarrow y^2 + 4x - 21 = 0 \quad \ldots(ii)$$

Subtracting (*i*) and (*ii*), we get

$$(x - y)(x + y) - 4(x - y) = 0$$

$$\Rightarrow \quad (x - y)(x + y - 4) = 0$$

$$\Rightarrow \quad x = y \text{ and } x + y = 4$$

Putting $x = y$ in (*i*), we get $x^2 + 4x - 21 = 0 \Rightarrow (x + 7)(x - 3) = 0$

$$\therefore \quad x = 3, \; x = -7$$

$$y = 3, \; y = -7$$

Again putting $y = 4 - x$ in eqn (*i*), we get

$$\therefore \quad x^2 + 4(4 - x) - 21 = 0 \Rightarrow x^2 - 4x - 5 = 0$$

$$\Rightarrow \quad (x - 5)(x + 1) = 0 \Rightarrow x = 5, \; x = -1$$

$$\therefore \quad y = 4 - 5 = -1, \; y = 4 - (-1) = 5.$$

Hence (3, 3), (5, –1), (–7, –7) and (–1, 5) may be possible extremum.

At $x = 3$, $y = 3$, we have $r = 18$; $s = 12$, $t = 18$

$$\therefore \quad rt - s^2 = 18 \times 18 - (12)^2 > 0 \text{ and } r = 18 > 0.$$

So there is minima at $x = 3$, $y = 3$, and the minimum value of u is

$(3)^3 + (3)^3 - 63(3 + 3) + 12(3)(3) = -216$.

At $x = 5$, $y = -1$, we have $r = 30$; $s = 12$, $t = -6$

$\therefore rt - s^2 = 30(-6) - (12)^2 < 0$, so there is neither maxima nor minima at $x = 5$, $y = -1$.

At $x = -7$, $y = -7$, we have $r = -42$; $s = 12$, $t = -42$

$\therefore rt - s^2 = (-42)(-42) - (12)^2 > 0$ and $r < 0$, so there is maxima at $x = -7$, $y = -7$ and its maximum value is

$(-7)^3 + (-7)^3 - 63(-7 - 7) + 12(-7)(-7) = 784$.

At $x = -1$, $y = 5$, we have $r = -6$; $s = 12$, $t = 30$

$\therefore rt - s^2 = (-6)(30) - (12)^2 < 0$, so there is neither maxima nor minima at $x = -1$, $y = -5$.

Example 2. Show that minimum value of $u = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ is $3a^2$.

Sol. $\dfrac{\partial u}{\partial x} = y - a^3 x^{-2}; \dfrac{\partial u}{\partial y} = x - a^3 y^{-2}, r = \dfrac{\partial^2 u}{\partial x^2} = 2a^3 x^{-3};$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1; \; t = \frac{\partial^2 u}{\partial y^2} = 2a^3 y^{-3}.$$

Now for maximum or minimum we must have $\dfrac{\partial u}{\partial x} = 0$, $\dfrac{\partial u}{\partial y} = 0$

So from $\dfrac{\partial u}{\partial x} = 0$, we get $y - a^3x^{-2} = 0$ or $x^2y = a^3$ ...(*i*)

and from $\dfrac{\partial u}{\partial y} = 0$, we get $x - a^3y^{-2} = 0$ or $xy^2 = a^3$ ...(*ii*)

Solving (*i*) and (*ii*), we get $x^2y = xy^2$ or $xy\ (x - y) = 0$

or $x = 0, y = 0$ and $x = y$.

From (*i*) and (*ii*), we find that $x = 0$ and $y = 0$ do not hold as it gives $a = 0$, which is against hypothesis.

$\therefore$ We have $x = y$ and from (*i*) we get $x^3 = a^3$ or $x = a$ and therefore, we have $x = a = y$.

This satisfies (*ii*) also. Hence it is a solution.

At $x = a = y$, we have $r = 2a^3a^{-3} = 2$, $s = 1$, $t = 2$

$\therefore$ $rt - s^2 = (2)\,(2) - 1^2 = 3 > 0$

Also $r = 2 > 0$. Hence, there is minima at $x = a = y$

$\therefore$ The minimum value of u

$= xy + (a^3/x) + (a^3/y)$

at $x = a = y$

$= a.a + (a^3/a) + (a^3/a) = a^2 + a^2 + a^2 = 3a^2$. **Hence proved.**

Example 3. Discuss the maximum or minimum values of u when $u = x^3 + y^3 - 3axy$.

(*U.P.T.U., 2004*)

Sol. $\dfrac{\partial u}{\partial x} = 3x^2 - 3ay; \dfrac{\partial u}{\partial y} = 3y^2 - 3ax; r = \dfrac{\partial^2 u}{\partial x^2} = 6x;$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -3a, \; t = \frac{\partial^2 u}{\partial y^2} = 6y.$$

Now for maximum or minimum, we must have $\dfrac{\partial u}{\partial x} = 0, \dfrac{\partial u}{\partial y} = 0$

So from $\dfrac{\partial u}{\partial y} = 0$, we get $x^2 - ay = 0$ ...(*i*)

and from $\dfrac{\partial u}{\partial y} = 0$, we get $y^2 - ax = 0$ ...(*ii*)

Solving (*i*) and (*ii*), we get $(y^2/a)^2 - ay = 0$

or $y^4 - a^3y = 0$ or $y\ (y^3 - a^3) = 0$ or $y = 0, a$.

Now from (*i*), we have when $y = 0$, $x = 0$, and when $y = a$, $x = \pm\ a$.

But $x = -a$, $y = a$, do not satisfy (*ii*), here are not solutions.

Hence the solutions are $x = 0$, $y = 0$; $x = a$, $y = a$;

At $x = 0$, $y = 0$, we have $r = 0$, $s = -3a$, $t = 0$.

$\therefore$ $rt - s^2 = 0 - (-3a)^2$ = negative and there is neither maximum nor minimum at $x = 0$, $y = 0$.

At $x = a$, $y = a$, we get $r = 6a$, $s = -3a$, $t = 6a$

$\therefore$ $rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 > 0$

Also $r = 6a > 0$ if $a > 0$ and $r < 0$ if $a < 0$.

Hence there is maximum or minimum according as $a < 0$ or $a > 0$. The maximum or minimum value of $u = -a^3$ according as $a < 0$ or $a > 0$.

Example 4. Determine the point where the function

$$u = x^2 + y^2 + 6x + 12 \text{ has a maxima or minima.}$$

Sol.

$$\frac{\partial u}{\partial x} = 2x + 6; \quad \frac{\partial u}{\partial y} = 2y$$

$$r \equiv \frac{\partial^2 u}{\partial x^2} = 2; \quad s \equiv \frac{\partial^2 u}{\partial x \partial y} = 0; \quad t \equiv \frac{\partial^2 u}{\partial y^2} = 2$$

Now for maxima or minima we must have $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0.$

From $\frac{\partial u}{\partial x} = 0$, we get $2x + 6 = 0$ or $x = -3$

From $\frac{\partial u}{\partial y} = 0$, we get $2y = 0$ or $y = 0$

Also at $x = -3$, $y = 0$, $r = 2$, $s = 0$, $t = 2$

$\therefore$ $rt - s^2 = 2(2) - (0)^2 = 4 > 0$ and $r = 2 > 0$

Hence, there is minima at $x = -3$, $y = 0$.

Example 5. A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction. *(U.P.T.U., 2005)*

Sol. $V = 32$ c.c.

Let length = l, breadth = b and height = h

Total surface area $S = 2lh + 2bh + lb$ ...(*i*)

$$S = 2(l + b)h + lb$$

Now volume $V = lbh = 32 \Rightarrow b = \frac{32}{lh}$ ...(*ii*)

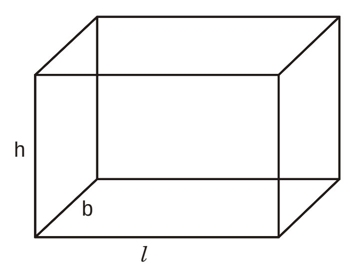


Fig. 2.7

Putting the value of '*b*' in equation (*i*)

$$S = 2\left(l + \frac{32}{lh}\right)h + l\left(\frac{32}{lh}\right)$$

$$S = 2lh + \frac{64}{l} + \frac{32}{h} \quad \text{...(iii)}$$

$$\therefore \quad \frac{\partial S}{\partial l} = 2h - \frac{64}{l^2}, \quad \frac{\partial S}{\partial h} = 2l - \frac{32}{h^2}$$

For minimum S, we get

$$\frac{\partial S}{\partial l} = 0 \Rightarrow 2h - \frac{64}{l^2} = 0 \Rightarrow h = \frac{32}{l^2} \quad ...(iv)$$

and
$$\frac{\partial S}{\partial h} = 0 \Rightarrow 2l - \frac{32}{h^2} = 0 \Rightarrow l = \frac{16}{h^2} \quad ...(v)$$

From (iv) and (v), we get

$$h = \frac{32 \times h^4}{256} \Rightarrow h^3 = 8 \Rightarrow h = 2$$

Putting $h = 2$, in equation (v), we get $l = \frac{16}{4} = 4$

From (ii) $\quad b = \frac{32}{4 \times 2} = 4$

Now, $\quad \frac{\partial^2 S}{\partial l^2} = \frac{128}{l^3} = \frac{128}{64} = 2 \Rightarrow r = 2 > 0$

and $\frac{\partial^2 S}{\partial l \partial h} = 2 \Rightarrow s = 2$ and $\frac{\partial^2 S}{\partial h^2} = \frac{64}{h^3} = \frac{64}{8} = 8 \Rightarrow t = 8$

$\therefore \quad rt - s^2 = 2 \times 8 - 4 = 12 > 0$

$\Rightarrow \quad rt - s^2 > 0$ and $r > 0$

Hence, S is minimum, for least material

$$l = 4,\ b = 4,\ h = 2.$$

Example 6. Examine the following surface for high and low points $z = x^2 + xy + 3x + 2y + 5$.

Sol. $\quad \frac{\partial z}{\partial x} = 2x + y + 3; \ \frac{\partial z}{\partial y} = x + 2$

$$r = \frac{\partial^2 z}{\partial x^2} = 2; \ s = \frac{\partial^2 z}{\partial x \partial y} = 1; \ t = \frac{\partial^2 z}{\partial y^2} = 0$$

For the maximum or minimum we must have

$$\frac{\partial z}{\partial x} = 0, \ \frac{\partial z}{\partial y} = 0.$$

From $\quad \frac{\partial z}{\partial x} = 0$, we get $2x + y + 3 = 0 \quad ...(i)$

From $\quad \frac{\partial z}{\partial y} = 0$, we get $x + 2 = 0. \quad ...(ii)$

From (ii), we get $x = -2$ and $\therefore$ from (i) $y = -3 + 4 = 1$.

Hence, the solution is $x = -2$, $y = 1$ and for these values we have $r = 2$, $s = 1$, $t = 0$.

$\therefore \quad rt - s^2 = (2)(0) - (1)^2 = -1 < 0.$

$\therefore \quad$ There is neither maximum nor minimum at $x = -2$, $y = 1$.

Example 7. Discuss the maximum and minimum values of 2 sin $\frac{1}{2}(x + y)$ cos $\frac{1}{2}(x - y)$ + cos $(x + y)$.

Sol. Let

$$u = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) + \cos (x + y)$$
$$= \sin x + \sin y + \cos (x + y)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \cos x - \sin (x + y); \quad \frac{\partial u}{\partial y} = \cos y - \sin (x + y)$$

$$r = \frac{\partial^2 u}{\partial x^2} = - \sin x - \cos (x + y); \; s = \frac{\partial^2 u}{\partial x \partial y} = - \cos (x + y);$$

$$t = \frac{\partial^2 u}{\partial y^2} = - \sin y - \cos (x + y)$$

For maximum or minimum, we must have $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$

From $\frac{\partial u}{\partial x} = 0$, we get $\cos x - \sin (x + y) = 0$ ...(*i*)

From $\frac{\partial u}{\partial y} = 0$, we get $\cos y - \sin (x + y) = 0$. ...(*ii*)

Solving (*i*) and (*ii*), we get $\cos x = \cos y$ which gives

$x = 2n\pi \pm y$, where n is any integer. In particular $x = y$.

When $x = y$, from (*i*), we get $\cos x - \sin 2x = 0$

or $\cos x (1 - 2 \sin x) = 0$ which gives $\cos x = 0$, $\sin x = \frac{1}{2}$

If $\sin x = \frac{1}{2}$, then $x = n\pi + (-1)^n . \frac{1}{6}\pi = y$, $\because x = y$

and for these value of x and y, we have

$$r = -\frac{1}{2} - \cos \left[2n\pi + (-1)^n . \frac{1}{3}\pi\right] < 0. \quad \textbf{(Note)}$$

Similarly, $t < 0$ and $s < 0$ and $r > s$, $t > s$.

$\therefore$ $rt - s^2 > 0$. Also $r < 0$.

Hence, there is a maximum when $x = n\pi + (-1)^n . \frac{1}{6}\pi = y$.

If $\cos x = 0$, then $x = 2n\pi \pm \frac{1}{2}\pi = y$, $\because x = y$

From here, we get $x = y = \pm \frac{1}{2}\pi$, $(3/2)\pi$, $(5/2)\pi$ etc.

If $x = \frac{1}{2}\pi = y$, then $r = -1 + 1 = 0$, $s = 1$, $t = 0$

$\therefore$ $rt - s^2 < 0$. Hence, there is neither maximum nor minimum at $x = \frac{1}{2}\pi = y$.

If $x = -\frac{1}{2}\pi = y$, then $r = 1 + 1 = 2 = t,\ \ s = 1$

$\therefore\ \ rt - s^2 = (2 \times 2) - 1 = 3 > 0$. Also $r > 0$

Hence, there is minimum at $x = -\frac{1}{2}\pi = y$. In a similar way we can discuss other values too.

Example 8. Show that the distance of any point (x, y, z) on the plane $2x + 3y - z = 12$, from the origin is given by

$$l = \sqrt{\left[x^2 + y^2 + (2x + 3y - 12)^2\right]}.$$

Hence find a point on the plane that is nearest to the origin.

Sol.

$$\begin{aligned} l &= \text{distance between } (x, y, z) \text{ and } (0, 0, 0) \\ &= \sqrt{\left[(x-0)^2 + (y-0)^2 + (z-0)^2\right]} = \sqrt{\left[x^2 + y^2 + z^2\right]} \\ &= \sqrt{\left[x^2 + y^2 + (2x + 3y - 12)^2\right]}, \quad \because z = 2x + 3y - 12 \end{aligned}$$

or $\quad l^2 = x^2 + y^2 + (2x + 3y - 12)^2 = u$ (say)

or $\quad l^2 = u = 5x^2 + 10y^2 + 12xy - 48x - 72y + 144 \quad$ ...(*i*)

$\therefore \quad \frac{\partial u}{\partial x} = 10x + 12y - 48;\ \frac{\partial u}{\partial y} = 20y + 12x - 72$

$r = \frac{\partial^2 u}{\partial x^2} = 10;\ s = \frac{\partial^2 u}{\partial y \partial x} = 12\ ;\ t = \frac{\partial^2 u}{\partial y^2} = 20$

$\therefore \quad rt - s^2 = (10)(20) - (12)^2 = 56 > 0$ and $r > 0$, so there is a minimum value of l.

Also $\quad \frac{\partial u}{\partial x} = 0 \Rightarrow 10x + 12y - 48 = 0$ or $5x + 6y = 24 \quad$ ...(*ii*)

and $\quad \frac{\partial u}{\partial y} = 0 \Rightarrow 20y + 12x - 72 = 0$ or $5y + 3x = 18 \quad$ ...(*iii*)

Solving (*ii*) and (*iii*), we get $x = \left(\frac{12}{7}\right),\ y = \left(\frac{18}{7}\right)$

Also $\quad 2x + 3y - z = 12$ or $2\left(\frac{12}{7}\right) + 3\left(\frac{18}{7}\right) - z = 12$

or $\quad 24 + 54 - 7z = 84$ or $7z = 24 + 54 - 84 = -6$ or $z = -\frac{6}{7}$

$\therefore$ The required point is $\left(\frac{12}{7}, \frac{18}{7}, -\frac{6}{7}\right)$.

Example 9. Discuss the maximum and minimum values of

$$x^4 + 2x^2y - x^2 + 3y^2.$$

Sol. Let $\quad u = x^4 + 2x^2y - x^2 + 3y^2$

Then $\quad \frac{\partial u}{\partial x} = 4x^3 + 4xy - 2x\ ;\quad \frac{\partial u}{\partial y} = 2x^2 + 6y;$

$$\therefore \qquad r = \frac{\partial^2 u}{\partial x^2} = 12x^2 + 4y - 2; \quad s = \frac{\partial^2 u}{\partial x \partial y} = 4x; \; t = \frac{\partial^2 u}{\partial y^2} = 6$$

For maximum and minimum, we must have $\frac{\partial u}{\partial x} = 0, \; \frac{\partial u}{\partial y} = 0.$

From $\frac{\partial u}{\partial x} = 0$, we get $2x(2x^2 + 2y - 1) = 0$ or $2x^2 + 2y - 1 = 0$ ...(*i*)

From $\frac{\partial u}{\partial y} = 0$, we get $2x^2 + 6y = 0$ or $x^2 + 3y = 0$ ...(*ii*)

Solving (*i*) and (*ii*), we get $4y + 1 = 0$ or $y = -\frac{1}{4}$

$\therefore$ From (*ii*), we get $x^2 = -3y = \left(\frac{3}{4}\right)$ or $x = \pm \frac{1}{2}\sqrt{3}$

$\therefore$ The solutions are $x = \frac{1}{2}\sqrt{3}, \; y = -\frac{1}{4}$ and $x = -\frac{1}{2}\sqrt{3}, \; y = -\frac{1}{4}$

When $x = \frac{1}{2}\sqrt{3}, \; y = -\frac{1}{4}$, we get

$$r = 12\left(\frac{3}{4}\right) + 4\left(-\frac{1}{4}\right) - 2 = 6, \; s = 4\left(\frac{1}{2}\sqrt{3}\right) = 2\sqrt{3}, \; t = 6$$

$\therefore$ $rt - s^2 = 6 \times 6 - \left(2\sqrt{3}\right)^2 > 0$. Also $r > 0$

$\therefore$ There is a minimum when $x = \frac{1}{2}\sqrt{3}, \; y = -\frac{1}{4}$

Again when $x = -\frac{1}{2}\sqrt{3}, \; y = -\frac{1}{4}$, we have $r = 6, \; s = -2\sqrt{3}, \; t = 6$

$\therefore$ $rt - s^2 = (6)\,(6) - \left(-2\sqrt{3}\right)^2 > 0$. Also $r > 0$.

Hence as before there is a minimum when

$$x = \frac{1}{2}\sqrt{3}, \; y = -\frac{1}{4}.$$

Example 10. Find the shortest distance between the lines

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and } \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}.$$

Sol. Let $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = \lambda \Rightarrow x = \lambda + 3, \; y = 5 - 2\lambda, \; z = 7 + \lambda$

Thus any point P on the line is $(3 + \lambda, 5 - 2\lambda, 7 + \lambda)$

and let $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = \mu \Rightarrow x = -1 + 7\mu, \; y = -1 - 6\mu, \; z = -1 + \mu$

The point Q is $(-1 + 7\mu, -1 - 6\mu, -1 + \mu)$

$\therefore$ Distance between these two lines is

$$D = \sqrt{(3+\lambda+1-7\mu)^2 + (5-2\lambda+1+6\mu)^2 + (7+\lambda+1-\mu)^2}$$

$$\Rightarrow \qquad D^2 = u(\text{Say}) = 6\lambda^2 + 86\mu^2 - 40\lambda\mu + 105.$$

$\therefore \qquad \dfrac{\partial u}{\partial \lambda} = 12\lambda - 40\mu, \; \dfrac{\partial u}{\partial \mu} = 172\mu - 40\lambda.$

But $\qquad \dfrac{\partial u}{\partial \lambda} = 0$ and $\dfrac{\partial u}{\partial \mu} = 0 \Rightarrow 12\lambda - 40\mu = 0,\ 172\mu - 40\lambda = 0.$

Solving these equations, we get $\lambda = 0,\ \mu = 0$

$$r = \frac{\partial^2 u}{\partial \lambda^2} = 12, \quad t = \frac{\partial^2 u}{\partial \mu^2} = 172, \quad \frac{\partial^2 u}{\partial \lambda \partial \mu} = s = -40$$

Now, $\qquad rt - s^2 = (12)\cdot(172) - (-40)^2 = 464 > 0$

$\Rightarrow \qquad rt - s^2 > 0$ and $r > 0$

Hence u occurs minimum value at $\lambda = 0$ and $\mu = 0$.

The shortest distance is given by

$$D = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116} = 2\sqrt{29}.$$

Example 11. The temperature T at any point (x, y, z) in space is $T(x, y, z) = Kxyz^2$ where K is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

(*U.P.T.U., 2008*)

Sol. $\qquad T = Kxyz^2 \qquad$ ...(*i*)

$$x^2 + y^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 - y^2$$

From (*i*) $\qquad T = Kxy\,(a^2 - x^2 - y^2)$

$\therefore \qquad \dfrac{\partial T}{\partial x} = Ky\,(a^2 - x^2 - y^2) - 2\,Kx^2y = Ky(a^2 - 3x^2 - y^2)$

Similarly, $\qquad \dfrac{\partial T}{\partial y} = Kx(a^2 - x^2 - 3y^2)$

for maximum and minimum value

$$\frac{\partial T}{\partial x} = 0 \text{ and } \frac{\partial T}{\partial y} = 0 \Rightarrow x = 0 \text{ and } y = 0$$

or $\qquad 3x^2 + y^2 = a^2$

$\qquad x^2 + 3y^2 = a^2$

Solving $\quad x = y = \pm \dfrac{a}{2}$

$$r = \frac{\partial^2 T}{\partial x^2} = -6\,Kxy, \quad s = \frac{\partial^2 T}{\partial x \partial y} = K\,(a^2 - 3x^2 - 3y^2)$$

and $\qquad t = \dfrac{\partial^2 T}{\partial y^2} = -6Kxy$

At (0, 0) $\qquad r = 0,\ s = Ka^2$ and $t = 0$

$\therefore \qquad rt - s^2 = 0.0 - Ka^2 = -Ka^2 < 0$

So, there is neither maximum nor minimum at $x = 0$ and $y = 0$

At $\qquad x = \dfrac{a}{2},\ y = \dfrac{a}{2}$ and $x = -\dfrac{a}{2},\ y = -\dfrac{a}{2}$

$$r = -\frac{6}{4}Ka^2 = -\frac{3}{2}Ka^2 < 0, \quad t = -\frac{3}{2}Kxy, \quad s = -\frac{a^2K}{2}$$

$$rt - s^2 = \frac{9}{4}K^2a^4 - \frac{a^4K^2}{4} = 2K^2a^4 > 0$$

$\therefore\ rt - s^2 > 0$ and $r < 0$

Hence, T has a maximum value at $x = \pm\, a/2$ and $y = \pm\, a/2$

The maximum value of $T = K \cdot \frac{a^2}{4}\left(\frac{a^2}{2}\right) = \frac{Ka^4}{8}$.

Example 12. Find the maximum and minimum values of the function $z = \sin x \sin y \sin (x + y)$.

Sol. Given $z = \sin x \sin y \sin (x + y)$

$$= \frac{1}{2}\ [2 \sin x \sin y] \sin (x + y)$$

$$= \frac{1}{2}\ [\cos (x - y) - \cos (x + y)] \sin (x + y)$$

$$= \frac{1}{4}\ [2 \sin (x + y) \cos (x - y) - 2 \sin (x + y) \cos (x + y)]$$

or
$$z = \frac{1}{4}\ [\sin 2x + \sin 2y - \sin(2x + 2y)]$$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{2}\ [\cos 2x - \cos (2x + 2y)]$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}[\cos 2y - \cos (2x + 2y)]$$

$$r = \frac{\partial^2 z}{\partial x^2} = -\sin 2x + \sin (2x + 2y) \qquad ...(A)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \sin (2x + 2y) \qquad ...(B)$$

$$t = \frac{\partial^2 z}{\partial y^2} = -\sin 2y + \sin (2x + 2y) \qquad ...(C)$$

For maximum or minimum, we must have $\frac{\partial z}{\partial x} = 0,\ \frac{\partial z}{\partial y} = 0$

From $\frac{\partial z}{\partial x} = 0$, we get $\cos 2x - \cos (2x + 2y) = 0$ ...(*i*)

From $\frac{\partial z}{\partial y} = 0$, we get $\cos 2y - \cos (2x + 2y) = 0$ ...(*ii*)

Solving (*i*) and (*ii*), we get $\cos 2x = \cos 2y$ which gives

$2x = 2n\pi \pm 2y$. In particular $2x = 2y$ or $x = y$

When $x = y$, from (*i*), we get $\cos 2x - \cos 4x = 0$

or $\cos 2x - (2 \cos^2 2x - 1) = 0 \qquad \because\ \cos 2\theta = 2 \cos^2 \theta - 1$

or $2 \cos^2 2x - \cos 2x - 1 = 0$

or $\cos 2x = \dfrac{1 \pm \sqrt{(1+8)}}{4} = \dfrac{1 \pm 3}{4} = 1, -\dfrac{1}{2}$

or $2x = 2n\pi \pm 0,\ 2m\pi \pm \dfrac{2\pi}{3}$, where m, n are zero or any integers

or $x = n\pi,\ m\pi \pm \dfrac{\pi}{3}$

In particular $x = \dfrac{\pi}{3}$

When $x = \dfrac{\pi}{3}$, we have $y = x = \dfrac{\pi}{3}$

and then $r = -\sin\dfrac{2\pi}{3} + \sin\dfrac{4\pi}{3}$, from (A)

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}\ ;$$

$$s = \sin\frac{4\pi}{3}, \text{ from } (B)$$

or $$s = -\frac{\sqrt{3}}{2}$$

and $$t = -\sin\frac{2\pi}{3} + \sin\frac{4\pi}{3}, \text{ from } (C)$$

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$\therefore \quad rt - s^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4}$$

$$= \text{positive.}$$

Thus at $x = \dfrac{\pi}{3} = y$, $rt - s^2 > 0$, $r < 0$, so there is a maximum at $x = \dfrac{\pi}{3} = y$.

Hence, maximum value $= \sin\dfrac{\pi}{3} \cdot \sin\dfrac{\pi}{3} \cdot \sin\left(\dfrac{\pi}{3} + \dfrac{\pi}{3}\right)$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}.$$

If we take $x = -\dfrac{\pi}{3}$, then $y = x = -\dfrac{\pi}{3}$

and $r = \sqrt{3}$, $s = \dfrac{1}{2}\sqrt{3}$, $t = \sqrt{3}$

$$\therefore \quad rt - s^2 = \frac{9}{4} > 0,\ r > 0$$

There is a minimum at $x = -\dfrac{\pi}{3} = y$.

Hence, the minimum value $= \sin\left(-\dfrac{\pi}{3}\right)\sin\left(-\dfrac{\pi}{3}\right)\sin\left\{\left(-\dfrac{\pi}{3}\right)+\left(-\dfrac{\pi}{3}\right)\right\}$

$$= -\sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}\ \sin\frac{2\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{8}.$$

Example 13. Find the local minima or local maxima of the function

$$f(x, y) = 4x^2 + y^2 - 4x + 6y - 15.$$

Sol. We have $f(x, y) = 4x^2 + y^2 - 4x + 6y - 15$

$$\frac{\partial f}{\partial x} = 8x - 4 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y + 6.$$

For stationary point $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\therefore$ $$8x - 4 = 0 \Rightarrow x = \frac{1}{2} \text{ and } 2y + 6 = 0 \Rightarrow y = -3$$

Now, the given function can be written as

$$f(x, y) = (2x - 1)^2 + (y + 3)^2 - 25 \qquad ...(i)$$

Since $(2x - 1)^2 \geq 0$ and $(y + 3)^2 \geq 0$.

From (*i*), we get

$f(x, y) \geq -25$ for all values of x and y. Hence $f(1/2, -3) = -25$ is a local minimum.

Example 14. Find the local minima or local maxima of the function

$$f(x, y) = -x^2 - y^2 + 2x + 2y + 16.$$

Sol. $$\frac{\partial f}{\partial x} = -2x + 2 = 0 \Rightarrow x = 1$$

$$\frac{\partial f}{\partial y} = -2y + 2 = 0 \Rightarrow y = 1$$

Now, $$f(x, y) = 18 - \{(x - 1)^2 + (y - 1)^2\} \qquad ...(i)$$

Since $(x - 1)^2 \geq 0$ and $(y - 1)^2 \geq 0$, from (*i*) we get $f(x, y) \leq 18$ for all values of x and y.

Hence $f(1, 1) = 18$ is a local maxima.

EXERCISE 2.3

1. Discuss the maximum values of u, where

$$u = 2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$$

$\left[\textbf{Ans. } x = \frac{a}{2}, y = \frac{a}{2}\right]$

2. Find the points (x, y) where the function $u = xy(1 - x - y)$ is maximum or minimum.

$\left[\textbf{Ans. } \text{Maxima at } x = \frac{1}{3}, y = \frac{1}{3}\right]$

3. Find the extrema of $f(x, y) = (x^2 + y^2)\, e^{(6x + 2x2)}$.

[**Ans.** minima at (0, 0) minimum value = 0 and at (–1, 0) min. value = e^{-4}, saddle point $(-\frac{1}{2}, 0)$].

4. If the perimeter of a triangle is constant prove that the area of this triangle is maximum when the triangle is equilateral.

   [**Hint**: $2s = a + b + c$, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$]

   [**Ans.** Maximum when $a = b = c = \dfrac{2s}{3}$]

5. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

   [**Hint:** $V = xyz$, diagonal of cubic $= \sqrt{x^2+y^2+z^2} = d \Rightarrow z = \sqrt{d^2-x^2-y^2}$ so

   $$V = xy\sqrt{d^2-x^2-y^2}\ \Big]$$

6. Find the shortest distance from origin to the surface $xyz^2 = 2$. [**Ans.** 2]
7. In a plane triangle ABC find the maximum value of $\cos A \cos B \cos C$. [**Ans.** 1/8]
8. Discuss the maximum or minimum values of u given by $u = x^3y^2(1 - x - y)$.

   [**Ans.** Maximum at $x = 0$, $y = 1/3$]

9. Find the maximum and minimum values of $u = 6xy + (47 - x - y)(4x + 3y)$.

   [**Ans.** Max. value of $u = 3384$]

10. Discuss the maxima and minima of the function

    $f(x, y) = \cos x, \cos y \cos (x + y)$ (*U.P.T.U., 2007*)

11. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. [**Ans.** 4, 8, 12]
12. Examine for extreme values: $u(x, y) = x^2 + y^2 + 6x + 12$. [**Ans.** Min. value = 3]

2.4 LAGRANGE'S* METHOD OF UNDETERMINED MULTIPLIERS

Let $\phi(x, y, z)$ is a function of three independent variables, where x, y, z are related by a known constraint $g(x, y, z) = 0$

Thus the problem is Extrema of

$$u = f(x, y, z) \quad \text{...}(i)$$

Subject to $$g(x, y, z) = 0 \quad \text{...}(ii)$$

For stationary point $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

$\therefore$ $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \quad \text{...}(iii)$$

From (*ii*) $$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz = 0 \quad \text{...}(iv)$$

Multiplying eqn. (*iv*) by λ and adding to (*iii*), we obtain

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}\right)dz = 0 \quad \text{...}(v)$$

*Joseph Louis Lagrange (1736–1813).

Since x, y, z are independent variables

$$\therefore \quad \frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x} = 0 \quad ...(vi)$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y} = 0 \quad ...(vii)$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z} = 0 \quad ...(viii)$$

On solving (*ii*), (*vi*), (*vii*) and (*viii*), we can find x, y, z and λ for which $f(x, y, z)$ has maximum or minimum.

Notes: 1. The Lagrange's method of undetermined multiplier's introduces an additional unknown constant λ known as Lagrange's multiplier.

2. Nature of stationary points cannot be determined by Lagranges method.

Example 1. Determine the maxima and minima of $x^2 + y^2 + z^2$ when $ax^2 + by^2 + cz^2 = 1$.

Sol. Let $\quad f(x, y, z) = x^2 + y^2 + z^2 \quad ...(i)$

and $\quad g(x, y, z) \equiv ax^2 + by^2 + cz^2 - 1 = 0 \quad ...(ii)$

From (*i*) $$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 2z$$

From (*ii*) $$\frac{\partial g}{\partial x} = 2ax, \frac{\partial g}{\partial y} = 2by, \frac{\partial g}{\partial z} = 2cz.$$

Now from Lagrange's equations, we get

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x} = 2x + \lambda \,.\, 2ax = 0 \Rightarrow 2x\,(1 + \lambda a) = 0 \Rightarrow x\,(1 + \lambda a) = 0 \quad ...(iii)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0 \Rightarrow 2y + \lambda \cdot 2by = 0 \Rightarrow 2y\,(1 + \lambda b) = 0 \Rightarrow y\,(1 + \lambda b) = 0 \quad ...(iv)$$

and $$\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z} = 0 \Rightarrow 2z + \lambda \cdot 2cz = 0 \Rightarrow 2z\,(1 + \lambda c) = 0 \Rightarrow z\,(1 + \lambda b) = 0 \quad ...(v)$$

Multiplying these equations by x, y, z respectively and adding, we get

$$x^2(1 + \lambda a) + y^2(1 + \lambda b) + z^2(1 + \lambda c) = 0$$

or $$(x^2 + y^2 + z^2) + \lambda\,(ax^2 + by^2 + cz^2) = 0 \quad ...(vi)$$

Using (*i*) and (*ii*) in above equation, we get

$$f + \lambda = 0 \Rightarrow \lambda = -f$$

Putting $\lambda = -f$ in equations (*iii*), (*iv*) and (*v*), we get

$$x\,(1 - fa) = 0,\ y\,(1 - fb) = 0,\ z\,(1 - fc) = 0$$

$$\Rightarrow \quad 1 - fa = 0,\ 1 - fb = 0,\ 1 - fc = 0$$

i.e., $$f = \frac{1}{a}, \frac{1}{b}, \frac{1}{c}.$$ These give the max. and min. values of f.

Example 2. Find the extreme value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$.

(*U.P.T.U., 2007*)

Sol. Let $\quad u = x^2 + y^2 + z^2 \quad ...(i)$

Given $\quad ax + by + cz = p. \quad ...(ii)$

For max. or min. from (*i*), we have

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 0. \qquad ...(iii)$$

Also from (*ii*), $a\,dx + b\,dy + c\,dz = 0.$ ...(*iv*)

Multiplying (*iv*) by λ and adding in (*iii*), we get $(x\,dx + y\,dy + z\,dz) + \lambda\,(a\,dx + b\,dy + c\,dz) = 0.$

Equating the coefficients of dx, dy and dz to zero, we get

$$x + \lambda a = 0,\ y + \lambda b = 0,\ z + \lambda c = 0 \qquad ...(v)$$

These are Lagrange's equations.

Multiplying these by x, y, z respectively and adding, we get

$$x\,(x + \lambda a) + y\,(y + \lambda b) + z\,(z + \lambda c) = 0$$

or $$(x^2 + y^2 + z^2) + \lambda\,(ax + by + cz) = 0$$

or $$u + \lambda p = 0 \text{ or } \lambda = -\,u/p.$$

$\therefore$ From (*v*), we get

$$x - \left(\frac{au}{p}\right) = 0,\ \ y - \left(\frac{bu}{p}\right) = 0,\ \ z - \left(\frac{cu}{p}\right) = 0$$

or $$\frac{x}{a} = \frac{u}{p} = \frac{y}{b} = \frac{z}{c} \text{ or } \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \qquad ...(vi)$$

From (*ii*), we get $a^2\left(\frac{x}{a}\right) + b^2\left(\frac{y}{b}\right) + c^2\left(\frac{z}{c}\right) = p$

or $$a^2\left(\frac{x}{a}\right) + b^2\left(\frac{x}{a}\right) + c^2\left(\frac{x}{a}\right) = p, \text{ from } (vi)$$

or $$(a^2 + b^2 + c^2)\left(\frac{x}{a}\right) = p \text{ or } x = \frac{ap}{a^2 + b^2 + c^2}$$

Similarly, $$y = \frac{bp}{a^2 + b^2 + c^2},\quad z = \frac{cp}{a^2 + b^2 + c^2}$$

These give the minimum value of u.

Hence minimum value of u is

$$u = \frac{a^2p^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2p^2}{(a^2 + b^2 + c^2)^2} + \frac{c^2p^2}{(a^2 + b^2 + c^2)^2}$$

$$= \frac{(a^2 + b^2 + c^2)p^2}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{(a^2 + b^2 + c^2)}.$$

Example 3. Find the maximum and minimum values of $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ where $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $lx + my + nz = 0.$

Sol. Let $$u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \qquad ...(i)$$

Given $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad ...(ii)$$

and $$lx + my + nz = 0 \qquad ...(iii)$$

From (*i*), (*ii*) and (*iii*), we get

$$\left(\frac{x}{a^4}\right)dx+\left(\frac{y}{b^4}\right)dy+\left(\frac{z}{c^4}\right)dz = 0 \qquad ...(iv)$$

$$\left(\frac{x}{a^2}\right)dx+\left(\frac{y}{b^2}\right)dy+\left(\frac{z}{c^2}\right)dz = 0 \qquad ...(v)$$

and $$l\,dx + m\,dy + n\,dz = 0 \qquad ...(vi)$$

Multiplying (*v*) and (*vi*) by λ_1, λ_2 and adding, we get

$$\left(\frac{x}{a^4}+\frac{\lambda_1 x}{a^2}+\lambda_2 l\right)dx + \left(\frac{y}{b^4}+\frac{\lambda_1 y}{b^2}+\lambda_2 m\right)dy + \left(\frac{z}{c^4}+\frac{\lambda_1 z}{c^2}+\lambda_2 n\right)dz = 0$$

Equating to zero the coefficients of dx, dy and dz, we get

$$\frac{x}{a^4}+\frac{\lambda_1 x}{a^2}+\lambda_2 l=0\,,\ \frac{y}{b^4}+\frac{\lambda_1 y}{b^2}+\lambda_2 m=0\,,\ \frac{z}{c^4}+\frac{\lambda_1 z}{c^2}+\lambda_2 n=0 \qquad ...(vii)$$

These are Lagrange's equations.

Multiplying these by x, y, z and adding, we get

$$\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) + \lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) + \lambda_2\,(lx + my + nz) = 0$$

or $u + \lambda_1 (1) + \lambda_2 (0) = 0$, using (*i*), (*ii*) and (*iii*)

or $$u + \lambda_1 = 0 \text{ or } \lambda_1 = -u$$

∴ From (*vii*), we have

$$\frac{x}{a^4}-\frac{ux}{a^2}+\lambda_2 l = 0,\quad \frac{y}{b^4}-\frac{uy}{b^2}+\lambda_2 m = 0,\quad \frac{z}{c^4}-\frac{uz}{c^2}+\lambda_2 n = 0$$

or $$x\,(1 - a^2u) = -la^4\lambda_2,\ y\,(1 - b^2u) = -mb^4\lambda_2,\ z\,(1 - c^2u) = -nc^4\lambda_2$$

or $$x = \frac{-la^4\lambda_2}{1-a^2u},\ y = \frac{-mb^4\lambda_2}{1-b^2u},\ z = \frac{-nc^4\lambda_2}{1-c^2u}$$

Substituting these values in (*iii*), we get

$$\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u} = 0$$

or $$\Sigma l^2a^4\,(1 - b^2u)\,(1 - c^2u) = 0$$

or $$\Sigma l^2a^4\,\{b^2c^2u^2 - (b^2 + c^2)\,u + 1\} = 0$$

or $$u^2\,(\Sigma l^2a^4b^2c^2) - u\,\{\Sigma l^2a^4\,(b^2 + c^2)\} + \Sigma l^2a^4 = 0$$

or $$a^2b^2c^2\,(l^2a^2 + m^2b^2 + n^2c^2)u^2 - a^2b^2c^2\left\{\Sigma l^2a^2\left(\frac{1}{c^2}+\frac{1}{b^2}\right)\right\}u + \Sigma l^2a^4 = 0$$

or $$(l^2a^2 + m^2b^2 + n^2c^2)\,u^2 - \left\{\Sigma l^2\left(\frac{a^2}{c^2}+\frac{a^2}{b^2}\right)\right\}u + \left(\frac{a^2l^2}{b^2c^2}+\frac{b^2m^2}{c^2a^2}+\frac{c^2n^2}{a^2b^2}\right) = 0,$$

which gives the max. and min. values of u.

Example 4. Find the minimum value of $x^2 + y^2 + z^2$ when $yz + zx + xy = 3a^2$.

Sol. Let $u = x^2 + y^2 + z^2$ ...(*i*)

Given $yz + zx + xy = 3a^2$. ...(*ii*)

For max. or min. from (*i*), we have

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 0 \qquad ...(iii)$$

Also from (*ii*), $(y\,dz + z\,dy) + (z\,dx + x\,dz) + (x\,dy + y\,dx) = 0$

$$(z + y)\,dx + (x + z)\,dy + (y + x)\,dz = 0. \qquad ...(iv)$$

Multiplying equation (*iv*), by λ and adding in (*iii*), we get

$$[x + \lambda(z + y)]\,dx + [y + \lambda(x + z)]\,dy + [z + \lambda(y + x)]\,dz = 0$$

Equating the coefficients of dx, dy, dz to zero, we get

$$x + \lambda(z + y) = 0,\ y + \lambda(x + z) = 0,\ z + \lambda(x + y) = 0 \qquad ...(v)$$

These are Lagrange's equations

Multiplying these by x, y, z respectively and adding, we get

$$[x^2 + \lambda x(z + y)] + [y^2 + \lambda y(x + z)] + [z^2 + \lambda z(y + x)] = 0$$

or $$(x^2 + y^2 + z^2) + 2\lambda(xy + yz + zx) = 0$$

or $$u + 2\lambda(3a^2) = 0 \text{ or } \lambda = -\frac{u}{6a^2}.$$

$\therefore$ From (*v*), we get $x = \dfrac{u(z+y)}{6a^2}$, $y = \dfrac{u(x+z)}{6a^2}$, $z = \dfrac{u(y+x)}{6a^2}$

or $$\frac{x}{z+y} = \frac{y}{x+z} = \frac{z}{x+y} = \frac{u}{6a^2}$$

or $$-6a^2x + uy + uz = 0,\ ux - 6a^2y + uz = 0,\ ux + uy - 6a^2z = 0$$

Eliminating x, y, z, we get

$$\begin{vmatrix} -6a^2 & u & u \\ u & -6a^2 & u \\ u & u & -6a^2 \end{vmatrix} = 0, \text{ which gives the max. and min. values of } u.$$

or $$\begin{vmatrix} -6a^2 & u+6a^2 & u+6a^2 \\ u & -6a^2-u & 0 \\ u & 0 & -6a^2-u \end{vmatrix} = 0, \quad C_2 \to C_2 - C_1\ C_3 \to C_3 - C_1$$

or $$(u + 6a^2)^2 \begin{vmatrix} -6a^2 & 1 & 1 \\ u & -1 & 0 \\ u & 0 & -1 \end{vmatrix} = 0 \text{ or } (u + 6a^2)^2 \begin{vmatrix} -6a^2 & 1 & 1 \\ u & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0,\ R_3 \to R_3 - R_2$$

or $(u + 6a^2)^2\,[-6a^2 - u(-1-1)] = 0$ | Expand with respect to first column

or $(u + 6a^2)^2\,[-6a^2 + 2u] = 0$ or $u = -6a^2, 3a^2$. But u cannot be equal to $-6a^2$, since sum of three squares (*viz.* x^2, y^2, z^2) from (*i*), cannot be negative. Hence $u = 3a^2$ gives max. or min. value of u.

Example 5. Find the minimum distance from the point (1, 2, 0) to the cone $z^2 = x^2 + y^2$. (*U.P.T.U., 2006*)

Sol. Let (x, y, z) be any point on the cone then distance from the point (1, 2, 0) is

$$D^2 = (x - 1)^2 + (y - 2)^2 + (z - 0)^2$$

Let $$u = (x - 1)^2 + (y - 2)^2 + z^2 \qquad ...(i)$$

Subject to $$x^2 + y^2 - z^2 = 0 \qquad ...(ii)$$

for minimum, from (*i*) and (*ii*), we get

$$du = (x - 1)dx + (y - 2)dy + zdz = 0 \quad ...(iii)$$

$$xdx + ydy - zdz = 0 \quad ...(iv)$$

Multiplying equation (*iv*) by λ and adding in (*iii*), we get

$$(x - 1)dx + (y - 2)dy + zdz + \lambda\,(xdx + ydy - zdz) = 0$$

$$\Rightarrow \{x(1+\lambda) - 1\}\,dx + \{y(1+\lambda) - 2\}\,dy + \{z(1-\lambda)\}\,dz = 0$$

$$\Rightarrow x(1+\lambda) - 1 = 0,\ y(1+\lambda) - 2 = 0,\ z(1-\lambda) = 0$$

$$\Rightarrow \quad x = \frac{1}{1+\lambda},\quad y = \frac{2}{1+\lambda},\quad \lambda = 1 \quad ...(v)$$

$$\therefore \quad x = \frac{1}{1+1} = \frac{1}{2},\quad y = \frac{2}{1+1} = 1$$

Putting the value of x and y in equation (*ii*), we get

$$\frac{1}{4} + 1 - z^2 = 0 \Rightarrow z^2 = \frac{5}{4} \Rightarrow z = \pm\frac{\sqrt{5}}{2}$$

Hence, the minimum distance from the point (1, 2, 0) is

$$D^2 = \left(\frac{1}{2} - 1\right)^2 + (1-2)^2 + \left(\frac{\sqrt{5}}{2}\right)^2 = \frac{1}{4} + 1 + \frac{5}{4} = \frac{10}{4}$$

or

$$D^2 = \frac{5}{2} \Rightarrow D = \sqrt{\frac{5}{2}}.$$

Example 6. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube. *(U.P.T.U., 2003)*

Sol. Let the length, breadth and height of solid are

$$l = 2x$$

$$b = 2y$$

$$h = 2z$$

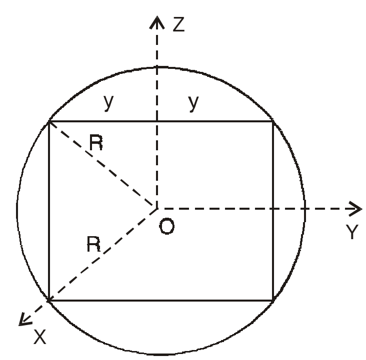


Fig. 2.8

∴ Volume of the solid $V = lbh = 2x \cdot 2y \cdot 2z$

$$\Rightarrow \quad V = 8xyz \quad ...(i)$$

Equation of the sphere

$$x^2 + y^2 + z^2 = R^2$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - R^2 = 0 \quad ...(ii)$$

For maximum differentiating (*i*), (*ii*), we get

$$dV = 8yzdx + 8xzdy + 8xydz = 0 \quad ...(iii)$$

and

$$2xdx + 2ydy + 2zdz = 0 \quad ...(iv)$$

Multiplying (*iv*) by λ and adding in (*iii*), we get

$$8yzdx + 8xzdy + 8xydz + \lambda\,(2xdx + 2ydy + 2zdz) = 0$$

$$\Rightarrow (2\lambda x + 8yz)dx + (2\lambda y + 8xz)dy + (2\lambda z + 8xy)dz = 0$$

Equating the coefficient of dx, dy and dz to zero, we get

$$\Rightarrow \lambda x = -4yz,\ \lambda y = -4xz,\ \lambda z = -4xy \quad ...(v)$$

These are Lagrange's equations

Multiplying equation (v) by x, y, z respectively, we get

$$\lambda x^2 = -4xyz,\ \lambda y^2 = -4xyz,\ \lambda z^2 = -4xyz$$

From these, we get

$$\lambda x^2 = \lambda y^2 = \lambda z^2$$

$$\Rightarrow \quad x^2 = y^2 = z^2$$

$$\Rightarrow \quad x = y = z$$

Thus, the rectangular solid is a cube. **Proved.**

Example 7. Use the method of Lagrange's multiplier to find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

(*U.P.T.U., 2002, 2000*)

Sol. Let

$$l = 2x$$
$$b = 2y$$
$$h = 2z$$

$$\therefore \quad V = 8xyz \qquad ...(i)$$

and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 = 0 \qquad ...(ii)$$

For largest volume, from (i) and (ii), we get

$$dV = yz\cdot dx + xz\cdot dy + xy\cdot dz = 0 \qquad ...(iii)$$

and

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz = 0 \qquad ...(iv)$$

Now, equation (iii) + λ × equation (iv), we get

$$\left(yz + \frac{\lambda}{a^2}x\right) dx + \left(xz + \frac{\lambda}{b^2}y\right) dy + \left(xy + \frac{\lambda}{c^2}z\right) dz = 0$$

$$\Rightarrow \quad yz + \frac{\lambda}{a^2}x = 0,\ \ xz + \frac{\lambda}{b^2}y = 0,\ \ xy + \frac{\lambda}{c^2}z = 0 \qquad ...(v)$$

Multiplying (v) with x, y, z respectively and adding then, we get

$$3xyz + \lambda\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right] = 0$$

$$\Rightarrow \quad 3xyz + \lambda = 0 \ \text{(using } ii)$$

$$\Rightarrow \quad \lambda = -3xyz$$

Putting the value of λ in any one of equation (v), we get

$$yz - 3xyz\cdot\frac{x}{a^2} = 0 \Rightarrow yz\left(1-\frac{3x^2}{a^2}\right) = 0$$

$$\Rightarrow \quad 1-\frac{3x^2}{a^2} = 0 \Rightarrow x = \frac{a}{\sqrt{3}},$$

Similarly, $\quad y = \frac{b}{\sqrt{3}},\ z = \frac{c}{\sqrt{3}}$

Hence, the largest volume $V = 8\cdot\frac{abc}{3\sqrt{3}}$.

Example 8. Determine the point on the paraboloid $z = x^2 + y^2$ which is closest to the point (3, – 6, 4).

Sol. Let (x, y, z) be any point on paraboloid nearest to the point (3, – 6, 4)

$$\therefore \quad D = \sqrt{(x-3)^2+(y+6)^2+(z-4)^2}$$

$$\Rightarrow \quad D^2 = (x-3)^2 + (y+6)^2 + (z-4)^2$$

Let $$u = (x-3)^2 + (y+6)^2 + (z-4)^2 \qquad ...(i)$$

Subject to $$x^2 + y^2 - z = 0 \qquad ...(ii)$$

For minimum distance, differentiating (*i*) and (*ii*), we get

$$du = 2(x-3)\,dx + 2(y+6)\,dy + 2(z-4)\,dz = 0 \qquad ...(iii)$$

and $$2xdx + 2ydy - dz = 0 \qquad ...(iv)$$

Now, equation (*iii*) + λ × (*iv*), we get

$$\{2x(1+\lambda) - 6\}\,dx + \{2y(1+\lambda) + 12\}\,dy + \{2z - \lambda - 8)\,dz = 0$$

$$\Rightarrow \quad x(1+\lambda) - 3 = 0,\ y(1+\lambda) + 6 = 0,\ 2z - (\lambda + 8) = 0$$

$$\Rightarrow \quad x = \frac{3}{1+\lambda},\ y = -\frac{6}{1+\lambda},\ z = \frac{(\lambda+8)}{2} \qquad ...(v)$$

Putting the values of x, y, z in equation (*ii*), we get

$$\frac{9}{(1+\lambda)^2} + \frac{36}{(1+\lambda)^2} - \frac{(\lambda+8)}{2} = 0$$

or $$\frac{45}{(1+\lambda)^2} - \frac{(\lambda+8)}{2} = 0$$

$$\Rightarrow \quad 90 - (\lambda+1)^2(\lambda+8) = 0$$

$$\Rightarrow \quad \lambda^3 + 10\lambda^2 + 17\lambda - 82 = 0$$

$$\Rightarrow \quad \lambda = 2$$

Hence $$x = 1,\ y = -2,\ z = 5.$$

Example 9. Find the maximum and minimum distances from the origin to the curve

$$3x^2 + 4xy + 6y^2 = 140.$$

Sol. Let (x, y) be any point on the curve

∴ distance from (0, 0) is given by

$$D^2 = x^2 + y^2 = u \text{ (say)} \qquad ...(i)$$

Subject to $$3x^2 + 4xy + 6y^2 - 140 = 0 \qquad ...(ii)$$

For maximum and minimum from (*i*) and (*ii*), we get

$$du = 2xdx + 2ydy = 0 \qquad ...(iii)$$

$$6xdx + 4dx\,y + 4xdy + 12ydy = 0 \qquad ...(iv)$$

Now, equation (*iii*) + λ × (*iv*), we get

$$\{x(2+6\lambda) + 4y\lambda\}\,dx + \{y(2+12\lambda) + 4x\lambda\}dy = 0$$

$$\Rightarrow \quad 2x + \lambda(6x + 4y) = 0 \qquad ...(v)$$

$$2y + \lambda(12y + 4x) = 0 \qquad ...(vi)$$

Multiplying the above equations by x, y respectively and adding, we get

$$2(x^2 + y^2) + 2\lambda(3x^2 + 4xy + 6y^2) = 0$$

$$\Rightarrow \quad 2u + 2\lambda(140) = 0 \qquad \text{(using } (i) \text{ and } (ii))$$

$$\therefore \quad \lambda = -\frac{u}{140}$$

Putting the value of λ in eqns. (v) and (vi), we get

$$2x - \frac{u}{140}(6x + 4y) = 0 \Rightarrow (140 - 3u)x - 2uy = 0$$

and

$$2y - \frac{u}{140}(12y + 4x) = 0 \Rightarrow -2ux + (140 - 6u)y = 0$$

This system has non-trivial solution if

$$\begin{vmatrix} 140-3u & -2u \\ -2u & (140-6u) \end{vmatrix} = 0$$

$$\Rightarrow \quad (140 - 3u)(140 - 6u) - 4u^2 = 0$$

$$\Rightarrow \quad 14u^2 - 1260u + (140)^2 = 0$$

$$u^2 - 90u - 1400 = 0$$

$$(u - 70)(u - 20) = 0$$

$$\Rightarrow \quad u = 70, 20$$

Thus, the maximum and minimum distances are

$$\sqrt{70}, \sqrt{20}. \qquad | \text{ As } D^2 = u$$

Example 10. A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.

Sol. Let part of square $= x$

and part of circle $= y \Rightarrow x + y = b$

$$\therefore \quad \text{side of square} = \frac{x}{4},$$

$$\text{radius of circle} = \frac{y}{2\pi} \quad | \text{ As } 2\pi r = y$$

$$\text{area of square} = \frac{x^2}{16}$$

$$\text{area of circle} = \frac{\pi y^2}{4\pi^2} = \frac{y^2}{4\pi}$$

Here, let

$$u = \text{sum of areas} = \frac{x^2}{16} + \frac{y^2}{4\pi} \qquad ...(i)$$

$$\text{Subject to } b = x + y \Rightarrow x + y - b = 0 \qquad ...(ii)$$

For minimum from (i) and (ii), we get

$$du = \frac{x}{8}dx + \frac{y}{2\pi}dy = 0 \qquad ...(iii)$$

and

$$dx + dy = 0 \qquad ...(iv)$$

Now, (*iii*) + λ × (*iv*), we get

$$\frac{xdx}{8}+\frac{ydy}{2\pi} + \lambda\,(dx + dy) = 0$$

$$\Rightarrow \quad \left(\frac{x}{8}+\lambda\right)dx+\left(\frac{y}{2\pi}+\lambda\right)dy = 0$$

$$\Rightarrow \quad \frac{x}{8}+\lambda = 0, \quad \frac{y}{2\pi}+\lambda = 0$$

$$\Rightarrow \quad x = -\,8\lambda,\ y = -\,2\pi\lambda \qquad ...(v)$$

Putting x and y in equation (*ii*), we get

$$-\,8\lambda - 2\pi\lambda = b \Rightarrow \lambda = -\,\frac{b}{8+2\pi}$$

Thus $$x = -\,8\lambda = \frac{8b}{8+2\pi},\ y = -\,2\pi\lambda = \frac{2\pi b}{8+2\pi}$$

∴ The least value of areas is, from (*i*)

$$u = \frac{x^2}{16}+\frac{y^2}{4\pi}$$

$$= \frac{64b^2}{16(8+2\pi)^2}+\frac{4\pi^2 b^2}{4\pi\,(8+2\pi)^2}$$

$$= \frac{b^2(\pi+4)}{4(\pi+4)^2} = \frac{b^2}{4(\pi+4)}.$$

Example 11. Find the dimension of rectangular box of maximum capacity whose surface area is given when (*a*) box is open at the top (*b*) box is closed. (*U.P.T.U., 2008*)

Sol. Let the length, breadth and height of box are x, y, z respectively.

So volume $$V = xyz \qquad ...(i)$$

There will be two surface area one for open and one for closed box

$$\therefore \quad nxy + 2yz + 2zx = S \text{ (say)} \qquad ...(ii)$$

or $$g(x, y, z) \equiv nxy + 2yz + 2zx - S = 0 \qquad ...(iii)$$

Here $n = 1$, when the box is open on the top

$n = 2$, when the box is closed.

The Lagrange's equations are

$$\frac{\partial V}{\partial x}+\lambda\frac{\partial g}{\partial x} = yz + \lambda(ny + 2z) = 0 \qquad ...(iv)$$

$$\frac{\partial V}{\partial y}+\lambda\frac{\partial g}{\partial y} = xz + \lambda(nx + 2z) = 0 \qquad ...(v)$$

$$\frac{\partial V}{\partial z}+\lambda\frac{\partial g}{\partial z} = xy + \lambda(2y + 2x) = 0 \qquad ...(vi)$$

Multiplying (*iv*), (*v*), (*vi*) by x, y, z respectively and adding, we get

$$3xyz + \lambda\ [2(nxy + 2yz + 2zx)] = 0$$

or $$3V + \lambda[2S] = 0 \ \Rightarrow\ \lambda = -\,\frac{3V}{2S} \qquad ...(vii)$$

Putting value of λ from (*vii*) in (*iv*), (*v*) and (*vi*), we get

$$yz - \frac{3V}{2S}(ny + 2z) = 0 \Rightarrow yz - \frac{3xyz}{2S}(ny + 2z) = 0$$

or $$nxy + 2xz = \frac{2S}{3} \qquad ...(viii)$$

Similarly $$nxy + 2yz = \frac{2S}{3} \qquad ...(ix)$$

$$2yz + 2zx = \frac{2S}{3} \qquad ...(x)$$

From (*viii*) and (*ix*), we get

$$x = y \qquad ...(xi)$$

and from (*ix*), (*x*), we get

$$ny = 2z \Rightarrow z = \frac{ny}{2} = \frac{nx}{2} \qquad ...(xii)$$

Putting (*xi*) and (*xii*) in equation (*ii*), we have

$$nx \cdot x + 2 \cdot x \cdot \frac{nx}{2} + 2 \cdot \frac{nx}{2} \cdot x = S \Rightarrow 3nx^2 = S$$

or $$x^2 = \frac{S}{3n}$$

(*a*) When box is open $n = 1$

$$\therefore \qquad x^2 = \frac{S}{3} \Rightarrow x = \sqrt{\frac{S}{3}}$$

Hence, the dimensions of the open box are $x = y = \sqrt{\frac{S}{3}}$ and $z = \frac{1}{2}\sqrt{\frac{S}{3}}$

(*b*) When box is closed $n = 2 \quad \therefore \quad x^2 = \frac{S}{6} \Rightarrow x = \sqrt{\frac{S}{6}}$

Hence, the dimensions of the closed box are

$$x = y = \sqrt{\frac{S}{6}} \quad \text{and} \quad z = \sqrt{\frac{S}{6}}.$$

EXERCISE 2.4

1. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere, $x^2 + y^2 + z^2 = 1$. (*U.P.T.U., 2001*) $\left[\textbf{Ans. } D_{\text{min}} = 12,\ D_{\text{max}} = 14\right]$
2. Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$. [*U.P.T.U. (C.O.), 2003*] $\left[\textbf{Ans. } D_{\text{min}} = 4.5706,\ D_{\text{max.}} = 21.6589\right]$
3. The temperature T at any point (x, y, z) in space is $T = 400\ xyz^2$. Find the highest temperature at the surface of a sphere $x^2 + y^2 + z^2 = 1$. $\left[\textbf{Ans. } T = 50\right]$

4. Find the maxima and minima of $x^2 + y^2 + z^2$ subject to the conditions : $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = 0$.

$$\left[\textbf{Ans. } \frac{l^2}{(au-1)} + \frac{m^2}{(bu-1)} + \frac{n^2}{(cu-1)} = 0\right]$$

5. Find the maximum value of $u = x^p y^q z^r$ when the variables x, y, z are subject to the condition $ax + by + cz = p + q + r$.

$$\left[\textbf{Ans. } u = \left(\frac{p}{a}\right)^p \left(\frac{q}{b}\right)^q \left(\frac{r}{c}\right)^r\right]$$

6. A rectangular box, which is open at the top has a capacity of 256 cubic feet. Determine the dimensions of the box such that the least material is required for the construction of the box. Use Lagrange's method of multipliers to obtain the solution.

[**Ans.** length = breadth = 8′, height = 4′]

7. Find the minimum value of $x^2 + y^2 + z^2$ subject to condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$.

[**Ans.** Minimum value = 27]

8. Determine the point in the plane $3x - 4y + 5z = 50$ nearest to the origin.

[**Ans.** (3, – 4, 5)]

9. Find the length and breadth of a rectangle of maximum area that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$.

[**Ans.** $l = \frac{3\sqrt{2}}{2}$, $b = \sqrt{2}$, area = 12]

10. Divide 24 into three parts such that the continued product of the first square of the second and the cube of the third may be maximum.

[**Ans.** 4, 8, 12, maximum value = $4 \cdot 8^2 \cdot 12^3$]

11. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid of revolution $4x^2 + 4y^2 + 9z^2 = 36$.

[**Ans.** Maximum volume = $16\sqrt{3}$]

12. Using the Lagrange's method of multipliers, find the largest product of the numbers x, y and z when $x^2 + y^2 + z^2 = 9$.

[**Ans.** $3\sqrt{3}$]

13. A torpedo has the shape of a cylinder with conical ends. For given surface area, show that the dimensions which give maximum value are $l = h = \frac{2}{\sqrt{5}} r$, where l is the length of the cylinder, r is the radius and h is the altitude of cone.

14. Find the dimensions of a rectangular box, with open top of given capacity (volume) such that the sheet metal (surface area) required is least.

[**Ans.** $x = y = 2z = (2V)^{1/3}$, V = volume]

15. Find the maximum and minimum values of $\sqrt{x^2 + y^2}$ when $13x^2 - 10xy + 13y^2 = 72$.

[**Ans.** Maximum = 3, minimum = 2]

16. A tent of given volume has a square base of side $2a$ and has its four sides of height b vertical and is surmounted by a pyramid of height h. Find the values of a and b in terms of h so that the canvas required for its construction be minimum.

[**Hint:** $V = 4a^2b + \frac{1}{3}(4a^2)h$, $S = 8ab + 4a\sqrt{a^2+h^2}$; $a = \frac{\sqrt{5}}{2}h$, $b = \frac{h}{2}$].

17. If x and y satisfy the relation $ax^2 + by^2 = ab$, prove that the extreme values of function $u = x^2 + xy + y^2$ are given by the roots of the equation $4(u - a)(u - b) = ab$.

18. Find the maximum value of $x^my^nz^p$ when $x + y + z = a$.

[**Ans.** $a^{m+n+b} \cdot m^m \cdot n^n \cdot p^p/(m + n + p)^{m+n+p}$]

19. Find minimum distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 30$. [**Ans.** 3]

20. Determine the perpendicular distance of the point (a, b, c) from the plane $lx + my + nz = 0$ by the Lagrange's method.

[**Ans.** Minimum distance $= \frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}}$]

OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer of the choices given below:

1. The Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ for the function $u = e^x \sin y$, $v = (x + \log \sin y)$ is (*U.P.T.U., 2008*)

(*i*) 1 (*ii*) 0

(*iii*) $\sin x \sin y - xy \cos x \cos y$ (*iv*) $\frac{e^x}{x}$

2. The Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for the function $u = 3x + 5y$, $v = 4x - 3y$ is

(*i*) 29 (*ii*) xy

(*iii*) $x^2y^2 - y^3$ (*iv*) -29

3. If $x = r\cos\theta$, $y = r\sin\theta$, $z = Z$, then $\frac{\partial(x,y,z)}{\partial(r,\theta,Z)}$ is

(*i*) $2r$ (*ii*) $r^2 - 5$

(*iii*) $\frac{5}{r}$ (*iv*) r

4. If $u = x\sin y$, $v = y\sin x$, then $\frac{\partial(u,v)}{\partial(x,y)}$ is

(*i*) $\sin x \sin y$ (*ii*) $\sin x \sin y - xy \cos x \cos y$

(*iii*) $\cos x \cos y - xy \sin y$ (*iv*) 0

5. If $u = 3x + 2y - z$, $v = x - y + z$, $w = x + 2y - z$, then $J(u, v, w)$ is

(*i*) 5 (*ii*) 0

(*iii*) – 2 (*iv*) $xy + 3x^3$

6. If in the area of an ellipse if a two per cent error is made in measuring the major and minor axis then the percentage error in area is

(*i*) 2% (*ii*) 3%

(*iii*) 5% (*iv*) 4%

7. The value of $(1.05)^{3.02}$ is

(*i*) 2.35 (*ii*) 1.25

(*iii*) 1.15 (*iv*) 0.57

8. If an error of 2% is made in measuring the sides of a rectangle, then what is the percentage of error in calculating its area?

(*i*) 1% (*ii*) 2%

(*iii*) 4% (*iv*) 8%

9. The relation among relative error of quotient, relative errors of dividend and the divisor (Take x = dividend, y = divisor, z = quotient) is

(*i*) $\frac{dz}{z} = \frac{dx}{x} + \frac{dy}{y}$ (*ii*) $\frac{dz}{z} < \frac{dx}{x} + 2\frac{dy}{y}$

(*iii*) $\frac{dz}{z} < \frac{dx}{x} - \frac{dy}{y}$ (*iv*) $\frac{dz}{z} = \frac{dx}{x} - \frac{dy}{y}$

10. If $u = x^2 + y^2 + 6x + 12$ then the stationary point is

(*i*) (– 3, 0) (*ii*) (– 3, 5)

(*iii*) (– 2, 6) (*iv*) (5, 6)

11. The extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$ are

(*i*) $f_{max} = 2$, $f_{min} = 1$ (*ii*) $f_{max} = 0$, $f_{min} = -2$

(*iii*) $f_{max} = 7$, $f_{min} = -5$ (*iv*) None of these

12. If $u = x^4 + 2x^2y - x^2 + 3y^2$ then $rt - s^2$ is equal to

(*i*) 24 (*ii*) 36

(*iii*) – 58 (*iv*) 14

B. Fill in the blanks:

1. If $u = u(r, s)$, $v = v(r, s)$ and $r = r(x, y)$, $s = s(x, y)$ then $\frac{\partial(u,v)}{\partial(x,y)}$ = ...

2. If $x = r\cos\theta$, $y = r\sin\theta$ then $\frac{\partial(x,y)}{\partial(r,\theta)}$ is ....

3. If $u = x(1 - y)$, $v = xy$ then $\frac{\partial(u,v)}{\partial(x,y)}$ = ...

4. $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)}$ = ...

C. Indicate True or False for the following statements:

1. (*i*) The functions u and v are said to be functionally independent if their Jacobian is not equal to zero.

(*ii*) If $f_1\,(u, v, x, y) = 0$ and $f_2\,(u, v, x, y) = 0$ then u, v are said to be implicit functions.

(*iii*) m functions of n variables are always functionally dependent when $m > n$.

(*iv*) If $x = r\cos\theta$, $y = r\sin\theta$ then $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r^2 - 2r + 1$.

2. (*i*) $\dfrac{dx}{x}$ represents relative error.

(*ii*) If $f(a, b) < f(a + h, b + k)$ then $f(a, b)$ is a maximum value.

(*iii*) If $f(a, b) > f(a + h, b + k)$ then $f(a, b)$ is a minimum value.

(*iv*) A point where function is neither maximum nor minimum is called saddle point.

3. (*i*) Nature of stationary points cannot be determined by Lagrange's method.

(*ii*) Solving $f_x = 0$ and $f_y = 0$ for stationary point.

(*iii*) Extremum is a point which is either a maximum or minimum.

(*iv*) Extrema occur only at stationary points. However, stationary points need not be extrema.

D. Match the Following:

1. (*U.P.T.U., 2008*)

	(*i*) Max^m	(*a*) $rt - s^2 = 0$
	(*ii*) Min^m	(*b*) $rt - s^2 < 0$
	(*iii*) Saddle point	(*c*) $rt - s^2 > 0,\ r > 0$
	(*iv*) Failure case	(*d*) $rt - s^2 > 0$, r < 0
2.	(*i*) δx or dx	(*a*) Local maximum
	(*ii*) $\dfrac{\delta x}{x}$ or $\dfrac{dx}{x}$	(*b*) Absolute error
	(*iii*) $100 \times \dfrac{dx}{x}$	(*c*) Relative error
	(*iv*) $f(x, y) \le f(a, b)$	(*d*) Percentage error

ANSWERS TO OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer:

1. (*ii*)	**2.** (*iv*)	**3.** (*iv*)
4. (*ii*)	**5.** (*iii*)	**6.** (*v*)
7. (*iii*)	**8.** (*iii*)	**9.** (*iv*)
10. (*i*)	**11.** (*i*)	**12.** (*i*)

B. Fill in the blanks:

1. $\frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$	**2.** r	**3.** x
4. 1		

C. True or False:

1. (*i*) *T*	(*ii*) *T*	(*iii*) *T*	(*iv*) *F*
2. (*i*) *T*	(*ii*) *F*	(*iii*) *F*	(*iv*) *T*
3. (*i*) *T*	(*ii*) *T*	(*iii*) *T*	(*iv*) *T*

D. Match the following:

1. (*i*) → (*d*) (*ii*) → (*c*) (*iii*) → (*b*) (*iv*) → (*a*)

2. (*i*) → (*b*) (*ii*) → (*c*) (*iii*) → (*d*) (*iv*) → (*a*)

❑❑❑

UNIT III

Matrices

3.0 INTRODUCTION

The term matrix was apparently coined by sylvester about 1850, but introduced first by Cayley in 1860. In this unit, we focus on matrix theory itself which theory will enable us to obtain additional important results regarding the solution of systems of linear algebraic equations.

One way to view matrix theory is to think in terms of a parallel with function theory. In mathematics, we first study numbers—the points on a real number axis. Then we study functions, which are mappings or transformations, from one real axis to another. For example, $f(x) = x^2$ maps the point $x = 2$, say on x-axis to the point $f = 4$ on f or y-axis. Just as functions act upon numbers, we shall see that matrices act upon vectors and are mappings from one vector space to another.

3.1 DEFINITION OF MATRIX

A matrix is a collection of numbers arranged in the form of a rectangular array. These numbers known as elements or entries are enclosed in brackets [ ] or ( ) or ‖ ‖.

Therefore a matrix A may be expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad ...(i)$$

The horizontal lines are called rows and vertical lines are called columns. The order of matrix A is $m \times n$ and is said to be a rectangular matrix.

3.1.1 Notation

Elements of matrix are located by the double subscript ij where i denotes the row and j the column. In view of subscript notation in (1), one also writes

$$A = [a_{ij}], \text{ where } i = 1, 2, ..., m \text{ and } j = 1, 2, ... n$$

3.2 TYPES OF MATRICES

There are following types of matrices:

(a) Rectangular matrix: A matrix in which the number of rows and columns are not equal, *i.e.*, $(m \neq n)$ is called a rectangular matrix, *e.g.*, $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & 1 \end{bmatrix}_{2 \times 3}$.

(b) Square matrix: A matrix in which the number of rows and columns are equal, *i.e.*, $(m = n)$ is called a square matrix, *e.g.*, $\begin{bmatrix} 5 & 2 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}_{3 \times 3}$.

(c) Row matrix: A matrix which has only single row and any number of columns is called a row matrix, *e.g.*, $[1 \;\; 2 \;\; 0 \;\; 5]_{1 \times 4}$

(d) Column matrix: A matrix which has only single column and any number of rows, *i.e.*, $(m \times 1)$ order is called column matrix, *e.g.*, $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 5 \end{bmatrix}_{4 \times 1}$.

(e) Null matrix or zero matrix: Any $m \times n$ matrix is called a null matrix if each of its elements is zero and is denoted by $O_{m \times n}$ or simply by O simply, *e.g.*, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(f) Diagonal matrix: A square matrix $A = [a_{ij}]$ is called diagonal matrix, if all the elements except principal diagonal are zero. Thus, for diagonal matrix $a_{ij} \neq 0$, $i = j$ and $a_{ij} = 0$, $i \neq j$, *e.g.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(g) Scalar matrix: Any diagonal matrix in which all its diagonal elements are equal to a scalar, say (K) is called a scalar matrix

Thus, $$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

i.e., $$A = [a_{ij}]_{n \times n} \text{ is a scalar matrix if}$$

$$a_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ K & \text{when } i = j \end{cases}$$

(h) Identity matrix (or unit matrix): Any diagonal matrix is called an identity matrix, if each of its diagonal elements is unity. Thus a matrix $A = \{a_{ij}\}_{n \times n}$ is called identity matrix iff $a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$. An identity matrix of order n is denoted by I or I_n.

Thus, $$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Symmetric matrix: A square matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric matrix if its (i, j)th element is equal to (j, i)th element. Thus matrix A is said to be symmetric matrix if $a_{ij} = a_{ji}$ $\forall$ i, j.

e.g., $$\begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & -5 \\ -4 & -5 & 3 \end{bmatrix}$$

(j) Skew symmetric matrix: A sqaure matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -\ a_{ji}$ $\forall$ i, j.

But for diagonal elements $a_{ii} = -\ a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$. This proves that every leading diagonal element of a skew symmetric matrix is zero.

Thus, $$\begin{bmatrix} 0 & 2 & 5 \\ -2 & 0 & 3 \\ -5 & -3 & 0 \end{bmatrix}$$ is a skew symmetric matrix.

(k) Triangular matrix: If every element above or below the leading diagonal of a square matrix is zero, the matrix is called a triangular matrix. It has the following two forms:

(*i*) *Upper triangular matrix:* A square matrix in which all the elements below the leading diagonal are zero *i.e.,* $a_{ij} = 0$; $i > j$ is called an upper triangular matrix.

e.g., $$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

(*ii*) *Lower triangular matrix:* A square matrix in which all the elements above the leading diagonal are zero *i.e.,* $a_{ij} = 0$; $i < j$ is called a lower triangular matrix.

e.g., $$\begin{bmatrix} 5 & 0 & 0 \\ 3 & 2 & 0 \\ 6 & 3 & 1 \end{bmatrix}$$

(l) Transpose of a matrix: The matrix is obtained by interchange the rows and columns of a given matrix A, is called the transpose of A and is denoted by A' or A^T *e.g.*,

If $$A = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 2 & 3 & 0 \\ 5 & 1 & 2 & 1 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \\ 6 & 0 & 1 \end{bmatrix}$$

(m) Conjugate of a matrix: The matrix obtained by replacing each element by its conjugate complex number of a given matrix A, is called the conjugate of A and is denoted by $\overline{A}$.

Thus, if $$A = \begin{bmatrix} 1+2i & -4i \\ 3 & 1-2i \end{bmatrix}, \text{ then } \overline{A} = \begin{bmatrix} 1-2i & 4i \\ 6 & 1+2i \end{bmatrix}$$

(n) Tranjugate or conjugate transpose of a matrix: The conjugate of a transposed matrix A or transpose of a conjugate matrix A is called a tranjugate matrix of A and is denoted by A^*.

Thus, if $$A = \begin{bmatrix} 2+3i & -5i \\ 6 & 1-2i \end{bmatrix}, \quad \text{then } A^* = \begin{bmatrix} 2-3i & 6 \\ 5i & 1+2i \end{bmatrix}$$

(o) Hermitian matrix: A square matrix $A = [a_{ij}]$ is said to be "Hermitian matrix" if its conjugate transpose matrix A^* is equal to itself *i.e.*, $A^* = A$.

Thus $A = [a_{ij}]$ is hermitian matrix if $a_{ij} = \bar{a}_{ji}$ $\forall$ i and j. Hence for diagonal elements, $a_{ii} = \bar{a}_{ii}$ *i.e.*, every leading diagonal element in a hermitian matrix is wholly real.

e.g., $$\begin{bmatrix} 1 & 3+i & i \\ 3-i & 2 & 5i \\ -i & -5i & 0 \end{bmatrix}$$

(p) Skew hermitian matrix: Any square matrix $A = [a_{ij}]$ is said to be a skew hermitian matrix if $A^* = -A$

Thus A is skew hermitian if $a_{ii} = -\bar{a}_{ji}$ $\forall$ i, j

For diagonal elements $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$

If $a_{ii} = x + iy$

and $\bar{a}_{ii} = x - iy$

then $a_{ii} + \bar{a}_{ii} = x + iy + x - iy = 2x$

or $x = 0$

Hence all the diagonal elements of a skew hermitian matrix are either zero or pure imaginary.

e.g., $$\begin{bmatrix} i & 3+i & 4+5i \\ -3+i & 0 & 2i \\ -4+5i & 2i & 0 \end{bmatrix}$$

(q) Nilpotent matrix: A square matrix A is said to be nilpotent of index p if p is the least positive integer such that $A^p = 0$. Thus, a square matrix A is said to be nilpotent of index 2, if

$$A^2 = 0; \textit{ e.g.}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ is a nilpotent matrix}$$

As $$A^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(r) Idempotent matrix: A square matrix A is said to be periodic of period p if p is the least positive integer such that $A^{p+1} = A$. If $p = 1$ so that $A^2 = A$, then A is called idempotent. Thus a square matrix A is said to be "idempotent" (or of period 1)

if $A^2 = A$.

e.g., $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$$

(s) Involutory matrix: A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

e.g.,
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(t) Orthogonal matrix: A square matrix A is said to be an orthogonal matrix, if $A'A = AA' = I$

(u) Unitary matrix: A square matrix A is said to be unitary matrix if $AA^* = A^*A = \text{I}$.

(*U.P.T.U., 2001, 2005*)

3.3 OPERATIONS ON MATRICES

3.3.1 Scalar Multiple of a Matrix

Consider a matrix $A = [a_{ij}]_{m \times n}$. Let k be any scalar belonging to a field over which A is defined. The scalar multiple of k and A, denoted by kA, is defined as

$$kA = [ka_{ij}]_{m \times n}$$

i.e., each element of A is multiplied by k.

If $k = -1$, then $(-1)\ A = [-a_{ij}]$

$(-1)\ A$ is denoted by $-A$ and is called the negative of matrix A.

$-A$ is also called "additive inverse of A".

Thus, if
$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \text{ then } 3A = \begin{bmatrix} 1\cdot 3 & 2\cdot 3 \\ -1\cdot 3 & 4\cdot 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & 12 \end{bmatrix}$$

3.3.2 Addition of Matrices

Any two matrices can be added if they are of the same order.

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

e.g.,
$$\text{Let } A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & -4 \\ 2 & 0 \end{bmatrix}, \text{ then } A + B = \begin{bmatrix} 7 & 1 \\ 5 & 1 \end{bmatrix}$$

3.3.3 Subtraction of Matrices

Any two matrices can be subtracted if they are of the same order.

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$; then $A - B = [a_{ij} - b_{ij}]_{m \times n}$

3.3.4 Properties of Addition of Matrices

(*i*) Commulative law
$$A + B = B + A$$

(*ii*) Associative law
$$(A + B) + C = A + (B + C)$$

(*iii*) Each matrix has an additive inverse

(*iv*) Cancellation law
$$A + B = A + C \Rightarrow B = C$$

3.3.5 Multiplication of Matrices

The product AB of two matrices A and B is only possible if the number of columns of A = number of rows of B. In the product AB, A is called the 'prefactor' and B is called the 'post factor'.

e.g.,

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

3.3.6 Properties of Multiplication of Matrices

(*i*) Associative law: $(AB)\, C = A\, (BC)$

(*ii*) Distributive law: $A(B + C) = AB + AC$.

3.4 TRACE OF MATRIX

If $A = [a_{ij}]_{n \times n}$ be a square matrix, then the sum of its diagonal elements is defined as the trace of the matrix, hence

$$\text{trace of } A = \sum_{i=1}^{\infty} a_{ii}$$

e.g.,

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}, \text{ the trace of } A = 1 + 2 + (-2) = 1.$$

3.5 PROPERTIES OF TRANSPOSE

(*i*) $(A')' = A$

(*ii*) $(KA)' = KA'$, K being scalar

(*iii*) $(A + B)' = A' + B'$

(*iv*) $(AB)' = B'A'$.

3.6 PROPERTIES OF CONJUGATE MATRICES

(*i*) $\left(\overline{KA}\right) = \overline{K}\,\overline{A}$, K being a scalar,

(*ii*) $\left(\overline{A+B}\right) = \overline{A} + \overline{B}$,

(*iii*) $\left(\overline{A\,B}\right) = \overline{A}\,\overline{B}$.

Example 1. Show that every square matrix can be uniquely expresses as the sum of a symmetric and a skew symmetric matrix.

Sol. Let A be any square matrix

Evidently $\quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

Taking $\quad P = \frac{1}{2}(A + A'),\ Q = \frac{1}{2}(A - A')$, we get

$$A = P + Q \qquad ...(i)$$

Now $\quad P' = \frac{1}{2}\left[(A + A')\right]' = \frac{1}{2}\left[A' + (A')'\right]$

$$= \frac{1}{2}(A' + A) = P$$

and $\quad Q' = \frac{1}{2}[(A - A')]' = \frac{1}{2}[A' - (A')']$

$$= \frac{1}{2}(A' - A) = -Q$$

$\Rightarrow \quad P' = P,\ Q' = -Q$

Hence P is symmetric and Q is skew symmetric.

This shows that a square matrix A is expressible as a sum of a symmetric and skew symmetric matrix.

Deduction: To prove that this representation is unique, let if possible $A = R + S$ be another representation of A, where R is symmetric and S is skew symmetric,

$\Rightarrow \quad R = R',\ S' = -S$

Now $\quad A = R + S$

$\Rightarrow \quad A' = (R + S)' \Rightarrow A' = R' + S'$

$\Rightarrow \quad A' = R - S \quad |\text{As } R = R',\ S' = -S$

$\therefore \quad A + A' = R + S + R - S = 2R$ or $R = \frac{1}{2}(A + A') = P$

Also $\quad A - A' = R + S - (R - S) = 2S$ or $S = \frac{1}{2}(A - A') = Q$

Thus $\quad R = P,\ S = Q$

This proves that the representation is unique.

Example 2. Every square matrix can be uniquely expressed as $P + iQ$, where P and Q are hermitian.

Sol. Let A be square matrix. Evidently

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$

$$= \frac{1}{2}(A + A^*) + i\left\{\frac{1}{2i}(A - A^*)\right\}$$

Taking $\quad P = \frac{1}{2}(A + A^*),\ Q = \frac{1}{2i}(A - A^*)$, we get

$$A = P + iQ \qquad ...(i)$$

so that $$P^* = \left[\frac{1}{2}(A + A^*)\right]^* = \frac{1}{2}(A^* + A^{**}) = \frac{1}{2}(A^* + A) = P,$$

$$Q^* = \left[\frac{1}{2i}(A - A^*)\right]^* = \left(\frac{1}{-2i}\right) A^* - A^{**} = -\frac{1}{2i}(A^* - A) = Q$$

Thus $P^* = P$, $Q^* = Q \Rightarrow P$ and Q are hermitian.

In this event (i) proves that A is expressible as $P + iQ$ where P and Q are hermitian.

Deduction: To prove that this representation is unique, let if possible. $A = R + iS$ be another representation where R and S are hermitian so that $R^* = R$, $S^* = S$

$$A^* = (R + iS)^* = R^* + \bar{i}S^* = R + (-i)S = R - iS$$

$$A + A^* = (R + iS) + (R - iS) = 2R$$

or $$R = \frac{1}{2}(A + A^*) = P$$

$$A - A^* = (R + iS) - (R - iS) = 2iS$$

$\Rightarrow$ $$S = \frac{1}{2i}(A + A^*) = Q$$

Finally, $R = P$, $S = Q$.

Hence the representation is unique.

Example 3. If A is unitary matrix, show that A' is also unitary.

Sol. $$AA^* = A^*A = I$$

$$(AA^*)^* = (A^*A)^* = I^* = I \quad (I^* = I)$$

$\Rightarrow$ $$(A^*)^* A^* = A^* (A^*)^* = \mathrm{I}$$

$$AA^* = A^* A = \mathrm{I} \qquad \left|\text{As } (A^*)^* = A\right.$$

$$(AA^*)' = (A^*A)' = (I)'$$

$$(A^*)' A' = A' (A^*)' = I$$

$\Rightarrow$ $$(A')^* \cdot A' = A' (A')^* = I$$

Hence, A' is a unitary matrix. **Proved.**

Example 4. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ show that $A^2 - 4A - 5I = 0$.

Sol. Here $$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

and $$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$\therefore \qquad A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-9 & 8-8 & 8-8 \\ 8-8 & 9-9 & 8-8 \\ 8-8 & 8-8 & 9-9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0. \quad \textbf{Proved.}$$

Example 5. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ prove that $A^K = \begin{bmatrix} 1+2K & -4K \\ K & 1-2K \end{bmatrix}$, K being any positive integer.

Sol. Let $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ and K be any positive integer

To prove $\quad A^K = \begin{bmatrix} 1+2K & -4K \\ K & 1-2K \end{bmatrix}$, we see that

$$A^1 = A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+2\cdot 1 & -4\cdot 1 \\ 1 & 1-2\cdot 1 \end{bmatrix}$$

This proves that result is true for $K = 1$

Now
$$A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 9-4 & -12+4 \\ 3-1 & -4+1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2\cdot 2 & -4\cdot 2 \\ 2 & 1-2\cdot 2 \end{bmatrix}$$

This proves that the result is true for $K = 2$.

Let us suppose that the result is true for $K = n$, so that

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}.$$

Now,
$$A^{n+1} = A^n A = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6n-4n & -4-8n+4n \\ 3n+1-2n & -4n-1+2n \end{bmatrix} = \begin{bmatrix} 1+2(n+1) & -4(n+1) \\ n+1 & 1-2(n+1) \end{bmatrix}$$

This proves that the result is true for $K = n + 1$, if it is true for $K = n$.

Also, we have shown that the result is true for $K = 1, 2$. Hence by mathematical induction the required result follows.

Example 6. Find the nature of the following matrices

$A + A^*$, AA^* and $A - A^*$. (*U.P.T.U., 2001*)

Sol. $A^* = A \rightarrow$ Hermitian matrix, $(A + A^*)^* = A^* + A \Rightarrow$ Hermitian

$(AA^*)^* = (A^*)^*\cdot A^* = A\cdot A^* \Rightarrow$ Hermitian matrix.

and $(A - A^*)^* = A^* - (A^*)^* = A^* - A = -(A - A^*) \Rightarrow$ skew symmetric matrix.

Example 7. Show that the matrix $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is a unitary matrix if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. (*U.P.T.U., 2005*)

Sol. $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \therefore A^* = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$

But for unitary matrix $AA^* = I$

$$\therefore \quad \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha^2 + \gamma^2 + \beta^2 + \delta^2 & \alpha\beta - i\alpha\delta + i\beta\gamma + \gamma\delta - \alpha\beta - i\beta\gamma + i\alpha\delta - \delta\gamma \\ \alpha\beta - i\beta\gamma + i\alpha\delta + \gamma\delta - \alpha\beta - i\alpha\delta + i\beta\gamma - \delta\gamma & \beta^2 + \delta^2 + \alpha^2 + \gamma^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$ **Proved.**

Example 8. Prove that the matrix $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary. (*U.P.T.U., 2001*)

Sol. Let
$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^* = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^* \cdot A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 + (1+1) & (1+i) - (1+i) \\ (1-i) - (1-i) & (1+1) + 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, A is a unitary matrix.

Example 9. Define a unitary matrix. If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ is a matrix, then show that $(I - N)(I + N)^{-1}$ is a unitary matrix where I is an identity matrix. (*U.P.T.U., 2000*)

Sol. A square matrix A is said to be unitary if $A^*A = I$.

Now
$$I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

and $$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = 1 - (-1 - 4) = 6$$

Let $$A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$\therefore$ $$A_{11} = 1, A_{12} = -(-1 + 2i) = 1 - 2i$$

$$A_{21} = -(1 + 2i) = -1 - 2i, A_{22} = 1$$

$\Rightarrow$ $$B = \begin{bmatrix} 1 & 1-2i \\ -1-2i & 1 \end{bmatrix} \text{ or } B' = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$\Rightarrow$ $$\text{Adj } A = \text{Adj } (I + N) = B' = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

and $$(I + N)^{-1} = \frac{\text{Adj}(I+N)}{|I+N|} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Now $$(I - N)(I + N)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = C \text{ (say)}$$

$\therefore$ $$C^* = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$\Rightarrow$ $$C^*C = \frac{1}{36}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$ **Hence Proved.**

Example 10. Show that $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is nilpotent of index 2.

Sol. Here $$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2-3 & 2+4-6 & 3+6-9 \\ 1+2-3 & 2+4-6 & 3+6-9 \\ -1-2+3 & -2-4+6 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Here A is nilpotent of index 2.

Example 11. If $A = [x\ y\ z]$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then find out ABC.

Sol.
$$AB = [x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz]$$

and
$$ABC = (AB)\,C = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= [(ax + hy + gz)x + (hx + by + fz)y + z(gx + fy + cz)]$$
$$= [ax^2 + by^2 + cz^2 + 2fy + 2zx + 2hxy].$$

Example 12. If e^A is defined as $I + A + \frac{A^2}{\lfloor 2} + \frac{A^3}{\lfloor 3} + \ldots$, show that

$$e^A = e^x \begin{bmatrix} \cos hx & \sin hx \\ \sin hx & \cos hx \end{bmatrix}, \text{ where } A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}.$$

Sol.
$$A^2 = \begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} 2x^2 & 2x^2 \\ 2x^2 & 2x^2 \end{bmatrix} = 2x^2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= 2x^2\,B \quad \left(say\ B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

Similarly, $A^3 = 2^2x^3\,B,\ A^4 = 2^3x^4\,B, \ldots$ etc.

$\therefore$
$$e^A = 1 + A + \frac{A^2}{\lfloor 2} + \frac{A^3}{\lfloor 3} + \ldots..$$
$$= I + xB + \frac{2x^2B}{\lfloor 2} + \frac{2^2x^3B}{\lfloor 3} + \ldots\ldots\ldots \quad | \text{ As } A = xB$$
$$= \frac{1}{2}\left(2I + (2x)B + \frac{(2x)^2}{\lfloor 2}B + \frac{(2x)^3}{\lfloor 3}B + \ldots..\right)$$
$$= \frac{1}{2}\begin{bmatrix} 2 + 2x + \frac{(2x)^2}{\lfloor 2} + \frac{(2x)^3}{\lfloor 3} + \ldots & 0 + 2x + \frac{(2x)^2}{\lfloor 2} + \frac{(2x)^3}{\lfloor 3} + \ldots \\ 0 + 2x + \frac{(2x)^2}{\lfloor 2} + \frac{(2x)^3}{\lfloor 3} + \ldots & 2 + 2x + \frac{(2x)^2}{\lfloor 2} + \frac{(2x)^3}{\lfloor 3} + \ldots \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} e^{2x} + 1 & e^{2x} - 1 \\ e^{2x} - 1 & e^{2x} + 1 \end{bmatrix} = e^x \begin{bmatrix} \frac{1}{2}\left(e^x + e^{-x}\right) & \frac{1}{2}\left(e^x - e^{-x}\right) \\ \frac{1}{2}\left(e^x - e^{-x}\right) & \frac{1}{2}\left(e^x + e^{-x}\right) \end{bmatrix}$$
$$= e^x \begin{bmatrix} \cos hx & \sin hx \\ \sin hx & \cos hx \end{bmatrix}.$$ **Hence Proved.**

3.7 SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be singular, if $|A| = 0$. If $|A| \neq 0$, then A is called a non-singular matrix or a regular matrix. (*U.P.T.U., 2008*)

3.8 ADJOINT OF A SQUARE MATRIX

Adjoint of A is obtained by first replacing each element of A by its cofactor in $|A|$ and then taking transpose of the new matrix or by first taking transpose of A and then replacing each element by its cofactor in the determinant of A.

$$\text{Let} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The cofactors of A in $|A|$ are as follows:

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$A_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, A_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$A_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\text{Let} \qquad B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \text{then Adj } A = B' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

3.8.1 Properties of Adjoint

If $A = [a_{ij}]$ is a square matrix of order n then

(*i*) adj A' = (adj A)′ (*ii*) adj A^* = (adj A)* (*iii*) adjoint of a symmetric (Hermitian) matrix is symmetric (Hermitian).

3.9 INVERSE OF A MATRIX (RECIPROCAL)

Consider only square matrices.

Inverse of a n-square matrix A is denoted by A^{-1} and is defined as follows:

$$AA^{-1} = A^{-1}A = I$$

where I is $n \times n$ unit matrix or $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

3.9.1 Properties of Inverse

(*i*) Inverse of A exists only if $|A| \neq 0$ *i.e.*, A is non-singular.

(*ii*) The inverse of a matrix is unique. If B and C are two inverses of the same matrix A then $(CA)\, B = C\, (AB)$, $IB = CI$ *i.e.*, $B = C$, so inverse is unique.

(*iii*) Inverse of a product is the product of inverses in the reverse order *i.e.*, $(AB)^{-1} = B^{-1} A^{-1}$, since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

(*iv*) For a diagonal matrix D with d_{ii} as diagonal elements, D^{-1} is a diagonal matrix with reciprocal $\frac{1}{d_{ii}}$ as the diagonal elements.

(*v*) Transportation and inverse are commutative *i.e.*,

$(A^{-1})^T = (A^T)^{-1}$, taking transpose of $AA^{-1} = A^{-1}A = I_n$

$(A^{-1})^T A^T = A^T (A^{-1})^T = I^T = I$ *i.e.*,

$(A^{-1})^T$ is the inverse of A^T or $(A^{-1})^T = (A^T)^{-1}$.

(*vi*) $(A^{-1})^{-1} = A$

Taking inverse of $(AA^{-1}) = I$,

$(AA^{-1})^{-1} = (A^{-1})^{-1} A^{-1} = I^{-1} = I = AA^{-1}$. Thus, $A = (A^{-1})^{-1}$.

Example 13. Find the inverse of matrix A, where

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Sol. Here we find A_{11}, A_{12}......... cofactors of A as follow:

$$A_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7,\ A_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -(12+1) = -13,\ A_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 10$$

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -(4-6) = 2,\ A_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -4+2 = -2,\ A_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1+2 = 3,\ A_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -(-1-6) = 7,\ A_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = 1-3 = -2$$

$$\text{Let} \quad B = \begin{bmatrix} -7 & -13 & 10 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix} \therefore \text{adj } A = (B)^T = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 10 & 2 & -2 \end{bmatrix}, |A| = 10$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 10 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -0\cdot 7 & 0\cdot 2 & 0\cdot 3 \\ -1\cdot 3 & -0\cdot 2 & 0\cdot 7 \\ 1 & 0\cdot 2 & -0\cdot 2 \end{bmatrix}.$$

Example 14. If A and B are n-rowed orthogonal (unitary) matrices, then AB and BA are also orthogonal (unitary).

Sol. (*i*) Let A and B be n-rowed orthogonal matrices

then $$A'A = AA' = I \text{ and } B'B = BB' = I$$

To prove AB and BA are orthogonal we have

$$(AB)'(AB) = (B'A')(AB) = B'(A'A)B = B'IB = B'B = I$$

$$(BA)'(BA) = (A'B')(BA) = A'(B'B)A = A'IA = A'A = I$$

Thus $$(AB)'(AB) = I \text{ and } (BA)'(BA) = I$$

This $\Rightarrow$ AB and BA are orthogonal.

(*ii*) Let A and B be unitary, then $A^*A = AA^* = I$ and $B^*B = BB^* = I$.

To prove AB and BA are unitary complete the proof yourself.

EXERCISE 3.1

1. Find the values of p, q, r, s, t and u if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix} \text{ so that } A + B - C = 0.$$

[**Ans.** $p = -2, q = 0, r = 4, s = -1, t = 9, u = 9$]

2. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$, then show that $A^3 = pI + qA + rA^2$.

3. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$, find $3A - 4B$. $\left[\textbf{Ans. } \begin{bmatrix} 2 & 1 & 27 \\ 0 & 1 & 3 \end{bmatrix}\right]$

4. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 3 & 1 & 2 \end{bmatrix}$, show that $6A^2 + 25A - 42I = 0$.

5. If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$.

Show that (*i*) $AB = AC$, (*ii*) $(B + C)A = BA + CA$.

6. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ prove that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

7. Find the product of the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 1 \\ -2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}.$$

$\left[\textbf{Ans. } \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}\right]$

8. If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then show that

$$(A_\alpha)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix} = A_{n\alpha},$$

where n is any positive integer. Also prove that A_α and A_β commute and that $A_\alpha A_\beta = A_{\alpha+\beta}$. Also prove that $A_\alpha A_{-\alpha} = I$.

9. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find BA and AB if they exist.

$\left[\textbf{Ans. } AB = \begin{bmatrix} 2 \\ 5 \end{bmatrix}; BA \text{ does not exist}\right]$

10. If $A = \begin{bmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, find $(AB)'$.

Hence verify that $(AB)' = B'A'$.

$\left[\textbf{Ans. } (AB)' = \begin{bmatrix} 10 & 3 & 4 \\ 11 & 3 & 2 \\ 22 & 6 & 4 \end{bmatrix}\right]$

11. Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & 3 \end{bmatrix}$ is idempotent.

12. Show that the matrix $A = \dfrac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ is orthogonal.

13. Prove that the matrix $A = \dfrac{1}{2}\begin{bmatrix} 1+i & i-1 \\ 1+i & 1-i \end{bmatrix}$ is unitary.

14. Express $\begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$ as sum of hermitian and skew hermitian matrices.

$\left[\textbf{Ans. } \dfrac{1}{2}\begin{bmatrix} -4 & 4-i & 4 \\ 4+i & 8 & 10 \\ 3-i & 2 & -4 \end{bmatrix} + \dfrac{1}{2}\begin{bmatrix} 6i & -2-i & 2i \\ 2-i & -10i & 0 \\ -1+i & 2i & 4i \end{bmatrix}\right]$

15. Show that $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is nilpotent.

16. Express given matrix A as sum of a symmetric and skew symmetric matrices.

$$A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix}.$$

$\left[\textbf{Ans. } \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}\right]$

17. Show that $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{bmatrix}$ is involutory.

18. Find the inverse of the matrix A.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix}.$$

$$\left[\textbf{Ans. } A^{-1} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}\right]$$

19. Prove that $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}^{-1} = \sec 2\alpha \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$

20. If ω is one the imaginary cube roots of unity and if

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}, \text{ then show that } A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}.$$

3.10 ELEMENTARY ROW AND COLUMN TRANSFORMATIONS

The following transformations on a given matrix are defined as elementary transformations:

(*i*) Inter-change of any two rows (columns).

(*ii*) Multiplication of any row (column) by any non-zero scalar k.

(*iii*) Addition to one row (column) of another row (column) multiplied by any non-zero scalar.

We will use the following notations to represent the elementary row (column) operations:

(*a*) R_{ij} (C_{ij}) or $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$) is used for the inter-change of ith and jth rows (columns).

(*b*) R_i (K) [C_i (K)] or $R_i \to KR_i$ ($C_i \to KC_i$) will denote the multiplication of the elements of the ith row (column) by a non-zero scalar K.

(*c*) R_{ij} (K) [C_{ij} (K)] or $R_i \to R_i + KR_j$ ($C_i \to C_i + KC_j$) is used for addition to the elements of ith row (column) the elements of jth row (column) multiplied by the constant K.

3.10.1 Elementary Matrices

The square matrices obtained from an identity or unit matrix by any single elementary transformation (*i*), (*ii*) or (*iii*) are called "elementary matrices".

3.10.2 Properties of Elementary Transformations

(*i*) Every elementary row (column) transformation on a matrix can be effected by pre-post multiplication by the corresponding elementary matrix of an appropriate order.

(*ii*) The inverse of an elementary matrix is an elementary matrix.

3.10.3 Equivalent Matrices

Two matrices A and B are said to be equivalent, denoted by $A \sim B$, if one matrix say A can be obtained from B by the application of elementary transformations.

3.10.4 Properties of Equivalent Matrices

(*i*) If A and B are equivalent matrices, then there exist non-singular matrices R and C such that $B = RAC$.

(*ii*) Every non-singular square matrix can be expressed as the product of elementary matrices.

(*iii*) If there exist a finite system of elementary matrices $R_1, R_2, \ldots, R_m$ such that $(R_m \ldots R_2 R_1) A = I$ and A is non-singular, than $A^{-1} = (R_m \ldots R_2 R_1) I$.

3.11 METHOD OF FINDING INVERSE OF A NON-SINGULAR MATRIX BY ELEMENTARY TRANSFORMATIONS

The property III gives a method of finding the inverse of a non-singular matrix A.

In this method, we write $A = IA$. Apply row transformations successively till A of *L.H.S.* becomes identity matrix I. Therefore, A reduces to I, I reduces to A^{-1}.

Example 1. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$.

Sol. Let $A = IA$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Applyirg $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + R_3$, we have

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$, $R_4 \to R_4 - R_2$, we have

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$, $R_4 \to 2R_4 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_4$, $R_2 \to R_2 - R_4$, $R_3 \to R_3 - 3R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ 6 & -8 & 10 & -6 \\ -2 & 2 & -3 & 2 \end{bmatrix} A$$

Applying $R_3 \to -\frac{1}{2} R_3$, $R_4 \to (-1) R_4$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} A$$

Hence
$$A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}.$$

Example 2. Find by the elementary row transformation inverse of the matrix.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

(*U.P.T.U., 2000, 2003*)

Sol. Let
$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Now
$$A = IA$$

$\therefore$
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + \dfrac{1}{2} R_3$, $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_3 \to \dfrac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

Therefore, $$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}.$$

Example 3. Find the inverse of the following matrix employing elementary transformations:

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}.$$ [*U.P.T.U. (C.O.), 2002*]

Sol. Let $A = IA$

$$\therefore \quad \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to 3R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 2 & -3 & 3 \end{bmatrix} A$$

$R_2 \to R_2 + 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 6 & -9 & 12 \\ 2 & -3 & 3 \end{bmatrix} A$$

$R_2 \to -\frac{1}{3} R_2$, $R_3 \to (-1) R_3$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

Hence $$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}.$$

Example 4. Find the inverse of the matrix

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

Sol. Let $A = IA$

$$\Rightarrow \quad \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - R_1$, we get

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & 1 & -1 & 1/2 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 11R_2,\ R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & -3 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ -7/2 & -11/2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -\frac{7}{2} & -\frac{11}{2} & 1 & 0 \end{bmatrix} A$$

$R_4 \to 2R_4$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -7 & -11 & 2 & 0 \end{bmatrix}$$

Applying $R_4 \to R_4 + 6R_3$, we get

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 3/2 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_2 \to R_2 - \frac{1}{2} R_4$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

Applying $R_1 \to R_1 + 3R_2$, we get

$$\begin{bmatrix} -1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -3 & -6 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_3$, we obtain

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -3 & -9 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_4$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 & -3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

In the last, we apply $R_1 \to (-1)\ R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

Therefore $$A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}.$$

Example 5. With the help of elementary operations, find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

Sol. Since $$A = IA$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to -\frac{1}{4}R_2$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/4 & -1/4 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + R_2$, $R_1 \to R_1 - 2R_2$, we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

Now applying $R_1 \to R_1 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

Hence $$A^{-1} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}.$$

EXERCISE 3.2

1. Find the inverse of the following matrices:

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 6 \end{bmatrix}.$$

$$\left[\textbf{Ans. } \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ 3 & -1 & 0 & 1 \end{bmatrix}\right]$$

2. $\begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 3 & 8 \end{bmatrix}$.

$$\left[\textbf{Ans. } \frac{1}{130}\begin{bmatrix} 4 & -42 & 20 \\ 20 & 50 & -30 \\ -9 & -3 & 20 \end{bmatrix}\right]$$

3. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

$$\left[\textbf{Ans. } \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}\right]$$

4. $\begin{bmatrix} 2 & 4 & 3 \\ 3 & 6 & 5 \\ 2 & 5 & 2 \\ 4 & 5 & 14 \end{bmatrix}$.

$$\left[\textbf{Ans. } \begin{bmatrix} -23 & 29 & -64/5 & -18/5 \\ 10 & -12 & 26/5 & 7/5 \\ 1 & -2 & 6/5 & 2/5 \\ 2 & -2 & 3/5 & 1/5 \end{bmatrix}\right]$$

5. $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } \dfrac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}\right]$

6. $\begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$. $\left[\textbf{Ans. } \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix}\right]$

7. $\begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } \begin{bmatrix} 0 & 1/4 & -1/2 \\ -1 & (3/4)\,i & (1/2)\,i \\ 0 & 1/4 & 1/2 \end{bmatrix}\right]$

8. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}\right]$

9. $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}\right]$

10. $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } \dfrac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}\right]$

11. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. $\left[\textbf{Ans. } \dfrac{1}{5}\begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 3 & -3 & 2 \end{bmatrix}\right]$

12. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\right]$

3.12 RANK OF A MATRIX

A positive number r is said to be the rank of matrix A if matrix A satisfies the following conditions.

(*i*) There exists at least one non-zero minor of order r.

(*ii*) Every minor of order $(r + 1)$ and higher, if any, vanishes.

The rank of matrix A is denoted by $\rho(A)$ or $r(A)$.

Or

The rank of a matrix or a linear map is the dimension of the image of the matrix or the linear map corresponding to the number of linearly independent rows or columns of the matrix or to the number of non-zero singular values of the map.

Result: (*a*) Rank of A and A' is same.

(*b*) For a rectangular matrix A of order $m \times n$, rank of $A \leq \min(m, n)$ *i.e.,* rank cannot be exceed the smaller of m and n.

Minor of *A* matrix. The determinant corresponding to any $r \times r$ submatrix of $m \times n$ matrix. A is called a minor of order r of the matrix A of order $m \times n$.

Example. If $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$, then $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$ are all minors of order 2 of A.

3.12.1 Normal Form

The normal form of matrix A of rank r is one of the forms

$$I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_r & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

where I_r is an identity matrix of order r. This form can be obtained by the application of both elementary row and column operations on any given matrix A.

3.12.2 Procedure to Obtain Normal Form

Consider $A_{m \times n} = I_{m \times m} \cdot A_{m \times n} \cdot I_{n \times n}$

Apply elementary row operations on A and on the prefactor $I_{m \times m}$ and apply elementary column operations on A and on the postfactor $I_{n \times n}$, such that A on the L.H.S. reduces to normal form. Then $I_{m \times m}$ reduces to $P_{m \times m}$ and $I_{n \times n}$ reduces to $Q_{n \times n}$; resulting in $N = PAQ$.

Here P and Q are non-singular matrices. Thus for any matrix of rank r, there exist non-singular matrices P and Q such that

$$\text{PAQ} = \text{N} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

3.12.3 Echelon Form

A matrix $A = [a_{ij}]$ is an echelon matrix or is said to be in echelon form, if the number of zeros preceding the first non-zero entry (known as distinguished elements) of a row increases row by row until only zero rows remain.

In row reduced echelon matrix, the distinguished elements are unity and are the only non-zero entry in their respective columns.

"The number of non-zero rows in an Echelon form is the rank".

Example 1. Find the rank of matrix

$$\begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}.$$ (*U.P.T.U., 2006*)

Sol. Let

$$A = \begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}$$

$R_1 \to R_1 + R_4$, $R_2 \to R_2 - R_3$, we get

$$A \sim \begin{bmatrix} 0 & 7 & -2 & 9 \\ 0 & -4 & -2 & -2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}$$

$R_3 \to R_3 + R_4$

$$\because \quad \sim \begin{bmatrix} 0 & 7 & -2 & 9 \\ 0 & -4 & -2 & -2 \\ 1 & 6 & 3 & 9 \\ -2 & 4 & 0 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 6 & 3 & 9 \\ 0 & -4 & -2 & -2 \\ 0 & 7 & -2 & 9 \\ -2 & 4 & 0 & 5 \end{bmatrix}$$

$R_4 \to R_4 + 2R_1$

$$A \sim \begin{bmatrix} 1 & 6 & 3 & 9 \\ 0 & -4 & -2 & -2 \\ 0 & 7 & -2 & 9 \\ 0 & 16 & 6 & 23 \end{bmatrix}$$

Now, $$|A| = \begin{vmatrix} -4 & -2 & -2 \\ 7 & -2 & -9 \\ 16 & 6 & 23 \end{vmatrix}$$ (Expanded w.r. to first column)

$$= -4(-46-54) + 2(161-144) - 2(42+32)$$

$$= 400 + 34 - 148 = 286$$

$\Rightarrow \quad |A| \neq 0$

Thus there is a non-singular minor of order 4.

Hence $\rho(A) = 4$.

Example 2. Find the rank of matrix A by echelon form.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}.$$ (*U.P.T.U., 2005*)

Sol. Applying $R_1 \leftrightarrow R_3$, we get

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Applying $R_3 \to R_3 - \frac{4}{5} R_2,\ R_4 \to R_4 - \frac{9}{5} R_2$, we get

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix}$$

In the last, we apply $R_4 \to R_4 - R_3$, we get

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the number of non-zero rows = 3

Therefore $\rho(A) = 3$

Example 3. Find the rank of following matrix:

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 16 \end{bmatrix}.$$

Sol. Applying $R_1 \to R_1 - R_2$ and then again $R_1 \to -R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 16 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1,\ R_3 \to R_3 - 5R_1,\ R_4 \to R_4 - 10R_1,\ R_5 \to R_5 - 15R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2, R_5 \to R_5 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here number of non-zero rows = 2

Therefore $\rho(A) = 2.$

Example 4. Reduce the matrix A to its normal form, where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$ and hence find the rank of A.

Sol. Applying $R_1 \leftrightarrow R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 3C_2$, $C_4 \to C_4 + C_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 2$.

Example 5. Reduce the matrix A to its normal form, when

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}. \qquad \textit{(U.P.T.U., 2001, 2004)}$$

Hence, find the rank of A.

Sol.
$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$C_3 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}$$

$R_3 \to R_3 - \frac{4}{5} R_2$, $R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 0 & 16/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -4 & 0 \\ 0 & 0 & 16/5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 4R_4$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 16/5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $R_4 \to R_4 - \dfrac{16}{5}\ R_3$, then $R_2 \to \dfrac{1}{5}\ R_2$ and $R_3 \to \dfrac{5}{16}\ R_3$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho\ (A) = 3.$$

Example 6. Find the non-singular matrices P and Q such that the normal form of A is PAQ where

$$A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4}.$$ Hence, find its rank.

Sol. Here we consider

$$A_{3\times 4} = I_{3\times 3} \cdot A_{3\times 4} \cdot I_{4\times 4} \quad \Big| \text{ As } A_{m\times n} = I_{m\times m} \cdot A_{m\times n} \cdot I_{n\times n}$$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ (pre), we get

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$ (pre)

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 6C_1$, $C_4 \to C_4 + C_1$ (post), we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2$, $C_4 \to C_4 - 2C_2$ (post)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, $I_2 = N = PAQ$, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and Rank of $A = 2$.

Example 7. Find non-singular matrices P and Q such that PAQ is the normal form where

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}_{3 \times 4}.$$

Sol. Here we consider

$$A_{3 \times 4} = I_{3 \times 3} \cdot A_{3 \times 4} \cdot I_{4 \times 4} \qquad \Big| \text{ As } A_{m \times n} = I_{m \times m} A_{m \times n} I_{n \times n}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$, $R_2 \to R_2 - 2R_1$ (pre), we get

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 6 & -9 & 6 \\ 0 & 4 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_1$, $C_3 \to C_3 - 2C_1$, $C_4 \to C_4 + C_1$ (post)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -9 & 6 \\ 0 & 4 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to \frac{1}{3} R_2, R_3 \to \frac{1}{2} R_3$ (pre)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4/3 & 1/3 & 0 \\ -1 & 0 & 1/2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ (pre)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{4}{3} & \frac{1}{3} & 0 \\ 1/3 & -1/3 & 1/2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$ (post)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4/3 & 1/3 & 0 \\ 1/3 & -1/3 & 1/2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2, C_4 \to C_4 + \frac{3}{2} C_2$ (post)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4/3 & 1/3 & 0 \\ 1/3 & -1/3 & 1/2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{2} R_2, R_3 \to (-1) R_3$ (pre)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/6 & 0 \\ -1/3 & 1/3 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow \qquad I_3 = PAQ$

Therefore, $P = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/6 & 0 \\ -1/3 & 1/3 & -1/2 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

And $\rho(A) = 3.$

Note: In such problems other Ans. is possible.

Example 8. Find the rank of

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 5 & 3 & 3 & 8 \\ 4 & 2 & 6 & -1 \end{bmatrix}.$$

Sol. Applying $C_1 \leftrightarrow C_2, C_3 \to \frac{1}{3} C_3$

$$A \sim \begin{bmatrix} 1 & 6 & 1 & 8 \\ 4 & 16 & 4 & 15 \\ 3 & 5 & 1 & 8 \\ 2 & 4 & 2 & -1 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_1, C_3 \to C_3 - C_1$, we have

$$\sim \begin{bmatrix} 1 & 4 & 0 & 8 \\ 4 & 8 & 0 & 15 \\ 3 & -1 & -2 & 4 \\ 2 & 0 & 0 & -1 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 4 & 0 & 8 \\ 2 & 0 & 0 & -1 \\ 3 & -1 & -2 & 4 \\ 2 & 0 & 0 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$, we get

$$\sim \begin{bmatrix} 1 & 4 & 0 & 8 \\ 2 & 0 & 0 & -1 \\ 1 & -1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow |A| = 0$ *i.e.*, minor of order 4 = 0

Next, we consider a minor of order 3

$$\therefore \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 0 & 0 \\ 1 & -1 & -2 \end{vmatrix} = 1\,(0 - 0) - 4\,(-4 - 0) + 0 = 16 \neq 0$$

$$\therefore \qquad \rho\,(A) = 3.$$

Example 9. Find the value of a such that the rank of A is 3, where

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ a & 2 & 2 & 2 \\ 9 & 9 & a & 3 \end{bmatrix}.$$

Sol. $$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ a & 2 & 2 & 2 \\ 9 & 9 & a & 3 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 9R_1$, we have

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ a-2 & 0 & 4 & 2 \\ 0 & 0 & a+9 & 3 \end{bmatrix}$$

Again $R_3 \to R_3 - 4R_2$, $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ a-2 & 0 & 0 & -2 \\ 0 & 0 & a+6 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & a+6 & 0 \\ a-2 & 0 & 0 & -2 \end{bmatrix}$$

Cases: (*i*) If $a = 2$, $|A| = 1 \cdot 0 \cdot 8 . (-2) = 0$, rank of $A = 3$.
(*ii*) If $a = -6$, no. of non-zero rows is 3, rank of $A = 3$.

Example 10. For which value of '*b*' the rank of the matrix.

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix} \text{ is 2.}$$ (*U.P.T.U., 2008*)

Sol. Since the rank of matrix A is 2 so the minor of 3rd order must be zero *i.e.*, $|A| = 0$.

Thus $$\begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{vmatrix} = 0$$

$$(30 - 26) - 5(0 - 2b) + 4(0 - 3b) = 0$$

$\Rightarrow$ $$4 + 10b - 12b = 0 \Rightarrow 4 - 2b = 0$$

Hence $$b = 2$$

EXERCISE 3.3

Find the rank of the following matrix by reducing normal form:

1. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$. (*U.P.T.U., 2001*) [**Ans.** $\rho(A) = 3$]

2. $\begin{bmatrix} 1 & 2 & -3 & 9 \\ 1 & 0 & -1 & 1 \\ 3 & -1 & 1 & -1 \\ -1 & 1 & 0 & 9 \\ 3 & 1 & 0 & 9 \end{bmatrix}$. [**Ans.** 4]

3. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$. [**Ans.** 2]

4. $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix}$. [**Ans.** 3]

5. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$. [**Ans.** 3] (*U.P.T.U. special exam., 2001*)

6. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$. [**Ans.** 3]

7. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$. [**Ans.** 3]

8. $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. [**Ans.** 3]

9. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$. [**Ans.** 3]

10. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$. [**Ans.** 2]

Find the rank of the following matrix.

11. $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$. [**Ans.** 3]

12. $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$. [**Ans.** 3]

13. $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$. [**Ans.** 4]

14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$. [**Ans.** 3]

15. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & 2 & 6 & -7 \end{bmatrix}$. [**Ans.** 2]

16. $\begin{bmatrix} 0 & i & -i \\ -i & 0 & -i \\ -3 & 1 & 0 \end{bmatrix}$. [**Ans.** 3]

17. $\begin{bmatrix} 3 & -1 & -2 \\ -6 & 2 & -4 \\ 3 & 1 & -2 \end{bmatrix}$. [**Ans.** 2]

18. $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$. [**Ans.** 1]

19. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$. [**Ans.** 2]

20. $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -15 \end{bmatrix}$. [**Ans.** 1]

Find the Echelon form of the following matrix and hence find the rank.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$. [**Ans.** 3]

22. $\begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix}$. [**Ans.** 2]

23. $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$. [**Ans.** 2]

24. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$. [**Ans.** 3]

25. $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$. [**Ans.** 3]

26. $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$. [**Ans.** 3]

27. $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$. [**Ans.** 3]

28. $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & -3 & 4 \end{bmatrix}$. [**Ans.** 2]

29. Determine the non-singular matrices P and Q such that PAQ is in the normal form for A. Hence find the rank of A.

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}.$$

$$\left[\textbf{Ans. } P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ \frac{1}{2} & -1/3 & 1/6 \end{bmatrix} Q = \begin{bmatrix} 1 & 4/17 & \frac{9}{119} & \frac{9}{217} \\ 0 & 1/7 & \frac{-1}{7} & \frac{-1}{7} \\ 0 & 0 & \frac{-1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}\right] \text{rank} = 2$$

(other forms are also possible)

30. $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$. (*U.P.T.U., 2002*)

$$\left[\textbf{Ans. } P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ \frac{-3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and rank} = 3 \right]$$

31. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$.

$$\left[\textbf{Ans. } P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

32. $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

$$\left[\textbf{Ans. } P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1/3 & -4/3 & -\frac{1}{3} \\ 0 & -1/6 & -5/6 & 7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ rank} = 2 \right]$$

3.13 SYSTEM OF LINEAR EQUATIONS (NON-HOMOGENEOUS)

Let us consider the following system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\left. \begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad &= \ldots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_n \end{aligned} \right\} \quad \ldots(i)$$

In matrix notation these equations can be put in the form

$$AX = B \quad \ldots(ii)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Augmented matrix: The augmented matrix $[A : B]$ or $\tilde{A}$ of system (*i*) is obtained by augmenting A by the column B.

i.e.,
$$\tilde{A} = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ a_{m1} & a_{m2} & & a_{mn} & \vdots & b_n \end{bmatrix}$$

3.13.1 Conditions for Solution of Linear Equations [*U.P.T.U., (C.O.), 2003*]

Consistent: If the ranks of A and augmented matrix $[A : B]$ are equal, then the system is said to be consistent otherwise inconsistent. There are following conditions for exist the solution of any system of linear equations :

(*i*) If $\rho(A) = \rho[A : B] = r = n$ (where n is the number of variables) then the system has a unique solution.

(*ii*) If $\rho(A) = \rho[A : B] = r < n$.

then the system has infinitely many solutions in terms of remaining $n - r$ unknowns which are arbitrary.

If $n - r = 1$ (then solution is one variable independent solution and let equal to K). $n - r = 2$ (then solution is two variable independent solution and let variables equal to K_1, and K_2) and so on.

Trivial solution: It is a solution where all x_i are zero *i.e.,* $x_1 = x_2 \ldots = x_n = 0$.

Example 1. Check the consistency of the following system of linear nonhomogeneous equations and find the solution, if exists: (*U.P.T.U., 2007*)

$$7x_1 + 2x_2 + 3x_3 = 16$$
$$2x_1 + 11x_2 + 5x_3 = 25$$
$$x_1 + 3x_2 + 4x_3 = 13.$$

Sol. Here,
$$A = \begin{bmatrix} 7 & 2 & 3 \\ 2 & 11 & 5 \\ 1 & 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 16 \\ 25 \\ 13 \end{bmatrix}$$

The augmented matrix
$$[A : B] = \begin{bmatrix} 7 & 2 & 3 & \vdots & 16 \\ 2 & 11 & 5 & \vdots & 25 \\ 1 & 3 & 4 & \vdots & 13 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_3$, we get

$$[A : B] \sim \begin{bmatrix} 1 & 3 & 4 & \vdots & 13 \\ 2 & 11 & 5 & \vdots & 25 \\ 7 & 2 & 2 & \vdots & 16 \end{bmatrix}$$

Again $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & \vdots & 13 \\ 0 & 5 & -3 & \vdots & -1 \\ 0 & -19 & -25 & \vdots & -75 \end{bmatrix}$$

$R_3 \to R_3 + \dfrac{19}{5} R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & \vdots & 13 \\ 0 & 5 & -3 & \vdots & -1 \\ 0 & 0 & \dfrac{-182}{5} & \vdots & -\dfrac{394}{5} \end{bmatrix}$$

$\Rightarrow \quad \rho(A) = \rho[A : B] = 3$

$\Rightarrow \quad r = n = 3. \therefore$ The system is consistent.

The given system has a unique solution.

Now from $\quad AX = B$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & -3 \\ 0 & 0 & \dfrac{-182}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \\ \dfrac{-394}{5} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ 5x_2 - 3x_3 \\ \dfrac{-182}{5} x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \\ \dfrac{-394}{5} \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 3x_2 + 4x_3 = 13 \quad \text{...(i)}$$

$$5x_2 - 3x_3 = -1 \quad \text{...(ii)}$$

$$\frac{182}{5} x_3 = \frac{394}{5} \quad \text{...(iii)}$$

On solving these equations, we get the final solution

$$x_1 = \frac{95}{91}, \ x_2 = \frac{100}{91}, \ x_3 = \frac{197}{91}.$$

Example 2. Test the consistency of following system of linear equations and hence find the solution. (*U.P.T.U., 2005*)

$$4x_1 - x_2 = 12$$

$$-x_1 + 5x_2 - 2x_3 = 0$$

$$-2x_2 + 4x_3 = -8$$

Sol. The augmented matrix $[A : B] = \begin{bmatrix} 4 & -1 & 0 & \vdots & 12 \\ -1 & 5 & -2 & \vdots & 0 \\ 0 & -2 & 4 & \vdots & -8 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_2$, we get

$$[A : B] \sim \begin{bmatrix} -1 & 5 & -2 & \vdots & 0 \\ 4 & -1 & 0 & \vdots & 12 \\ 0 & -2 & 4 & \vdots & -8 \end{bmatrix}$$

$R_2 \to R_2 + 4R_1$

$$\sim \left[\begin{array}{ccc:c} -1 & 5 & -2 & 0 \\ 0 & 19 & -8 & 12 \\ 0 & -2 & 4 & -8 \end{array}\right]$$

$R_1 \to -R_1,\ R_3 \to R_3 + \dfrac{2}{19} R_2,$

$$\sim \left[\begin{array}{ccc:c} 1 & -5 & 2 & 0 \\ 0 & 19 & -8 & 12 \\ 0 & 0 & \frac{60}{19} & \frac{-128}{19} \end{array}\right]$$

$\therefore \qquad \rho(A) = \rho\,[A : B] = 3$

i.e., $\qquad r = n = 3$ (The system is consistent).

Hence, there is a unique solution

$$\Rightarrow \begin{bmatrix} -1 & -5 & 2 \\ 0 & 19 & -8 \\ 0 & 0 & \frac{60}{19} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ \frac{-128}{19} \end{bmatrix}$$

$$\Rightarrow \qquad x_1 - 5x_2 + 2x_3 = 0 \qquad ...(i)$$

$$19x_2 - 8x_3 = 12 \qquad ...(ii)$$

$$\frac{60x_3}{19} = \frac{-128}{19} \qquad ...(iii)$$

$$\Rightarrow \qquad x_3 = -\frac{32}{15}$$

putting the value of x_3 in equation (*ii*), we get

$$19x_2 - 8\left(-\frac{32}{15}\right) = 12$$

$$\Rightarrow \qquad 19x_2 = 12 - \frac{256}{15} = -\frac{76}{15}$$

$$\Rightarrow \qquad x_2 = \frac{-76}{15 \times 19} = -\frac{4}{15}$$

and putting the values of x_1, x_2 in equation (*i*), we get

$$x_1 - 5\left(-\frac{4}{15}\right) + 2\left(-\frac{32}{15}\right) = 0$$

$$x_1 + \frac{20}{15} - \frac{64}{15} = 0$$

$$\Rightarrow \qquad x_1 - \frac{44}{15} = 0$$

$$\Rightarrow \qquad x_1 = \frac{44}{15}$$

Hence, $\qquad x_1 = \dfrac{44}{15},\ x_2 = -\dfrac{4}{15}$ and $x_3 = -\dfrac{32}{15}$.

Example 3. Solve

$$\begin{aligned} 2x_1 - 2x_2 + 4x_3 + 3x_4 &= 9 \\ x_1 - x_2 + 2x_3 + 2x_4 &= 6 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 3 \\ x_1 - x_2 + x_4 &= 2 \end{aligned}$$

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & -2 & 4 & 3 & \vdots & 9 \\ 1 & -1 & 2 & 2 & \vdots & 6 \\ 2 & -2 & 1 & 2 & \vdots & 3 \\ 1 & -1 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 2 & -2 & 4 & 3 & \vdots & 9 \\ 2 & -2 & 1 & 2 & \vdots & 3 \\ 1 & -1 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_4 \to R_4 - R_1,\ R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 0 & -1 & \vdots & -3 \\ 0 & 0 & -3 & -2 & \vdots & -9 \\ 0 & 0 & -2 & -1 & \vdots & -4 \end{bmatrix}$$

$R_2 \to (-1)\ R_2,\ R_3 \to (-1)\ R_3,\ R_4 \to (-1)\ R_4$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 0 & 1 & \vdots & 3 \\ 0 & 0 & 3 & 2 & \vdots & 9 \\ 0 & 0 & 2 & 1 & \vdots & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_4$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 0 & 1 & \vdots & 3 \\ 0 & 0 & 1 & 1 & \vdots & 5 \\ 0 & 0 & 2 & 1 & \vdots & 4 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 1 & 1 & \vdots & 5 \\ 0 & 0 & 0 & 1 & \vdots & 3 \\ 0 & 0 & 2 & 1 & \vdots & 4 \end{bmatrix}$$

$R_4 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 1 & 1 & \vdots & 5 \\ 0 & 0 & 2 & 1 & \vdots & 4 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 1 & 1 & \vdots & 5 \\ 0 & 0 & 0 & -1 & \vdots & -6 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 1 & 1 & \vdots & 5 \\ 0 & 0 & 0 & 1 & \vdots & 6 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} (R_3 \to -R_3)$$

$R_4 \to R_4 - R_3$ and then $R_4 \to (-1)\ R_4$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 2 & \vdots & 6 \\ 0 & 0 & 1 & 1 & \vdots & 5 \\ 0 & 0 & 0 & 1 & \vdots & 6 \\ 0 & 0 & 0 & 0 & \vdots & 3 \end{bmatrix}$$

Hence, $\rho(A) = 3$ and $\rho\,[A : B] = 4 \Rightarrow \rho(A) \neq \rho\,[A : B]$.

So the given system is inconsistent and therefore it has no solution.

Example 4. Investigate for what values of λ, μ the equations

$$x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu$$

have (*i*) no solution (*ii*) a unique solution (*iii*) an infinity of solutions. (*U.P.T.U., 2001*)

Sol. The augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 10 \\ 1 & 2 & \lambda & \vdots & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \lambda - 1 & \vdots & \mu - 6 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \lambda - 3 & \vdots & \mu - 10 \end{bmatrix}$$

(*i*) For no solution $\rho(A) \neq \rho\,[A; B]$ it is only possible when $\lambda = 3$.

(*ii*) For unique solution $\rho(A) = \rho[A \vdots B]$ it is only possible when $\lambda - 3 \neq 0$ *i.e.*, $\lambda \neq 3$ and $\mu \neq 10$.

(*iii*) For infinite number of solutions $\rho(A) = \rho[A : B] = r < n$ it is only possible when $\lambda = 3$ and $\mu = 10$.

Example 5. Show that the equations

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

are consistent and solve them.

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 14 \\ 1 & 4 & 7 & \vdots & 30 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 3 & 6 & \vdots & 24 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Hence, $\rho(A) = \rho[A : B] = 2$

i.e., $r = 2 < 3 \ (n = 3)$

$\therefore$ $n - r = 3 - 2 = 1$ (one variable independent solution).

The system is consistent and have infinitely solutions.

Now $AX = B$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$$x + y + z = 6 \qquad \ldots(i)$$
$$y + 2z = 8 \qquad \ldots(ii)$$

Let $z = k$

Putting $z = k$ in (*ii*), we get

$$y + 2k = 8 \Rightarrow y = 8 - 2k$$

From (*i*) $x + 8 - 2k + k = 6 \Rightarrow x = k - 2$

Therefore, $x = k - 2$, $y = 8 - 2k$ and $z = k$. **Ans.**

Example 6. Solve

$$\begin{aligned} 3x + 3y + 2z &= 1 \\ x + 2y &= 4 \\ 10y + 3z &= -2 \\ 2x - 3y - z &= 5 \end{aligned}$$

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 3 & 2 & : & 1 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1,\ R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix}$$

$R_3 \to R_3 + \frac{10}{3}R_2,\ R_4 \to R_4 - \frac{7}{3}R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 0 & \frac{29}{3} & : & \frac{-116}{3} \\ 0 & 0 & \frac{-17}{3} & : & \frac{68}{3} \end{bmatrix}$$

$R_3 \to \frac{3}{29}R_3,\ R_4 \to \frac{3}{17}R_4$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 0 & 1 & : & -4 \\ 0 & 0 & -1 & : & 4 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 0 & 1 & : & -4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\Rightarrow \quad \rho(A) = \rho[A : B] = 3$

i.e., $\quad r = 3 = n =$ number of variables.

Hence, the system is consistent and has unique solution.

Now, $\quad AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -4 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x + 2y = 4 \quad \text{...}(i)$$

$$-3y + 2z = -11 \quad \text{...}(ii)$$

$$z = -4 \quad \text{...}(iii)$$

On solving (*i*) and (*ii*), we get $x = 2$, $y = 1$. Hence, $x = 2$, $y = 1$ and $z = -4$.

Example 7. Apply the matrix method to solve the system of equations

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

[*U.P.T.U., 2003; U.P.T.U. (C.O.), 2003*]

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 2 & : & 1 \\ 2 & -2 & 3 & : & 2 \\ 1 & -1 & 1 & : & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 5 & : & -8 \\ 0 & -6 & 5 & : & -4 \\ 0 & -3 & 2 & : & -4 \end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7}R_2$, $R_4 \to R_4 - \frac{3}{7}R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 5 & : & -8 \\ 0 & 0 & \frac{5}{7} & : & \frac{20}{7} \\ 0 & 0 & -\frac{1}{7} & : & -\frac{4}{7} \end{bmatrix}$$

$$R_4 \to R_4 + \frac{1}{5} R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 5 & : & -8 \\ 0 & 0 & \frac{5}{7} & : & \frac{20}{7} \\ 0 & 0 & 0 & & 0 \end{bmatrix}$$

$\Rightarrow$ $\rho(A) = \rho[A : B] = 3 =$ number of variables.

Hence, the system is consistent and has a unique solution.

Now, $AX = B$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & \frac{5}{7} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ \frac{20}{7} \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x + 2y - z = 3 \qquad ...(i)$$

$$-7y + 5z = -8 \qquad ...(ii)$$

$$\frac{5}{7} z = \frac{20}{7} \Rightarrow z = 4$$

From (i) and (ii), we get, $x = -1$, $y = 4$

$\Rightarrow$ $x = -1$, $y = 4$, $z = 4$.

3.14 SYSTEM OF HOMOGENEOUS EQUATIONS

If in the set of equations (1) of (3.13), $b_1 = b_1 = ... = b_n = 0$, the set of equation is said to be homogeneous.

Result 1: If $r = n$, *i.e.*, the rank of coefficient matrix is equal to the number of variables, then there is always a trivial solution ($x_1 = x_2 = ... = x_n = 0$).

Result 2: If $r < n$, *i.e.*, the rank of coefficient matrix is smaller than the number of variables, then there exist a non-trivial solution.

Result 3: For non-trivial solution always $|A| = 0$.

Example 8. Solve the following system of homogeneous equations:

$$x + 2y + 3z = 0$$

$$3x + 4y + 4z = 0$$

$$7x + 10y + 12z = 0$$

Sol. Here, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -8 \\ 0 & -4 & -9 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows rank $(A) = 3$ = number of unknowns. Hence, the given system has a trivial solution *i.e.*, $x = y = z = 0$.

Example 9. Solve

$$\begin{aligned} x + y - 2z + 3w &= 0 \\ x - 2y + z - w &= 0 \\ 4x + y - 5z + 8w &= 0 \\ 5x - 7y + 2z - w &= 0 \end{aligned}$$

Sol. The coefficient matrix A is

$$A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$, $R_5 \to R_5 - 5R_1$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 4R_2$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho\,(A) = 2 < 4\ (n = 4)$, so there exist a non-trivial solution.

Now, $AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad x + y - 2z + 3w = 0 \qquad ...(i)$$

$$- 3y + 3z - 4w = 0 \qquad ...(ii)$$

Choose $z = k_1$, $w = k_2$, then from (*ii*) and (*i*), we get

$$- 3y + 3k_1 - 4k_2 = 0 \Rightarrow y = k_1 - \frac{4}{3}k_2$$

and $$x + k_1 - \frac{4}{3}k_2 - 2k_1 + 3k_2 = 0 \Rightarrow x = k_1 - \frac{5}{3}k_2.$$

where k_1 and k_2 are arbitrary constants.

Example 10. Find the value of λ such that the following equations have unique solution.

$\lambda x + 2y - 2z - 1 = 0$, $4x + 2\lambda y - z - 2 = 0$, $6x + 6y + \lambda z - 3 = 0$ (*U.P.T.U., 2003*)

Sol. We have

$$\lambda x + 2y - 2z = 1$$
$$4x + 2\lambda y - z = 2$$
$$6x + 6y + \lambda z = 3$$

The coefficient matrix A is

$$A = \begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix}$$

For unique solution $|A| \neq 0$

$$\therefore \qquad \begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} = \lambda^3 + 11\lambda - 30 \neq 0$$

$$\Rightarrow \qquad (\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 0 \Rightarrow \lambda \neq 2.$$

Example 11. Determine b such that the system of homogeneous equations (*U.P.T.U., 2008*)

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

has (*i*) trivial solution and (*ii*) non-trivial solution.

Sol. (*i*) For trivial solution, $|A| \neq 0$

$$\therefore \qquad |A| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{vmatrix} \neq 0$$

$$\Rightarrow \quad 2(b - 9) - (b - 12) + 2(3 - 4) \neq 0$$

$$\Rightarrow \qquad 2b - 18 - b + 12 - 2 \neq 0 \Rightarrow b - 8 \neq 0 \Rightarrow b \neq 8.$$

(*ii*) For non-trivial solution, $|A| = 0$

$$\Rightarrow \qquad b - 8 = 0 \Rightarrow b = 8.$$

3.15 GAUSSIAN ELIMINATION METHOD

Gaussian elimination method is an exact method which solves a given system of equations in n unknowns by transforming the coefficient matrix into an upper triangular matrix and then solve for the unknowns by back substitution.

Example 12. Solve the system of equations:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 9 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 + 2x_3 &= 8 \end{aligned}$$

(*U.P.T.U., 2006*)

by Gaussian elimination method.

Sol. The augmented matrix is:

$$[A : B] = \begin{bmatrix} 2 & 3 & 1 & : & 9 \\ 1 & 2 & 3 & : & 6 \\ 3 & 1 & 2 & : & 8 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 2 & 3 & 1 & : & 9 \\ 3 & 1 & 2 & : & 8 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -5 & : & -3 \\ 0 & -5 & -7 & : & -10 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -5 & : & -3 \\ 0 & 0 & 18 & : & 5 \end{bmatrix}$$

which is upper triangular form

$\therefore \quad 18x_3 = 5 \Rightarrow x_3 = \dfrac{5}{18}$

and $\quad x_1 + 2x_2 + 3x_3 = 6 \quad$ ...(*i*)

$\quad - x_2 - 5x_3 = - 3 \quad$ ...(*ii*)

From (*ii*) $\quad x_2 + \dfrac{25}{18} = 3 \Rightarrow x_2 = 3 - \dfrac{25}{18} = \dfrac{29}{18}$

again from (*i*), we have

$$x_1 + 2 \times \frac{29}{18} + 3 \times \frac{5}{18} = 6 \Rightarrow x_1 + \frac{73}{18} = 6$$

$\Rightarrow \quad x_1 = 6 - \dfrac{73}{18} = \dfrac{35}{18}$

Hence, $\quad x_1 = \dfrac{35}{18},\ x_2 = \dfrac{29}{18}$ and $x_3 = \dfrac{5}{18}$.

Example 13. Solve by Gaussian elimination method

$$\begin{aligned} 10x_1 - 7x_2 + 3x_3 + 5x_4 &= 6 \\ -6x_1 + 8x_2 - x_3 - 4x_4 &= 5 \\ 3x_1 + x_2 + 4x_3 + 11x_4 &= 2 \\ 5x_1 - 9x_2 - 2x_3 + 4x_4 &= 7 \end{aligned}$$

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ -6 & 8 & -1 & -4 & : & 5 \\ 3 & 1 & 4 & 11 & : & 2 \\ 5 & -9 & -2 & 4 & : & 7 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 3 & 1 & 4 & 11 & : & 2 \\ -6 & 8 & -1 & -4 & : & 5 \\ 5 & -9 & -2 & 4 & : & 7 \end{bmatrix}$$

$R_2 \to R_2 - \dfrac{3}{10} R_1,\ R_3 \to R_3 + \dfrac{6}{10} R_1,\ R_4 \to R_4 - \dfrac{1}{2} R_1$

$$\sim \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & \frac{31}{10} & \frac{31}{10} & \frac{19}{2} & : & \frac{1}{5} \\ 0 & \frac{19}{5} & \frac{4}{5} & -1 & : & \frac{43}{5} \\ 0 & -\frac{11}{2} & -\frac{7}{2} & \frac{3}{2} & : & 4 \end{bmatrix}$$

$R_2 \to \dfrac{10}{31} R_2,\ R_3 \to \dfrac{5}{19} R_2,\ R_3 \to -\dfrac{2}{11} R_3$

$$\sim \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 1 & 1 & \frac{95}{31} & : & \frac{2}{31} \\ 0 & 1 & \frac{4}{19} & -\frac{5}{19} & : & \frac{43}{19} \\ 0 & 1 & \frac{7}{11} & -\frac{3}{11} & : & \frac{-8}{11} \end{bmatrix}$$

$R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 1 & 1 & \frac{95}{31} & : & \frac{2}{31} \\ 0 & 0 & -\frac{15}{19} & -\frac{1960}{589} & : & \frac{1295}{589} \\ 0 & 0 & -\frac{4}{11} & -\frac{1138}{341} & : & -\frac{270}{341} \end{bmatrix}$$

$R_3 \to -\frac{19}{15}R_3,\ R_4 \to -\frac{11}{4}R_4$

$$\sim \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 1 & 1 & \frac{95}{31} & : & \frac{2}{31} \\ 0 & 0 & 1 & \frac{392}{93} & : & -\frac{259}{93} \\ 0 & 0 & 1 & \frac{569}{62} & : & \frac{135}{62} \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 1 & 1 & \frac{95}{31} & : & \frac{2}{31} \\ 0 & 0 & 1 & \frac{392}{93} & : & -\frac{259}{93} \\ 0 & 0 & 0 & \frac{923}{186} & : & \frac{923}{186} \end{bmatrix}$$

Hence, the coefficient matrix is an upper triangular form

$$\therefore \quad \frac{923}{186}x_4 = \frac{923}{186} \Rightarrow x_4 = 1$$

$$x_3 + \frac{392}{93}x_4 = -\frac{259}{93} \Rightarrow x_3 + \frac{392}{93} = -\frac{259}{93}$$

or $$x_3 = -\frac{259}{93} - \frac{392}{93} = -\frac{651}{93} = -7$$

and $$x_2 + x_3 + \frac{95}{31}x_4 = \frac{2}{31} \Rightarrow x_2 - 7 + \frac{95}{31} = \frac{2}{31}$$

$$\Rightarrow \quad x_2 - \frac{122}{31} = \frac{2}{31} \Rightarrow x_2 = \frac{2}{31} + \frac{122}{31} = \frac{124}{31} = 4$$

Again $10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$

$\Rightarrow\ 10x_1 - 7 \times 4 + 3 \times (-7) + 5 \times 1 = 6 \Rightarrow 10x_1 - 28 - 21 + 5 = 6$

$\Rightarrow \quad 10x_1 - 44 = 6 \Rightarrow 10x_1 = 50 \Rightarrow x_1 = 5$

Therefore, $x_1 = 5$, $x_2 = 4$, $x_3 = -7$ and $x_4 = 1$.

3.15.1 Gauss-Jordan Elimination Method

Apply elementary row operations on both A and B such that A reduces to the normal form. Then the solution is obtained.

Example 14. Solve by Gauss-Jordan elimination method:

$$2x_1 + x_2 + 3x_3 = 1$$
$$4x_1 + 4x_2 + 7x_3 = 1$$
$$2x_1 + 5x_2 + 9x_3 = 3$$

Sol. Here $\quad A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

∴ Augmented matrix

$$[A : B] = \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 4 & 4 & 7 & : & 1 \\ 2 & 5 & 9 & : & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 0 & 2 & 1 & : & -1 \\ 0 & 4 & 6 & : & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 0 & 2 & 1 & : & -1 \\ 0 & 0 & 4 & : & 4 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$, $R_2 \to \frac{1}{2}R_2$, $R_3 \to \frac{1}{4}R_3$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & : & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & : & -\frac{1}{2} \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \to R_1 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 0 & \frac{5}{4} & : & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & : & -\frac{1}{2} \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \to R_1 - \frac{5}{4}R_3,\ R_2 \to R_2 - \frac{1}{2}R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -\frac{1}{2} \\ 0 & 1 & 0 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Hence, the matrix A is in normal form

$\therefore \quad x_1 = -\frac{1}{2},\ x_2 = -1,\ x_3 = 1.$

Example 15. Solve by Gauss-Jordan elimination method:

$$\begin{aligned} 2x_1 + 5x_2 + 2x_3 - 3x_4 &= 3 \\ 3x_1 + 6x_2 + 5x_3 + 2x_4 &= 2 \\ 4x_1 + 5x_2 + 14x_3 + 14x_4 &= 11 \\ 5x_1 + 10x_2 + 8x_3 + 4x_4 &= 4 \end{aligned}$$

Sol. $$[A : B] = \begin{bmatrix} 2 & 5 & 2 & -3 & : & 3 \\ 3 & 6 & 5 & 2 & : & 2 \\ 4 & 5 & 14 & 14 & : & 11 \\ 5 & 10 & 8 & 4 & : & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 2 & 5 & 2 & -3 & : & 3 \\ 1 & 1 & 3 & 5 & : & -1 \\ 1 & -1 & 9 & 12 & : & 9 \\ 1 & 5 & -6 & -10 & : & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_4,\ R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 5 & -6 & -10 & : & -7 \\ 1 & -1 & 9 & 12 & : & 9 \\ 1 & 1 & 3 & 5 & : & -1 \\ 2 & 5 & 2 & -3 & : & 3 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 5 & -6 & -10 & : & -7 \\ 0 & -6 & 15 & 22 & : & 16 \\ 0 & -4 & 9 & 15 & : & 6 \\ 0 & -5 & 14 & 17 & : & 17 \end{bmatrix}$$

$R_2 \to R_2 - R_4$, then $R_2 \to -R_2$, $R_3 \to R_3 - 4R_2$, then again $R_3 \to -R_3$, $R_4 \to R_4 - 5R_2$, then also again $R_4 \to -R_4$

$$\sim \begin{bmatrix} 1 & 5 & -6 & -10 & : & -7 \\ 0 & 1 & -1 & -5 & : & 1 \\ 0 & 0 & -5 & 5 & : & -10 \\ 0 & 0 & 9 & -8 & : & 22 \end{bmatrix}$$

$R_3 \to -\frac{1}{5}R_3$, then $R_4 \to R_4 - 9R_3$

$$\sim \begin{bmatrix} 1 & 5 & -6 & -10 & : & -7 \\ 0 & 1 & -1 & -5 & : & 1 \\ 0 & 0 & 1 & -1 & : & 2 \\ 0 & 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$R_3 \to R_3 + R_4$, $R_2 \to R_2 + 5R_4$, $R_1 \to R_1 + 10R_4$

$$\sim \begin{bmatrix} 1 & 5 & -6 & 0 & : & 33 \\ 0 & 1 & -1 & 0 & : & 21 \\ 0 & 0 & 1 & 0 & : & 6 \\ 0 & 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$R_2 \to R_2 + R_3$, $R_1 \to R_1 + 6R_3$

$$\sim \begin{bmatrix} 1 & 5 & 0 & 0 & : & 69 \\ 0 & 1 & 0 & 0 & : & 27 \\ 0 & 0 & 1 & 0 & : & 26 \\ 0 & 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & -66 \\ 0 & 1 & 0 & 0 & : & 27 \\ 0 & 0 & 1 & 0 & : & 6 \\ 0 & 0 & 0 & 1 & : & 4 \end{bmatrix}$$

Hence, the coefficient matrix is in normal form

$\therefore \quad x_1 = -66,\ x_2 = 27,\ x_3 = 6$ and $x_4 = 4$.

Example 16. Find the values of λ for which the equations

$$3x + y - \lambda z = 0;\ 4x - 2y - 3z = 0;\ 2\lambda x + 4y + \lambda z = 0$$

have a non-trivial solution. Obtain the most general solutions in each case.

Sol. The coefficient matrix is

$$A = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}$$

for non-trivial solution $|A| = 0$

$$\therefore \quad |A| = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} = 0 \Rightarrow 3(-2\lambda + 12) - (4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$$

$$\Rightarrow \quad -6\lambda + 36 - 10\lambda - 16\lambda - 4\lambda^2 = 0$$

$$\Rightarrow \quad -4\lambda^2 - 32\lambda + 36 = 0 \Rightarrow \lambda^2 + 8\lambda - 9 = 0$$

$$\Rightarrow \quad \lambda^2 + 9\lambda - \lambda - 9 = 0 \Rightarrow (\lambda - 1)(\lambda + 9) = 0$$

$$\Rightarrow \quad \lambda = 1, -9.$$

Case I. If $\lambda = 1$,

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$$

$R_2 \to R_2 - \frac{4}{3}R_1$, $R_3 \to R_3 - \frac{2}{3}R_1$

$$\sim \begin{bmatrix} 3 & 1 & -1 \\ 0 & \frac{-10}{3} & -\frac{5}{3} \\ 0 & \frac{10}{3} & \frac{5}{3} \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 3 & 1 & -1 \\ 0 & -\frac{10}{3} & -\frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

This shows $\rho(A) = 2 < 3$ $(n = 3)$

$\therefore$ Let $z = k$

$$-\frac{10y}{3} - \frac{5z}{3} = 0 \quad \Rightarrow 2y + k = 0 \Rightarrow y = -\frac{k}{2}$$

and $$3x + y - z = 0 \quad \Rightarrow 3x - \frac{k}{2} - k = 0 \Rightarrow x = \frac{k}{2}.$$ **Ans.**

Case II. If $\lambda = -9$

$$A = \begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}$$

Solve itself like case I.

$$x = -\frac{3k}{2}, \; y = -\frac{9k}{2}, \; z = k.$$

EXERCISE 3.4

Examine whether the following systems of equations are consistent. If consistent solve.

1.
$$x + y + z = 6$$
$$2x + 3y - 2z = 2$$
$$5x + y + 2z = 13$$
(U.P.T.U., 2000) [**Ans.** $x = 1, y = 2, z = 3$]

2.
$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$

$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

[**Ans.** $x = 2$, $y = 1$ and $z = -4$]

3.
$$x_1 + 2x_2 - x_3 = 3$$
$$3x_1 - x_2 + 2x_3 = 1$$
$$2x_1 - 2x_2 + 3x_3 = 2$$
$$x_1 - x_2 + x_3 = -1$$

(U.P.T.U., 2002) [**Ans.** $x_1 = -1$, $x_2 = 4$, $x_3 = 4$]

4.
$$x + y + z = 7$$
$$x + 2y + 3z = 8$$
$$y + 2z = 6$$

[**Ans.** inconsistent; no solution]

5.
$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

[**Ans.** $x = (7 - 6k)/11$, $y = (3 + k)/11$, $z = k$]

6.
$$-x_1 + x_2 + 2x_3 = 2$$
$$3x_1 - x_2 + x_3 = 6$$
$$-x_1 + 3x_2 + 4x_3 = 4$$

[**Ans.** $x_1 = 1$, $x_2 = -1$, $x_3 = 2$]

7. Find the values of a and b for which the system has (*i*) no solution, (*ii*) unique solution and (*iii*) infinitely many solution.

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + az = b$$

[**Ans.** (*i*) $a = 5$, $b \neq 9$; (*ii*) $a \neq 5$, b any value; (*iii*) $a = 5$, $b = 9$]

8. Discuss the solutions of the system of equations for all values of λ.

$x + y + z = 2$, $2x + y - 2z = 2$, $\lambda x + y + 4z = 2$.

[**Ans.** Unique solution if $\lambda \neq 0$; infinite number of solutions if $\lambda = 0$]

9.
$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

[**Ans.** $x = k - 2$, $y = 8 - 2k$, $z = k$]

10.
$$2x - y + z = 7$$
$$3x + y - 5z = 13$$
$$x + y + z = 5$$

[**Ans.** $x = 4$, $y = 1$, $z = 0$]

Solve the following homogeneous equations:

11.
$$x + 2y + 3z = 0$$
$$2x + y + 3z = 0$$
$$3x + 2y + z = 0$$

[**Ans.** $x = y = z = 0$]

12.
$$x + y + 3z = 0$$
$$x - y + z = 0$$
$$x - 2y = 0$$
$$x - y + z = 0$$

[**Ans.** $x = -2k$, $y = -k$, $z = k$]

13.
$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

[**Ans.** $x = y = z = 0$]

14.
$$4x + 2y + z + 3w = 0$$
$$6x + 3y + 4z + 7w = 0$$
$$2x + y + w = 0$$
[**Ans.** $x = k_1,\ y = -2k_1 - k_2,\ z = -k_2,\ w = k_2$]

15. For what values of λ the given equations will have a non-trivial solution.
$$x + 2y + 3z = \lambda x$$
$$2x + 3y + z = \lambda x$$
$$3x + y + 2z = \lambda y$$
[**Ans.** $\lambda = 6$]

16. Find the value of 'a' so that the following system of homogeneous equations have exactly 2 linearly independent solutions.
$$ax_1 - x_2 - x_3 = 0$$
$$-x_1 + ax_2 - x_3 = 0$$
$$-x_1 - x_2 + ax_3 = 0$$
[**Ans.** $a = -1$]

17. Apply the test of rank to examine if the following equations are consistent:
$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$
and if consistent, find the complete solution. [**Ans.** $x = y = z = 2$]

18. Show that the equations
$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$
have no solutions unless $a + b + c = 0$, in which case they have infinitely many solutions. Find their solutions $a = 1$, $b = 1$ and $c = -2$. [**Ans.** $x = k - 1,\ y = k - 1,\ z = k$]

19. Show that the system of equations $x + 2y - 2w = 0$, $2x - y - w = 0$, $x + 2y - w = 0$. $4x - y + 3z - w = 0$ do not have a non-trivial solution.

20. Show that the homogeneous system of equations $x + y \cos\gamma + z \cos\beta = 0$, $x \cos\gamma + y + z \cos\alpha = 0$, $x \cos\beta + y \cos\gamma + z = 0$, has non-trivial solution if $\alpha + \beta + \gamma = 0$.

Solve the following system of equations by Gaussian elimination method:

21.
$$x_1 + 2x_2 - x_3 = 3$$
$$2x_1 - 2x_2 + 3x_3 = 2$$
$$3x_1 - x_2 + 2x_3 = 1$$
$$x_1 - x_2 + x_3 = -1$$
[**Ans.** $x_1 = 1,\ x_2 = 4,\ x_3 = 4$]

22.
$$2x_1 + x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + 3x_3 = 18$$
$$x_1 + 4x_2 + 9x_3 = 16$$
[**Ans.** $x_1 = 7,\ x_2 = -9,\ x_3 = 5$]

23.
$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
[**Ans.** $x_1 = 3,\ x_2 = 2,\ x_3 = 1$]

24.
$$\begin{aligned} x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 6x_3 &= -12 \\ 3x_1 - x_2 - x_3 &= 4 \end{aligned}$$
$\left[\textbf{Ans. } x_1 = \frac{117}{71},\ x_2 = -\frac{81}{71},\ x_3 = \frac{148}{71}\right]$

25.
$$\begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 6x_1 - 6x_2 + 6x_3 + 12x_4 &= 36 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1 \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10 \end{aligned}$$
[**Ans.** $x_1 = 2$, $x_2 = 1$, $x_3 = -1$, $x_4 = 3$]

26.
$$\begin{aligned} 2x_1 - 7x_2 + 4x_3 &= 9 \\ x_1 + 9x_2 - 6x_3 &= 1 \\ -3x_1 + 8x_2 + 5x_3 &= 6 \end{aligned}$$
[**Ans.** $x_1 = 4$, $x_2 = 1$, $x_3 = 2$]

Solve by Gauss-Jordan method:

27.
$$\begin{aligned} x - 3y - 8z &= -10 \\ 3x + y &= 4 \\ 2x + 5y + 6z &= 13 \end{aligned}$$
[**Ans.** $x = y = z = 1$]

28.
$$\begin{aligned} x + y + z &= 6 \\ 2x + 3y - 2z &= 2 \\ 5x + y + 2z &= 13 \end{aligned}$$
[**Ans.** $x = 1$, $y = 2$, $z = 3$]

29.
$$\begin{aligned} 3x + y + 2z &= 3 \\ 2x - 3y - z &= -3 \\ x + 2y + z &= 4 \end{aligned}$$
[**Ans.** $x = 1$, $y = 2$, $z = -1$]

30.
$$\begin{aligned} 4x + 3y + 3z &= -2 \\ x + z &= 0 \\ 4x + 4y + 3z &= -3 \end{aligned}$$
[**Ans.** $x = 1$, $y = 1$, $z = -1$]

31. Test the consistency and solve $2x - 3y + 7z = 5$, $3x + y - 3z = 13$, $2x + 19y - 47z = 32$.
[**Ans.** Inconsistent]

Solve the following system by any method:

32.
$$\begin{aligned} 2x + 6y + 7z &= 0 \\ 6x + 20y - 6z &= -3 \\ 6y - 18z &= -1 \end{aligned}$$
[**Ans.** Inconsistent]

33.
$$\begin{aligned} 4x - y + 6z &= 16 \\ x - 4y - 3z &= -16 \\ 2x + 7y + 12z &= 48 \\ 5x - 5y + 3z &= 0 \end{aligned}$$
$\left[\textbf{Ans. } x = \frac{-9k}{5} + \frac{16}{3},\ y = \frac{-6k}{5} + \frac{16}{3},\ z = k\right]$

34.
$$\begin{aligned} 2x_1 + x_2 + 5x_3 + x_4 &= 5 \\ x_1 + x_2 - 3x_3 - 4x_4 &= -1 \\ 3x_1 + 6x_2 - 2x_3 + x_4 &= 8 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 &= 2 \end{aligned}$$
$\left[\textbf{Ans. } x_1 = 2,\ x_2 = \frac{1}{5},\ x_3 = 0,\ x_4 = \frac{4}{5}\right]$

35.
$$\begin{aligned} 5x + 3y + 7z &= 4 \\ 3x + 26y + 2z &= 9 \\ 7x + 2y + 11z &= 5 \end{aligned}$$
$\left[\textbf{Ans. } x = \frac{7}{11},\ y = \frac{3}{11},\ z = 0\right]$

3.16 LINEAR DEPENDENCE OF VECTORS

The set of vectors* (row or column matrices) $X_1, X_2, \ldots X_n$ is said to be **linearly dependent** if there exist scalars $a_1, a_2, \ldots a_n$ not all zero such that

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n = O \qquad [O \text{ is null matrix}]$$

3.16.1 Linear Independence of Vectors

If the set of vectors is not linearly dependent then it is said to be linearly independent.

i.e., if every relation of the type

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n = O$$

$$\Rightarrow \quad a_1 = a_2 = \ldots = a_n = 0.$$

Example 1. Show that the vectors (3, 1, – 4), (2, 2, – 3) and (0, – 4, 1) are linearly dependent.

Sol. Let $X_1 = (3, 1, -4)$, $X_2 = (2, 2, -3)$, $X_3 = (0, -4, 1)$

Now, $\quad a_1X_1 + a_2X_2 + a_3X_3 = O \quad$ | linear dependence

$$a_1(3, 1, -4) + a_2(2, 2, -3) + a_3(0, -4, 1) = (0, 0, 0)$$

$$\Rightarrow \quad (3a_1 + 2a_2, a_1 + 2a_2 - 4a_3, -4a_1 - 3a_2 + a_3) = (0, 0, 0)$$

$$\Rightarrow \quad 3a_1 + 2a_2 = 0, a_1 + 2a_2 - 4a_3 = 0, -4a_1 - 3a_2 + a_3 = 0$$

The system of equations is homogeneous.

Now the coefficient matrix is

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & -4 \\ -4 & -3 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & -4 \\ 3 & 2 & 0 \\ -4 & -3 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 + 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & -4 & 12 \\ 0 & 5 & -15 \end{bmatrix}$$

$R_2 \rightarrow -\frac{1}{4}R_2, R_3 \rightarrow \frac{1}{5}R_3$

$$\sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

---

* An ordered set of n numbers belonging to a field F and denoted by $\mathbf{X} = [x_1, x_2, \ldots x_n]$ or $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called an n-dimensional vector over F.

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad \rho(A) = 2$ and $n = 3$

Let $\quad a_3 = k$

and $\quad a_2 - 3a_3 = 0 \Rightarrow a_2 - 3k = 0 \Rightarrow a_2 = 3k$

Again $a_1 + 2a_2 - 4a_3 = 0 \Rightarrow a_1 + 6k - 4k = 0 \Rightarrow a_1 = -2k$

Therefore, $\quad a_1 = -2k,\ a_2 = 3k,\ a_3 = k.$

Hence, a_1, a_2 and a_3 cannot be zero otherwise there is a trivial solution which is impossible.

So the vectors X_1, X_2 and X_3 are linearly dependent and the relation is

$$-2kX_1 + 3kX_2 + kX_3 = O$$

$$\Rightarrow \quad 2X_1 - 3X_2 - X_3 = O.$$

Example 2. Find the value of λ for which the vectors (1, – 2, λ), (2, – 1, 5) and (3, – 5, 7λ) are linearly dependent. (*U.P.T.U., 2006*)

Sol. Let $X_1 = (1, -2, \lambda)$, $X_2 = (2, -1, 5)$, $X_3 = (3, -5, 7\lambda)$

Now $\quad a_1X_1 + a_2X_2 + a_3X_3 = 0 \quad$ | For linear dependence

$\Rightarrow \quad a_1 (1, -2, \lambda) + a_2 (2, -1, 5) + a_3 (3, -5, 7\lambda) = (0, 0, 0)$

$$\Rightarrow \quad \left.\begin{aligned} a_1 + 2a_2 + 3a_3 &= 0 \\ -2a_1 - a_2 - 5a_3 &= 0 \\ \lambda a_1 + 5a_2 + 7\lambda a_3 &= 0 \end{aligned}\right\} \quad ...(i)$$

The system is homogeneous

∴ For non-trivial solution* $|A| = 0$

$$\Rightarrow \quad |A| = \begin{vmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7\lambda \end{vmatrix} = 0$$

$= \ (-7\lambda + 25) - 2(-14\lambda + 5\lambda) + 3(-10 + \lambda) = 0$

$= \ -7\lambda + 25 + 18\lambda - 30 + 3\lambda = 0$

$= \ 14\lambda - 5 = 0 \Rightarrow \lambda = \dfrac{5}{14}.$

Example 3. Show that the vectors [0, 1, – 2], [1, – 1, 1] [1, 2, 1] form a linearly independent set.

Sol. Let $X_1 = [0, 1, -2]$, $X_2 = [1, -1, 1]$ $X_3 = [1, 2, 1]$

Also suppose $a_1X_1 + a_2X_2 + a_3X_3 = O$

$\Rightarrow \quad a_1 [0, 1, -2] + a_2 [1, -1, 1] + a_3 [1, 2, 1] = O$

$\Rightarrow \quad \{a_2 + a_3,\ a_1 - a_2 + 2a_3,\ -2a_1 + a_2 + a_3] = [0, 0, 0]$

$$\Rightarrow \quad a_2 + a_3 = 0$$

$$a_1 - a_2 + 2a_3 = 0$$

* For linear dependence the solution must be non-trivial otherwise it will be linear independence.

$$-2a_1 + a_2 + a_3 = 0$$

Now $$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 5 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} = \rho(A) = 3$$

Now $$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$a_1 - a_2 + 2a_3 = 0$$
$$a_2 + a_3 = 0$$
$$a_3 = 0$$

$\Rightarrow$ $$a_1 = a_2 = a_3 = 0$$

All a_1, a_2 and a_3 are zero. Therefore, they are linearly independent.

Example 4. Examine the vectors

$$X_1 = [1, 1, 0]^T, X_2 = [3, 1, 3]^T, X_3 = [5, 3, 3]^T$$

are linearly dependent.

Sol. Here $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $X_3 = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$

Now $$a_1X_1 + a_2X_2 + a_3X_3 = O$$

$\Rightarrow$ $$a_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + a_3\begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} a_1 + 3a_2 + 5a_3 \\ a_1 + a_2 + 3a_3 \\ 0 + 3a_2 + 3a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$a_1 + 3a_2 + 5a_3 = 0$$
$$a_1 + a_2 + 3a_3 = 0$$
$$0.a_1 + 3a_2 + 3a_3 = 0$$

$$\therefore \quad A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$

$R_1 \to R_2 - R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \end{bmatrix}$$

$R_3 \to R_3 + \dfrac{3}{2}R_2$

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $\rho(A) = 2$ but $n = 3$ so let $a_3 = k$.

and
$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 + 3a_2 + 5a_3 = 0 \quad \text{...}(i)$$

$$a_2 + a_3 = 0 \quad \text{...}(ii)$$

$$\Rightarrow \quad a_2 + k \Rightarrow a_2 = -k$$

From (i) $a_1 + 3(-k) + 5k = 0 \Rightarrow a_1 = 2k$

Since all a_1, a_2, a_3 are not zero.

Therefore, they are linearly dependent

And the relation is

$$2kX_1 - kX_2 + kX_3 = O$$

$$\Rightarrow \quad 2X_1 - X_2 + X_3 = O.$$

Example 5. Show that the vectors [2, 3, 1, –1], [2, 3, 1, – 2], [4, 6, 2, – 3] are linearly independent.

Sol. Consider the relation $a_1X_1 + a_2X_2 + a_3X_3 = O$

$\Rightarrow \quad a_1[2, 3, 1, -1] + a_2 [2, 3, 1, -2] + a_3 [4, 6, 2, -3] = O$

$$\Rightarrow \quad 2a_1 + 2a_2 + 4a_3 = 0$$

$$3a_1 + 3a_2 + 6a_3 = 0$$

$$a_1 + a_2 + 2a_3 = 0$$

$$a_1 + 2a_2 + 3a_3 = 0$$

$$\therefore \quad A = \begin{bmatrix} 2 & 2 & 4 \\ 3 & 3 & 6 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{bmatrix}, R_1 \leftrightarrow R_4, R_2 \leftrightarrow R_3 \quad \sim \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 3 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & -3 & 0 \\ 0 & -2 & 0 \end{bmatrix},\ R_3 \to R_3 - 3R_2,\ R_4 \to R_4 - 2R_2 \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_4 \to R_4 - \dfrac{2}{3} R_3$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 3 \text{ and } n = 3$$

$\therefore$ The system has a trivial solution

Hence, $a_1 = a_2 = a_3 = 0$

i.e., they are linearly independent.

EXERCISE 3.5

1. Examine the following vectors for linear dependence and find the relation if it exists.
   $X_1 = (1, 2, 4),\ X_2 = (2, -1, 3),\ X_3 = (0, 1, 2),\ X_4 = (-3, 7, 2)$. *(U.P.T.U., 2002)*
   [**Ans.** Linearly dependent, $9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$]
2. Show the vectors $X_1 = [1, 2, 1]$, $X_2 = [2, 1, 4]$, $X_3 = [4, 5, 6]$ and $X_4 = [1, 8, -3]$ are linearly independent?
3. Show that the vectors [1, 2, 3], [3, –2, 1] , [1, – 6, – 5] are linearly dependent.
4. If the vectors (0, 1, a), (1, a, 1), (a, 1, 0) are linearly dependent, then find the value of a.
   [**Ans.** $a = 0, \sqrt{2}, -\sqrt{2}$]
5. Examine for linear dependence [1, 0, 2, 1], [3, 1, 2, 1] [4, 6, 2, – 4], [–6. 0, – 3, –4] and find the relation between them, if possible.
   [**Ans.** Linear dependent and the relation is $2X_1 - 6X_2 + X_3 - 2X_4 = 0$]
6. Show that the vectors $X_1 = [2, i, -i]$. $X_2 = [2i, -1, 1]$, $X_3 = [1, 2, 3]$ are linearly dependent.
7. If $X_1 = [1, 1, 2]$, $X_2 = [2, -1, -6]$, $X_3 = [13, 4, -4]$ prove that $7X_1 + 3X_2 - X_3 = 0$.
8. Show that the vectors [3, 1, – 4], [2, 2, – 3] form a linearly independent set but [3, 1, – 4], [2, 2, – 3] and [0, – 4, 1] are linearly dependent.

3.17 EIGEN VALUES AND EIGEN VECTORS

Introduction: At the start of 20th century, Hilbert studied the eigen values. He was the first to use the German word eigen to denote eigen value and eigen vectors in 1904. The word *eigen* mean—**own characteristic or individual.**

More formally in a vector space, a vector function A (matrix) defined if each vector X of vector space, there corresponds a unique vector $Y = AX$. So here we consider a linear transform $Y = AX$

transforms X into a scalar multiple of itself say λ so $AX = \lambda X$ which is called **Eigen value equation**.

In this section, we study the problem

$$AX = \lambda X \qquad \text{...}(i)$$

where A is a $n \times n$ matrix, X is an unknown $n \times 1$ vector and λ is an unknown scalar. From equation (*i*) it can be understand that AX is a scalar multiple of X say λX. Geometrically each vector on the line through the origin determined by X gets mapped back onto the same line under multiplication by A.

Geometrical representation: The eigen value equation means that under the transformation, a eigen vector experience only changes in magnitude and sign. The direction of AX is the same as that of X.

Here A acts to stretch the vector X, not change its direction. So X is an given vector of A.

The eigen value determines the amount, the eigen vector is scaled under the linear transformation. For example, the eigen value $\lambda = 2$ means that the eigen vector is doubled in length and point in the same direction. The eigen value $\lambda = 1$, means that the eigen vector is unchanged, while an eigen value $\lambda = -1$ means that the eigen vector is reversed in direction.

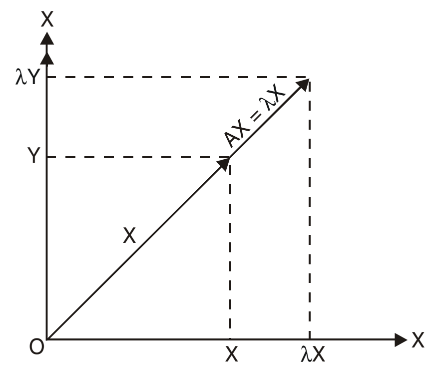


Fig. 3.1

Thus, the eigen value λ is simply the amount of "stretches" or "shrinks" to which a vector is subjected when transformed by A.

3.17.1 Characteristic Equation *(U.P.T.U., 2007)*

If we re-express (*i*) as $AX = \lambda IX$ (where I is an identity matrix).

or
$$AX - \lambda IX = O$$

$$\Rightarrow \quad (A - \lambda I)X = O \qquad \text{...}(ii)$$

Which is homogeneous system of n equations in the n variables $x_1, x_2 \ldots x_n$. The system (*ii*) must have non-trivial solutions otherwise $X = 0$ (which is impossible).

$\therefore$ For non-trivial solution the coefficient matrix $(A - \lambda I)$ will be singular

$$\Rightarrow \quad |A - \lambda I| = 0 \qquad \text{...}(iii) \quad |\text{singular matrix } |A| = 0$$

Expansion of the determinant gives an algebraic equation in λ, known as the "characteristic equation" of A. The determinant $|A - \lambda I|$ is called characteristic polynomial of A.

3.17.2 Characteristic Roots or Eigen Values [*U.P.T.U. (C.O.), 2003, 2007*]

The roots of characteristic equation are called characteristic roots or eigen values.

3.17.3 Eigen Vectors [*U.P.T.U. (C.O.), 2003, 2007*]

The corresponding non-zero vector X is called characteristic eigen vector.

Notes: **1.** If there is one linearly independent solution and two eigen values are same then there will be same eigen vectors of both eigen values.

2. If there is two linearly independent solution and two eigen values are same then there will be different eigen vectors of each eigen value.

3.17.4 Properties of Eigen Values and Eigen Vectors

1. If A is real, its eigen values are real or complex conjugate in pairs.
2. Determinant of A = product of eigen values of A.
3. A and A^T has same eigen values.
4. A^{-1} exists iff 0 is not an eigen value of A, eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

(*U.P.T.U., 2008*)

5. Eigen vector cannot correspond to two distinct characteristic values.
6. **Eigen values of diagonal,** upper triangular or lower triangular matrices are the principal diagonal elements.
7. KA (scalar multiples) has eigen values $K\lambda_i$.
8. A^m has eigen values λ^m.
9. Two vectors X and Y are said to be orthogonal if $X^TY = Y^TX = 0$. (*U.P.T.U., 2008*)

Theorem 1. The latent roots of a Hermitian matrix are all real [*U.P.T.U. (C.O.), 2003*]

Proof. We have $AX = \lambda X$ ...(*i*)

To prove that λ is a real number, we have to prove $\lambda = \bar{\lambda}$

From (*i*)

$$X^*(AX) = X^*(\lambda X) \Rightarrow X^*AX = \lambda X^*X$$

$$\Rightarrow \quad (X^*AX)^* = (\lambda X^*X)^* \Rightarrow X^*A^*X = \bar{\lambda} X^*X, \; (X^{**} = X)$$

$$\Rightarrow \quad X^*AX = \bar{\lambda} X^*X \; (A^* = A)$$

$$\Rightarrow \quad X^*\lambda X = \bar{\lambda} X^*X \Rightarrow \lambda X^*X = \bar{\lambda} X^*X$$

$$\Rightarrow \quad (\lambda - \bar{\lambda}) X^*X = 0 \Rightarrow \lambda = \bar{\lambda}.$$ **Proved.**

Theorem 2. The latent roots of a skew-Hermitian matrix are either zero or purely imaginary. [*U.P.T.U. (C.O.), 2003*]

Proof. Let A be a skew-Hermitian matrix so that $A^* = -A$.

Now we are to prove that $\lambda = 0$ or purely imaginary number

Here $(iA)^* = \bar{i} A^* = -iA^* = -i(-A) = iA$ (As $A^* = -A$)

Hence, iA is a Hermitian matrix.

Let λ be an eigen value relative to the eigen vector X of A, then

$$AX = \lambda X \qquad \ldots(i)$$

$$\Rightarrow \quad iAX = i\lambda X \Rightarrow (iA)X = (i\lambda)X$$

$\Rightarrow$ $i\lambda$ is an eigen root relative to the vector X of Hermitian matrix iA.

$\Rightarrow$ $i\lambda$ is a real number, for eigen values of a Hermitian matrix are all real.

$\Rightarrow$ $\lambda = 0$ or purely imaginary number. **Proved.**

Theorem 3. The characteristic roots of a unitary matrix are of unit modulus. [*U.P.T.U. Special Exam., 2001*]

Proof. Let A be unitary matrix so that $A^*A = I$.

We have $AX = \lambda X$ ...(*i*)

To prove $|\lambda| = 1$

From (*i*), we have

$$(AX)^* = (\lambda X)^* \Rightarrow X^*A^* = \bar{\lambda} X^* \qquad ...(ii)$$

Pre-multiplying (*i*) by (*ii*),

$$(X^*A^*)(AX) = (\bar{\lambda} X^*)(\lambda X)$$

$$\Rightarrow \quad X^*(A^*A)X = \bar{\lambda}\lambda X^*X$$

$$\Rightarrow \quad X^*IX = |\lambda|^2 X^*X \Rightarrow (1 - |\lambda|^2) X^*X = 0$$

$\Rightarrow 1 - |\lambda|^2 = 0 \Rightarrow |\lambda|^2 = 1$ or $|\lambda| = 1$. **Proved.**

Theorem 4. Prove that the product of all eigen values of A is equal to the determinant (A).

(*U.P.T.U., 2004*)

Proof. We have $AX = \lambda IX \Rightarrow (A - \lambda I)X = 0$

For non-trivial solution $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots\ a_{1n} \\ \vdots & & \\ a_{n1} & a_{n2} & \dots a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1)^n \left\{\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \dots + b_n\right\} = 0$$

$$\Rightarrow \quad (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) = 0$$

putting $\lambda = 0$, we get

$$|A| = \lambda_1, \lambda_2, \lambda_3 \dots \lambda_n. \textbf{ Proved.}$$

Theorem 5. The latent roots of real symmetric matrix are all real.

Proof. Let A be a real symmetric matrix so that

$$\bar{A} = A,\ A' = A.$$

The $A^* = (\bar{A})' = A' = A$ or $A^* = A$, meaning thereby, A is a Hermitian matrix. Hence, the latent roots of A are all real, by Theorem 1.

Theorem 6. The characteristic roots of an idempotent matrix are either zero or unity.

Proof. Let A be idempotent matrix so that $A^2 = A$.

Let $$AX = \lambda X \qquad ...(i)$$

premultiplying by A on equation (*i*), we get

$$A(AX) = A(\lambda X) = \lambda(AX)$$

$$\Rightarrow \quad (AA)X = \lambda(\lambda X) \Rightarrow A^2X = \lambda^2 X \text{ or } AX = \lambda^2 X \qquad (\text{As } A^2 = A)$$

$$\Rightarrow \quad \lambda X = \lambda^2 X$$

$$\Rightarrow \quad (\lambda^2 - \lambda)X = 0 \Rightarrow \lambda^2 - \lambda = 0 \qquad (\text{As } X \neq 0)$$

$$\Rightarrow \quad \lambda = 0, 1. \textbf{ Proved.}$$

Example 1. Show that 0 is a characteristic root of a matrix if the matrix is singular.

(*U.P.T.U., 2008*)

Sol. The characteristic equation is

$$|A - \lambda I| = 0$$

or $$|A| - \lambda |I| = 0$$

As $$|A| = 0 \text{ (singular matrix)}$$

$$\therefore \qquad 0 = \lambda \, |I| = 0 \Rightarrow \lambda = 0$$

Conversely, if $\lambda = 0$ then $|A| = 0$. **Proved.**

Example 2. Find the characteristic equation of the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$. Also find the eigen values and eigen vectors of this matrix. (*U.P.T.U., 2006*)

Sol. Let
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Now,
$$A - \lambda I = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix}$$

$\therefore$ Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)\left\{(2-\lambda)^2 - 2\right\} - 2\{0+1\} + 2\{0+(2-\lambda)\} = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2 - 4\lambda + 2) - 2 + 4 - 2\lambda = 0$$
$$\Rightarrow \quad \lambda^2 - 4\lambda + 2 - \lambda^3 + 4\lambda^2 - 2\lambda + 2 - 2\lambda = 0$$
$$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$
$$\Rightarrow \quad \lambda^2(\lambda - 1) - 4\lambda(\lambda - 1) + 4(\lambda - 1) = 0$$
$$\Rightarrow \quad (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$$

Hence, $\lambda = 1, 2, 2$. These are eigen values of A.

Now, we consider the relation

$$AX = \lambda X \Rightarrow AX = \lambda IX$$
$$\Rightarrow \quad (A - \lambda I)X = 0 \qquad \text{...}(i)$$

Taking $\lambda = 1$, from (1), we get

$$\begin{bmatrix} 1-1 & 2 & 2 \\ 0 & 2-1 & 1 \\ -1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 - 2R_2,\ R_1 \to (-1)\ R_1$

$$\sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \qquad x_1 - 2x_2 - x_3 = 0$$

$$x_2 + x_3 = 0$$

Let $x_3 = k$ then $x_2 = -k$

and $x_1 + 2k - k = 0 \Rightarrow x_1 = -k$

$$\therefore \qquad X_1 = \begin{bmatrix} -k \\ -k \\ k \end{bmatrix} = -k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Taking $\lambda = 2$, from (1), we get

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1,\ R_1 \to -R_1$

$$\sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to -\dfrac{1}{2} R_2$ and then $R_3 \to R_3 - R_2$, we get

$$\sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Here the solution is one variable linearly independent.

$$\Rightarrow \qquad x_1 - 2x_2 - x_3 = 0$$

$$x_3 = 0$$

Let $x_2 = k$ then $x_1 - 2k - 0 = 0$

$$\Rightarrow \qquad x_1 = 2k$$

$$\text{Hence,} \qquad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus only one eigen vector X_2 corresponds to the repeated eigen value $\lambda = 2$.

Hence the eigen values are 1, 2, 2

eigen vectors $$X_1 = -k\begin{bmatrix}1\\1\\-1\end{bmatrix},\ X_2 = X_3 = k\begin{bmatrix}2\\1\\0\end{bmatrix}$$

or $$X_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix},\ X_2 = X_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$$

Example 3. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\0 & 0 & 1\end{bmatrix}$$

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \qquad \begin{vmatrix}2-\lambda & 1 & 1\\1 & 2-\lambda & 1\\0 & 0 & 1-\lambda\end{vmatrix} = (1-\lambda)(\lambda-1)(\lambda-3) = 0$$

Thus $\lambda = 1, 1, 3$ are the eigen values of A.

Now, we consider the relation $(A - \lambda I)X = 0$

For $\lambda = 3$,

$$\begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \qquad \Bigg|\ \text{let } X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$, then $R_1 \to -R_1$ on coefficient matrix, we get

$$\sim \begin{bmatrix}1 & -1 & -1\\0 & 0 & 2\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

Again $R_3 \to R_3 + R_2$, $R_2 \to \frac{1}{2}R_2$

$$\sim \begin{bmatrix}1 & -1 & -1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

$$\Rightarrow \qquad x_1 - x_2 - x_3 = 0$$

$$x_3 = 0 \left| \begin{matrix} r = 2, n = 3 \\ n - r = 1 \end{matrix} \right.$$

Suppose $x_2 = k$, then $x_1 = k$

Here, $$X_1 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Here $r = 1$, $n = 3 \Rightarrow n - r = 3 - 1 = 2$.

$\therefore$ There is two variables linearly independent solution

Let $x_2 = k_1$, $x_3 = k_2$ and $x_1 + x_2 + x_3 = 0$

$\Rightarrow$ $x_1 + k_1 + k_2 = 0 \Rightarrow x_1 = -(k_1 + k_2)$

$\Rightarrow$ $$X_2 = \begin{bmatrix} -(k_1 + k_2) \\ k_1 \\ k_2 \end{bmatrix}.$$

Since the vectors are linearly independent so, we choose k_1 and k_2 as follows:

(*a*) If we suppose $x_2 = k_1 = 0$, $x_3 = k_2$ (any arbitrary)

Then $$X_2 = \begin{bmatrix} -k_2 \\ 0 \\ k_2 \end{bmatrix} = -k_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(*b*) If we suppose $x_2 = k_1$ (any arbitrary) and $x_3 = k_2 = 0$

Then $$X_3 = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} = -k_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Hence the eigen values are $\lambda = 1, 2, 2$

Eigen vectors are $$X_1 = k\begin{bmatrix}1\\1\\0\end{bmatrix};\ X_2 = -k_2\begin{bmatrix}1\\0\\-1\end{bmatrix};\ X_3 = -k_1\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

or $$X_1 = \begin{bmatrix}1\\1\\0\end{bmatrix};\ X_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix};\ X_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

Example 4. Find the eigen values and eigen-vectors of matrix

$$A = \begin{bmatrix}3 & 1 & 4\\0 & 2 & 6\\0 & 0 & 5\end{bmatrix}.$$ [*U.P.T.U., 2004 (C.O.), 2002*]

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix}3-\lambda & 1 & 4\\0 & 2-\lambda & 6\\0 & 0 & 5-\lambda\end{vmatrix} = (3-\lambda)\{(2-\lambda)(5-\lambda)\} - 0 + 0 = 0$$

$$\therefore \quad \lambda = 2, 3, 5$$

Now, we consider the relation $(A - \lambda I)X = 0$ ...(*i*)

For $\lambda = 2$,

$$\begin{bmatrix}3-2 & 1 & 4\\0 & 2-2 & 6\\0 & 0 & 5-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \qquad \Bigg| X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1 & 1 & 4\\0 & 0 & 6\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

$R_3 \to R_3 - \dfrac{1}{2}R_2$ on coefficient matrix

$$\sim \begin{bmatrix}1 & 1 & 4\\0 & 0 & 6\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + x_2 + 4x_3 = 0 \quad \text{and } x_3 = 0$$

Let $x_2 = k$ then $x_1 + k + 0 = 0 \Rightarrow x_1 = -k$

$$\therefore \quad X_1 = \begin{bmatrix}-k\\k\\0\end{bmatrix} = -k\begin{bmatrix}1\\-1\\0\end{bmatrix} \text{ or } \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

For $\lambda = 3$, from (*i*), we get

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1$ on coefficient matrix

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 - \frac{1}{5}R_2$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \quad x_2 + 4x_3 = 0$ and $10x_3 = 0 \Rightarrow x_3 = 0$

$\Rightarrow \quad x_2 + 0 = 0 \Rightarrow x_2 = 0$, let $x_1 = K$

$$\therefore \quad X_2 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Again, for $\lambda = 5$, we get

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x_1 - x_2 - 4x_3 = 0$

$3x_2 - 6x_3 = 0$

Let $\quad x_3 = k$, then $3x_2 - 6k = 0$

$\Rightarrow \quad x_2 = 2k$ and $2x_1 - 2k - 4k = 0 \Rightarrow x_1 = 3k$

$$\therefore \quad X_3 = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Hence the eigen values are $\lambda = 2, 3, 5$

And eigen vectors are $\quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$

Example 5. Determine the latent roots and the corresponding vector of the matrix,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 + R_3$, we get

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

Now, $C_3 \to C_3 + C_2$ gives

$$(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\{(6-\lambda)(2-\lambda+2)-8\} = 0$$

$$\therefore \quad (2-\lambda)(\lambda^2 - 10\lambda + 16) = 0 \Rightarrow (2-\lambda)(2-\lambda)(\lambda-8) = 0$$

$$\Rightarrow \quad \lambda = 2, 2, 8$$

Now, consider the relation

$$(A - \lambda I)X = 0 \qquad ...(i)$$

For $\lambda = 8$, the equation (*i*), gives

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$ on coefficient matrix

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ and $R_1 \to \dfrac{-1}{2}R_1$, $R_2 \to \dfrac{-1}{3}R_2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 - x_3 = 0$$

$$x_2 + x_3 = 0$$

Let $x_3 = k$, then $x_2 = -k$ and $x_1 = 2k$

$$\therefore \qquad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$, the equation (*i*) gives

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{1}{2}R_1$, $R_3 \to R_3 - \frac{1}{2}R_1$, we get

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $r = 1$, $n = 3 \Rightarrow n - r = 3 - 1 = 2$ (Two linearly independent)

Let $\qquad x_2 = k_1$ and $x_3 = k_2$

and we have $\qquad 4x_1 - 2x_2 + 2x_3 = 0$

$$\Rightarrow \qquad 4x_1 - 2k_1 + 2k_2 = 0$$

$$\Rightarrow \qquad x_1 = \frac{1}{2}(k_1 - k_2)$$

$$\therefore \qquad X_2 = \begin{bmatrix} k_1 \\ k_2 \\ \frac{1}{2}(k_1 - k_2) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2k_1 \\ 2k_2 \\ (k_1 - k_2) \end{bmatrix}$$

Since the vectors are linearly independent so, we choose, k_1 and k_2 as follows:

(*i*) Let $x_2 = k_1 = 0$ and $x_3 = k_2$ (arbitrary)

then $$X_2 = \frac{1}{2}\begin{bmatrix} 0 \\ 2k_2 \\ -k_2 \end{bmatrix} = \frac{k_2}{2}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

(*ii*) Let $x_3 = k_2 = 0$ and $x_2 = k_1$ (arbitrary)

then $$X_3 = \frac{1}{2}\begin{bmatrix} 2k_1 \\ 0 \\ k_1 \end{bmatrix} = \frac{k_1}{2}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence $\lambda = 8, 2, 2$

$$X_1 = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \; X_2 = \frac{k_2}{2}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \; X_3 = \frac{k_1}{2}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Example 6. Find the eigen values of matrix A.

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

Sol. The characteristic equation of B is, where

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0$$

or $$(\lambda - 3)^2 (\lambda + 3) = 0 \Rightarrow \lambda = 3, 3, -3$$

So the eigen values of $A = \frac{1}{3}B = 1, 1, -1.$

Example 7. Show that the matrix

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

has less than three independent eigen vectors. Is it possible to obtain a similarity transformation that will diagonalise this matrix. (*U.P.T.U., 2001*)

Sol. The characteristic equation is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

or $$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

$\Rightarrow$ $$(\lambda - 2)^2 (\lambda - 3) = 0 \Rightarrow \lambda = 2, 2, 3$$

Now, consider the relation

$$(A - \lambda I)X = 0 \qquad \text{...}(i)$$

For $\lambda = 3$, we have

$$\begin{bmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \to \frac{1}{5}R_1,\ R_2 \to -\frac{1}{2}R_2$, we have

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 - 3R_1$ on coefficient matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$
$$x_1 + 3x_2 + 2x_3 = 0$$
$$0x_1 + 2x_2 + x_3 = 0$$

Solving these equations by cross-multiplication

$$\frac{x_1}{3-4} = -\frac{x_2}{1-0} = \frac{x_3}{2-0}$$

$\Rightarrow$
$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$\Rightarrow$
$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ The proportional values are

$$x_1 = 1,\ x_2 = 1,\ x_3 = -2$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

For $\lambda = 2$, we have, from equation (*i*)

$$\begin{bmatrix} 3-2 & 10 & 5 \\ -2 & -3-2 & -4 \\ 3 & 5 & 7-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Applying $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$ on coefficient matrix

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to \dfrac{R_2}{3}$, $R_3 \to -\dfrac{1}{5}R_3$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

This shows that the solution is one linearly independent

$$\Rightarrow \qquad x_1 + 10x_2 + 5x_3 = 0$$

$$5x_2 + 2x_3 = 0$$

$$\frac{x_1}{-5} = -\frac{x_2}{2} = \frac{x_3}{5}$$

or

$$\frac{x_1}{5} = \frac{x_2}{2} = -\frac{x_3}{5}$$

Taking proportional values, $x_1 = 5$, $x_2 = 2$, $x_3 = -5$

$$\therefore \qquad X_2 = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$

And the third eigen vector corresponding to repeated root $\lambda_3 = 2 = \lambda_2$ will be same because the solution is not two linearly independent. Hence the vectors X_2 and X_3 are not linearly independent. Similarity transformation is not possible.

Therefore $\lambda = 2, 2, 3$ and $X_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. $X_2 = X_3 = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$.

Example 8. Verify the statement that the sum of the elements in the diagonal of a matrix is the sum of the eigen values of the matrix.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \qquad \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(-2-\lambda)[(1-\lambda)(-\lambda)-12]-2(-2\lambda-6)-3[-4+(1-\lambda)] = 0$

$\Rightarrow$ $(-2-\lambda)(\lambda^2-\lambda-12)+4(\lambda+3)+3(\lambda+3) = 0$

$\Rightarrow$ $-2\lambda^2+2\lambda+24-\lambda^3+\lambda^2+12\lambda+4\lambda+12+3\lambda+9 = 0$

$\Rightarrow$ $-\lambda^3-\lambda^2+21\lambda+45 = 0$

$\Rightarrow$ $\lambda^3+\lambda^2-21\lambda-45 = 0.$

This is cubic equation in λ and hence it has 3 roots *i.e.,* there are three eigen values.

The sum of the eigen values $= -\left(\dfrac{\text{coefficient of } \lambda^2}{\text{coefficient of } \lambda^3}\right) = -1.$

The sum of the elements on the diagonal of the matrix A

$= -2+1+0 = -1$. Hence the result.

Example 9. Find the eigen values and corresponding eigen vectors of the matrix,

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$ (*U.P.T.U., 2008*)

Sol. The characteristic equation is $|A-\lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (5+\lambda)(2+\lambda)-4 = 0$$

$$\Rightarrow \quad \lambda^2+7\lambda+6 = 0 \quad \Rightarrow \quad (\lambda+6)(\lambda+1) = 0$$

$$\therefore \quad \lambda = -1, -6$$

Now, we consider the relation $(A-\lambda I)X = 0$ ...(*i*)

For $\lambda = -1$

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 + 2R_1$, on coefficient matrix

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad -2x_1 + x_2 = 0$$ ...(*ii*)

Let $x_2 = k$

$\therefore$ From (*i*) $x_1 = \dfrac{k}{2}$

$$X_1 = \begin{bmatrix} k/2 \\ k \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = -6$, from (*i*), we get

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_1$, on coefficient matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + 2x_2 = 0 \quad \text{...}(iii)$$

Let $x_2 = k \quad \therefore \quad x_1 = -2k$

$$X_2 = \begin{bmatrix} -2k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence, the eigen values are $\lambda = -1, -6$

And eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

EXERCISE 3.6

Find the eigen values and eigen vectors of:

1. $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. $\left[\textbf{Ans. } 2, -1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$

2. $\begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}$. $\left[\textbf{Ans. } 10, -10, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$

3. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } -2, 3, 6, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]$

4. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. $\left[\textbf{Ans. } 0, 3, 15, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right]$

5. $\begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$. $\left[\textbf{Ans. } -1, 1, 2, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right]$

6. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. $\left[\textbf{Ans. } 2, 2, 2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$

7. $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. $\left[\textbf{Ans. } -1, \pm i, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1+i \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1-i \\ 1 \\ 1 \end{bmatrix}\right]$

8. $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. $\left[\textbf{Ans. } 2, 3, 5, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right]$

9. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. $\left[\textbf{Ans. } 5, -3, -3, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\right]$

10. $\begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$. $\left[\textbf{Ans. } 5, 2, 2, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}\right]$

11. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. $\left[\textbf{Ans. } -2, 3, 6, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]$

12. $\begin{bmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{bmatrix}$. $\left[\textbf{Ans. } 0, -1, 2, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right]$

13. $\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}$. $\left[\textbf{Ans. } 0, 3, -3, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\right]$

14. $\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$. $\left[\textbf{Ans. } 0, 1, 2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right]$

15. $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. $\left[\textbf{Ans. } -1, -1, 3, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right]$

16. $\begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$. $\left[\textbf{Ans. } -2, 9, -18, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right]$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. $\left[\textbf{Ans. } 0, 0, 14, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right]$

18. $\begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$. $\left[\textbf{Ans. } \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = 0,\ \lambda = 2, 1, 1, 1 \quad \begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -4 \\ -5 \end{bmatrix}\right]$

19. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$. $\left[\textbf{Ans.}\ 2, 2, -2, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}\right]$

20. Find the eigen values of A^5 when

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}.$$ $\left[\textbf{Ans.}\ 3^5, 4^5, 1^5\right]$

21. If A and B are $n \times n$ matrices and B is a non-singular matrix prove that A and $B^{-1}AB$ have same eigen values.

[**Hint :** Characteristic polynomial of $B^{-1}AB = |B^{-1}AB - \lambda I| = |B^{-1}AB - B^{-1}(\lambda I)B|$
$= |B^{-1}(A - \lambda I)B| = |B^{-1}|\,|A - \lambda I|\,|B| = |B^{-1}|\,|B|\,|A - \lambda I| = |B^{-1}B|\,|A - \lambda I|$
$= |I|\,|A - \lambda I| = |A - \lambda I|$ = characteristic polynomial of A].

22. Find the sum and product of the eigen values.

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$ $\left[\textbf{Ans.}\ \text{Sum} = 2 + 1 + 2 = 5, \text{product} = |A| = 21\right]$

3.18 CAYLEY-HAMILTON THEOREM

STATEMENT

Any square matrix A satisfies its own characteristic equation. *(U.P.T.U., 2005)*

i.e., if $C_0 + C_1\lambda + C_2\lambda^2 + ... + C_n\lambda^n$ is the characteristic polynomial of degree n in λ then $C_0 I + C_1 A + C_2 A^2 + ... + C_n A^n = 0$.

Proof: Let A be a square matrix of order n.

Let $|A - \lambda I| = C_0 + C_1\lambda + C_2\lambda^2 + ... + C_n\lambda^n$ ...(*i*)

be the characteristic polynomial of A.

Now, the elements of *adj* $(A - \lambda I)$ are cofactors of the matrix $(A - \lambda I)$ and are polynomials in λ of degree not exceeding $n - 1$.

Thus *adj* $(A - \lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + ... + B_{n-1}\lambda^{n-1}$ ...(*ii*)

where B_i are n-square matrices whose elements are the functions of the elements of A and independent of λ.

We know that $(A - \lambda I)$. *adj* $(A - \lambda I) = |A - \lambda I|\, I$

From (*i*) and (*ii*), we have

$(A - \lambda I)(B_0 + B_1\lambda + B_2\lambda^2 + ... + B_{n-1}\lambda^{n-1}) = (C_0 + C_1\lambda + C_2\lambda^2 + ... + C_n\lambda^n)I$ ...(*iii*)

Equating the like powers of λ on both sides of (*iii*), we get

$$AB_0 = C_0 I$$
$$AB_1 - B_0 = C_1 I$$

$$\begin{aligned} \vdots \quad & \vdots \quad \vdots \\ AB_{n-1} - B_{n-2} &= C_{n-i} I \\ -B_{n-1} &= C_n I \end{aligned}$$

Pre-multiplying the above equations by $I, A, A^2, \ldots A^n$ respectively and adding; we get

$$C_0 I + C_1 A + C_2 A^2 + \ldots + C_n A^n = 0 \qquad \ldots(iv)$$

Since all the terms on the L.H.S. cancel each other. Thus A satisfies its own characteristic equation. **Proved.**

3.18.1 Inverse by Cayley-Hamilton Theorem

We have $C_0 I + C_1 A + C_2 A^2 + \ldots + C_n A^n = 0$ ...(*i*)

Multiplying equation (*i*) by A^{-1}, we get

$$C_0 A^{-1} + C_1 I + C_2 A + \ldots + C_n A^{n-1} = 0$$

$$\Rightarrow \quad C_0 A^{-1} = -[C_1 I + C_2 A + \ldots + C_n A^{n-1}]$$

$$\Rightarrow \quad A^{-1} = -\frac{1}{C_0}[C_1 I + C_2 A + \ldots + C_n A^{n-1}]$$

Note: A^{-1} exists only if $C_0 = |A|$.

Definition. An expression of the form $C_0 + C_1\lambda + C_2\lambda^2 +, + C_n\lambda^n$ where $C_0, C_1, \ldots, C_n$ are square matrices of the same order and $C_n \neq 0$ is called a "matrix polynomial of degrees n".

Example 1. Verify the Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Also find its inverse using this theorem. (*U.P.T.U., 2006*)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)[(4-\lambda)(6-\lambda) - 25] - 2[2(6-\lambda) - 15] + 3[10 - 3(4-\lambda)] = 0$$

$$\Rightarrow \quad \lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$$

Cayley-Hamilton theorem is verified if A satisfies the above characteristic equation *i.e.*

$$A^3 - 11A^2 - 4A + 1 = 0 \qquad \ldots(i)$$

Now,
$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

Verification, $$A^3 - 11A^2 - 4A + I = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A^3 - 11A^2 - 4A + 1 = 0$$

Hence theorem verified.

To find A^{-1} : Multiplying equation by A^{-1}, we get

$$A^2 - 11A - 4I + A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = -A^2 + 11A + 4I$$

$$\Rightarrow \quad A^{-1} = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

Example 2. Express $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$ as a quadratic polynomial in A, where A is square matrix given in Example 1. Find B as well as A^4.

Sol. $$B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$$
$$= A^5 (A^3 - 11A^2 - 4A + I) + A (A^3 - 11A^2 - 4A + I) + A^2 + A + I$$
$$= A^5 (0) + A(0) + A^2 + A + I \quad | \text{As } A^3 - 11A^2 - 4A + I = 0 \text{ in Example 1}$$
$$= A^2 + A + I$$

Thus, $$B = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 16 & 27 & 34 \\ 27 & 50 & 61 \\ 34 & 61 & 77 \end{bmatrix}.$$

To find A^4 : We have

$$A^3 - 11A^2 - 4A + I = 0$$

$$\Rightarrow \quad A^3 = 11A^2 + 4A - I$$

Multiplying both sides by A

$$A^4 = 11A^3 + 4A^2 - IA$$

$$A^4 = 11\begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} + 4\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1782 & 3211 & 4004 \\ 3211 & 5786 & 7215 \\ 4004 & 7215 & 8997 \end{bmatrix}.$$

Example 3. Find the characteristic equation of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$ and hence also find A^{-1} by Cayley-Hamilton theorem. (*U.P.T.U., 2003, 2004, 2005*)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda)[\lambda^2 - 4\lambda + 4 - 2] + [-2 + \lambda + 1] + [2 - 2 + \lambda] = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

By Cayley-Hamilton theorem $A^3 - 6A^2 + 8A - 3I = 0.$

Multiplying by A^{-1} on both sides, we get

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{3}(A^2 - 6A + 8I)$$

Now,
$$A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 6 \\ -5 & 7 & -5 \\ 6 & -9 & 7 \end{bmatrix}$$

$\therefore$
$$A^2 - 6A + 8I = \begin{bmatrix} 6 & -6 & 6 \\ -5 & 7 & -5 \\ 6 & -9 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

Thus,
$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix}.$$

Example 4. Evaluate A^{-1}, A^{-2}, A^{-3} if

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

Sol. Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

By Cayley-Hamilton theorem, we have

$$A^3 - 4A^2 - A + 4I = 0$$

$\Rightarrow \quad A^2 - 4A - I + 4A^{-1} = 0$ (multiplying by A^{-1})

$$A^{-1} = \frac{1}{4}[-A^2 + 4A + I] \qquad ...(i)$$

Now, $$A^2 = \begin{bmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{bmatrix}$$

$\therefore$ $$-A^2 + 4A + I = \begin{bmatrix} -16 & -18 & -18 \\ -5 & -7 & -6 \\ 5 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 16 & 24 & 24 \\ 4 & 12 & 8 \\ -4 & -16 & -12 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

Thus, $$A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

Multiplying equation (1) by A^{-1}, we get

$$A^{-2} = \frac{1}{4}\left[-A + 4I + IA^{-1}\right] = \frac{1}{4}\begin{bmatrix} 1/4 & -1/2 & -9/2 \\ -5/4 & 5/2 & -3/2 \\ 5/4 & 3/2 & 11/2 \end{bmatrix}$$

Similarly, $$A^{-3} = \frac{1}{4}\ [-I + 4A^{-1} + A^{-2}]$$

$$= \frac{1}{64}\begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}.$$

Example 5. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and hence find A^{-1}. (*U.P.T.U., 2008*)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow -(1 - \lambda^2) - 4 = 0$$

or $$\lambda^2 - 5 = 0$$

By Cayley-Hamilton theorem $A^2 - 5 = 0$ ...(*i*)

Now $$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I$$

Putting value of A^2 in equation (*i*), we get

$$A^2 - 5 = 5I - 5I = 0$$

Hence, the Cayley-Hamilton theorem is verified.

To find A^{-1} : Multiplying equation (*i*) by A^{-1}, we get

$$A - 5A^{-1} = 0 \quad \Rightarrow \quad A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Example 6. Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies its characteristic equation. Also find A^{-1}.

Sol.
$$|A - \lambda I| = \begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$$

i.e., $\quad -\lambda[\lambda^2 + a^2] - c[\lambda c - ab] - b[ac + b\lambda] = 0$

i.e., $\quad \lambda^3 + \lambda(a^2 + b^2 + c^2) = 0$

By Cayley-Hamilton theorem, $A^3 + A\,(a^2 + b^2 + c^2) = 0$ ...(*i*)

Now,
$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -(c^2+b^2) & ab & ac \\ ab & -(c^2+a^2) & bc \\ ac & bc & -(b^2+a^2) \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} -(b^2+c^2) & ab & ac \\ ab & -(c^2+a^2) & bc \\ ac & bc & -(b^2+a^2) \end{bmatrix}$$

or
$$A^3 = \begin{bmatrix} 0 & -c\left(a^2+b^2+c^2\right) & b\left(a^2+b^2+c^2\right) \\ c\left(a^2+b^2+c^2\right) & 0 & -a\left(a^2+b^2+c^2\right) \\ -b\left(a^2+b^2+c^2\right) & a\left(a^2+b^2+c^2\right) & 0 \end{bmatrix} = -\left(a^2+b^2+c^2\right)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

or $\quad A^3 = -(a^2 + b^2 + c^2)\,A$ ...(*ii*)

Using (*ii*) in equation (*i*), we get

$A^3 + A\,(a^2 + b^2 + c^2) = -(a^2 + b^2 + c^2)\,A + A\,(a^2 + b^2 + c^2) = 0$

To find A^{-1}: Multiplying equation (*i*) A^{-2} on both sides, we get

$A + A^{-1}\,(a^2 + b^2 + c^2) = 0$

$$\Rightarrow \quad A^{-1} = -\frac{1}{\left(a^2+b^2+c^2\right)} \cdot A$$

Hence,
$$A^{-1} = -\frac{1}{\left(a^2+b^2+c^2\right)}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}.$$

EXERCISE 3.7

1. Find the characteristic equation of the matrix A

$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$. Hence find A^{-1}. (*U.P.T.U., 2001*)

$$\left[\textbf{Ans.}\ \text{Characteristic equation : } \lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0, A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}\right]$$

2. Find the characteristic equation of the matrix

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Verify Cayley-Hamilton theorem and hence (*U.P.T.U., 2002*)

evaluate the matrix equation $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

$$\left[\textbf{Ans.}\ \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0, A^2 + A + I, \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}\right]$$

3. Verify Cayley-Hamilton theorem for A and hence find A^{-1}

(*i*) $\begin{bmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ $\left[\textbf{Ans.}\ A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix}\right]$

(*ii*) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\left[\textbf{Ans.}\ A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\right]$

(*iii*) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ $\left[\textbf{Ans.}\ A^{-1} = -\frac{1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}\right]$

4. Find A^{-1} if $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$. $\left[\textbf{Ans.}\ A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}\right]$

5. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence

or otherwise obtain the value of A^{-1}. $\left[\textbf{Ans.}\ A^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$

6. Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$.

$$\left[\textbf{Ans. } A^{-1} = \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}\right]$$

7. If $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ find A^{-1}.

$$\left[\textbf{Ans. } A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right]$$

8. Find $B = A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ if $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

$$\left[\textbf{Ans. } \lambda^2 - 4\lambda + 5 = 0, B = 5I - 4A = \begin{bmatrix} 1 & -8 \\ 4 & -7 \end{bmatrix}\right]$$

9. Find A^{-1} if $A = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

$$\left[\textbf{Ans. } A^{-1} = \frac{1}{11}\begin{bmatrix} -1 & 7 & -24 \\ 2 & -3 & 4 \\ 2 & -3 & 15 \end{bmatrix}\right]$$

10. Show that the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ satisfies the equation $A(A - I)(A + 2I) = 0$.

3.19 DIAGONALIZATION OF A MATRIX

A square matrix 'A' is said to be diagonalisable if there exists another non-singular square matrix P such that $P^{-1}AP$ is a diagonal matrix. In other words, A is diagonalisable if A is similar to a diagonal matrix.

Similar matrix: Two matrices A and B are said to be similar if there exists a non-singular matrix P such that $B = P^{-1}AP$.

This transformation of A to B is known as similarity transformation.

Note: Similar matrices A and B have same eigen values. Further, if X is an eigen vector of A then $Y = P^{-1}X$ is an eigen vector of the matrix B.

3.19.1 Working Rule

The matrix P is called a model matrix for A. There are following steps to obtain diagonal form of matrix A.

Step 1: First of all obtain eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ of the matrix A.

Step 2: If there exists a pair of distinct complex eigen values then A is not diagonalisable over the field R of real numbers.

Step 3: If there does not exist a set of n linearly independent eigen vectors, then A is not diagonalisable.

Step 4: If there exist n linearly independent eigen vectors.

$$X_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix}, X_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} ...., X_n = \begin{bmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nn} \end{bmatrix}$$

then let,
$$P = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \vdots & \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}$$

and diagonal matrix $D = P^{-1} AP$.

3.19.2 Powers of Matrix *A*

Consider
$$D = P^{-1} AP$$

$\Rightarrow$
$$D^2 = (P^{-1} AP) (P^{-1} AP) = P^{-1} A (PP^{-1}) AP$$
$$= P^{-1} A. IAP = P^{-1} AAP = P^{-1} A^2 P$$

Similarly,
$$D^3 = P^{-1} A^3 P, ..., D^n = P^{-1} A^n P$$

We can obtain A^n pre-multiplying by P and post-multiplying by P^{-1}

$\Rightarrow$
$$P D^n P^{-1} = P (P^{-1} A^n P) P^{-1} = (PP^{-1}) A^n (PP^{-1})$$
$$= IA^nI = A^n$$

$\therefore$
$$\boxed{A^n = PD^n P^{-1}}$$

Example 1. Diagonalize the matrix $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. (*U.P.T.U., 2006*)

Sol. The characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (\lambda + 1) (\lambda - 3) (\lambda - 4) = 0 \Rightarrow \lambda = -1, 3, 4$

Consider the equation

$$(A - \lambda I) X = 0$$

For, $\lambda = -1$,

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_2$ on coefficient matrix, we get

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \to R_1 - R_2,\ R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $$x_1 + 3x_2 - x_3 = 0$$
$$x_3 = 0$$

Let, $x_2 = k$, then $x_1 = -3k$

$\therefore$ $$X_1 = \begin{bmatrix} -3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

For, $\lambda = 3$

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $$-2x_1 + 6x_2 + x_3 = 0$$
$$x_1 - x_2 + 0 \cdot x_3 = 0$$

$$\frac{x_1}{1} = \frac{-x_2}{-1} = \frac{x_3}{-4} \text{ or } \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-4}$$

$\Rightarrow$ $x_1 = 1,\ x_2 = 1,\ x_3 = -4$

$\therefore$ $$X_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

For, $\lambda = 4$

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 6 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + 3R_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \to R_1 + R_2,\ R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$
$$x_1 - 2x_2 + x_3 = 0$$
$$0x_1 + 0x_2 + x_3 = 0$$

$\Rightarrow$
$$\frac{x_1}{-2} = -\frac{x_2}{1} = \frac{x_3}{0}$$

or
$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{0} \Rightarrow x_1 = 2,\ x_2 = 1,\ x_3 = 0$$

$\therefore$
$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus,
$$P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

To obtain P^{-1}: $|P| = -3\,(4) - 1\,(0) + 2\,(-4) = -20$

matrix of cofactors
$$B = \begin{bmatrix} 4 & 0 & -4 \\ -8 & 0 & -12 \\ -1 & 5 & -4 \end{bmatrix}$$

$\therefore$
$$adj\ P = B' = \begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}$$

and
$$P^{-1} = \frac{adj\,P}{|P|} = -\frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

Diagonalisation:

$$D = P^{-1} AP$$

$$D = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -15 \\ 16 & 48 & 16 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -20 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 80 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Example 2. Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form.

(*U.P.T.U., 2001, 2004*)

Sol. $$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

On solving, we get $\lambda = 1, +\sqrt{5}, -\sqrt{5}$

For, $\lambda = 1$ $\quad (A - \lambda I)\, X = 0$

$$\Rightarrow \quad \begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$

$$0x_1 + 4x_2 + 0x_3 = 0$$

$$\frac{x_1}{-4} = -\frac{x_2}{0} = \frac{x_3}{4} \text{ or } \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\Rightarrow \quad x_1 = 1,\ x_2 = 0,\ x_3 = -1$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\quad \lambda = \sqrt{5}$

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2-\sqrt{5} & 1 \\ -1-\sqrt{5} & 2 & -2 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + (1 + \sqrt{5})\, R_1,\ R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2-\sqrt{5} & 1 \\ 0 & \sqrt{5}-1 & \sqrt{5}-1 \\ 0 & 1-\sqrt{5} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + R_2$, and $R_2 \to \dfrac{1}{\sqrt{5}-1}\, R_2$

$$\begin{bmatrix} 1 & 2-\sqrt{5} & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + (2 - \sqrt{5})\, x_2 + x_3 = 0$$

$$0x_1 + x_2 + x_3 = 0$$

Solving by cross-multiplication

$$\frac{x_1}{1-\sqrt{5}} = -\frac{x_2}{1} = \frac{x_3}{1}$$

or

$$\frac{x_1}{\sqrt{5}-1} = \frac{x_2}{1} = \frac{x_3}{-1} \Rightarrow x_1 = \sqrt{5}-1,\ x_2 = 1,\ x_3 = -1$$

$$\therefore \quad X_2 = \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$$

Similarly, the eigen vector corresponding $\lambda = -\sqrt{5}$

$$X_3 = \begin{bmatrix} \sqrt{5}+1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ The modal matrix

$$P = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

To obtain P^{-1}:

$$\begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 0 & \sqrt{5}-2 & \sqrt{5}+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} P$$

$R_1 \to R_1 - \left(\sqrt{5}-1\right) R_2$

$$\begin{bmatrix} 1 & 0 & 2\sqrt{5} \\ 0 & 1 & -1 \\ 0 & \sqrt{5}-2 & \sqrt{5}+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-\sqrt{5} & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} P$$

$R_3 \to R_3 - \left(\sqrt{5}-2\right) R_2$

$$\begin{bmatrix} 1 & 0 & 2\sqrt{5} \\ 0 & 1 & -1 \\ 0 & 0 & 2\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 1-\sqrt{5} & 0 \\ 0 & 1 & 0 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix} P$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix} P$$

$R_3 \to \dfrac{1}{2\sqrt{5}} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ \dfrac{1}{2\sqrt{5}} & \dfrac{2-\sqrt{5}}{2\sqrt{5}} & \dfrac{1}{2\sqrt{5}} \end{bmatrix} P$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -1 \\ \dfrac{1}{2\sqrt{5}} & \dfrac{2+\sqrt{5}}{2\sqrt{5}} & \dfrac{1}{2\sqrt{5}} \\ \dfrac{1}{2\sqrt{5}} & \dfrac{2-\sqrt{5}}{2\sqrt{5}} & \dfrac{1}{2\sqrt{5}} \end{bmatrix} P$$

$\therefore$ $$P^{-1} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}$$

Now, $$D = P^{-1}AP = \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}.$$

Example 3. Diagonalise the matrix $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ and hence find A^5.

Sol. The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \qquad \begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{bmatrix} = 0$$

On simplification, we get

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$$\Rightarrow \qquad \lambda(\lambda - 1)(\lambda - 2) = 0$$

$$\therefore \qquad \lambda = 0, 1, 2$$

The eigen values are real and distinct and hence A is diagonalisable.

Now, consider the relation $(A - \lambda I) X = 0$

For $\lambda = 0$

$$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \qquad 11x_1 - 4x_2 - 7x_3 = 0 \qquad | \text{ Take any two equations.}$$

$$7x_1 - 2x_2 - 5x_3 = 0$$

$$10x_1 - 4x_2 - 6x_3 = 0$$

Taking the first-two equations, we get

$$\frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{6} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$, we have

$$\begin{bmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \qquad 10x_1 - 4x_2 - 7x_3 = 0$$

$$7x_1 - 3x_2 - 5x_3 = 0$$

$$10x_1 - 4x_2 - 7x_3 = 0$$

Taking the first-two equations, we get

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$$\therefore \qquad X_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

Similarly,

$$\text{For} \qquad \lambda = 2,\ X_3 = \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix}$$

The three vectors are linearly independent,

$$\text{Hence the modal matrix} \qquad P = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

$$\therefore \qquad P^{-1} = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\text{Now,} \qquad D = P^{-1}AP = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\text{Next} \qquad A^5 = PD^5P^{-1}$$

$$\text{But,} \qquad D^5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & 2^5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

$$\therefore \qquad A^5 = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 191 & -64 & -127 \\ 97 & -32 & -65 \\ 190 & -64 & -126 \end{bmatrix}.$$

Example 4. Prove that the matrix $A = \begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$ is not diagonalisable.

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \qquad \begin{vmatrix} -\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \qquad \lambda^3 - 3\lambda^2 + 4 = 0 \Rightarrow (\lambda + 1)(\lambda - 2)^2 = 0$

$\therefore \qquad \lambda = -1, 2, 2.$ Here the eigen values are real.

Let X be an eigen vector corresponding to $\lambda = 2$

$\therefore \qquad (A - 2I)\, X = 0$

$$\Rightarrow \quad \begin{bmatrix} -2 & -2 & -2 \\ -1 & -1 & 2 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \to \frac{-1}{2} R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1,\ R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to \frac{1}{3} R_2$ and $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \qquad r = 2, n = 3 \Rightarrow n - r = 1$

there exist only one linearly independent eigen vector corresponding $\lambda = 2$.

Hence, there does not exist three linearly independent eigen vectors. Thus A is not diagonalisable.

Example 5. Find a matrix P which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, verify $P^{-1} AP = D$ where D is the diagonal matrix. (*U.P.T.U., 2008*)

Sol. The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (4 - \lambda)(3 - \lambda) - 2 = 0$$

$\Rightarrow \qquad \lambda^2 - 7\lambda + 10 = 0$

$\Rightarrow \qquad (\lambda - 2)(\lambda - 5) = 0 \quad \therefore \quad \lambda = 2, 5$

Now, consider the relation $(A - \lambda I)\, X = 0$

For $\lambda = 2$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,\ R_2 \to R_2 - 2R_1 \quad \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow \qquad 2x_1 + x_2 = 0,$ Here let $x_2 = k,$ so $x_1 = -\frac{k}{2}$

$$\therefore \qquad X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-k}{2} \\ k \end{bmatrix} = -\frac{k}{2}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

For $\lambda = 5$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,\ R_2 \to R_2 + 2R_1 \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow -x_1 + x_2 = 0$. Here again let $x_2 = k$ so $x_1 = k$

$$\therefore \qquad X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The modal matrix $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

Verification: $\quad |P| = 1 + 2 = 3$

$$\therefore \qquad P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -4 & 5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D \text{ (diagonal matrix).}$$

Thus $P^{-1}AP = D$ verified.

3.20 APPLICATION OF MATRICES TO ENGINEERING PROBLEMS

Matrices have various engineering applications, they can be used to characterize connections in electrical networks, in nets of roads, in production processes, etc., as follows:

3.20.1 Nodal Incidence Matrix

The network in figure consists of 6 branches (connections) and 4 nodes (point where two or more branches together). One node is the reference node (grounded node, whose voltage is zero). We number the other nodes and number and direct the branches. This we do arbitrarily. The network can now we described by a matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{bmatrix} +1 \text{ if branch } j \text{ leaves node } (i) \\ -1 \text{ if branch } j \text{ enters node } (i) \\ 0 \text{ if branch } j \text{ does not touch node } (i) \end{bmatrix}$$

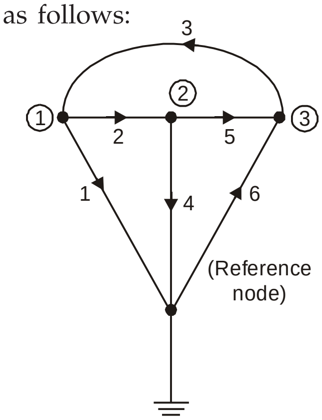


Fig. 3.2

A is called the nodal incidence matrix of the network. The matrix for the network in figure is given below:

Branch	1	2	3	4	5	6
Node 1	1	–1	–1	0	0	0
Node 2	0	1	0	1	1	0
Node 3	0	0	1	0	–1	–1

3.20.2 Mesh Incidence Matrix

A network can also be characterized by the mesh incidence matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} +1 \text{ if branch } j \text{ is in mesh } \boxed{i} \text{ and has the same orientation} \\ -1 \text{ if branch } j \text{ is in mesh } \boxed{i} \text{ and has the opposite orientation} \\ 0 \text{ if branch } j \text{ is not in mesh } \boxed{i} \end{cases}$$

And a mesh is a loop with no branch in its interior (or in its exterior). Here, the meshes are numbered and directed (oriented) in an arbitrary fashion. Show that for the network in Figure. The matrix M has the given form, where row 1 corresponds to mesh 1, etc.

$$M = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

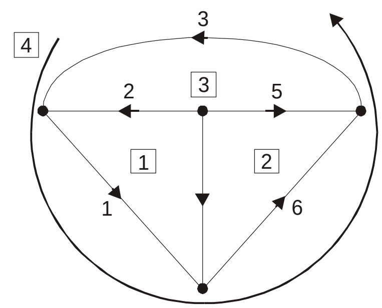


Fig. 3.3

Example 1. There is a circuit (electrical network) in figure given below. Find the currents i_1, i_2 and i_3 respectively.

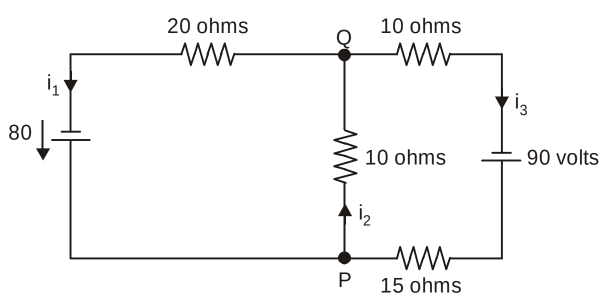


Fig. 3.4

Sol. We label the currents as shown, choosing directions arbitrarily; if a current will come out negative, this will simply mean that the current flows against the direction of our arrow. The current entering each battery will be the same as the current leaving it. The equations for the currents results from Kirchhoff's laws.

Kirchhoff's Current Law (KCL): At any point of a circuit, the sum of the inflowing currents equals the sum of the out flowing currents.

Kirchhoff's Voltage Law (KVL): In any closed loop, the sum of all voltage drops equals the impressed electromotive force.

Node P gives the first equation, node Q the second, the right loop the third, and the left loop the fourth, as indicated in the figure.

Node P : $i_1 - i_2 + i_3 = 0$ (*i*)

Node Q : $-i_1 + i_2 - i_3 = 0$ (*ii*)

Right loop : $10\ i_2 + 25\ i_3 = 90$ (*iii*)

Left loop : $20\ i_1 + 10\ i_2 = 80$ (*iv*)

We solve these equations by matrix method

$$\text{Augmented matrix } [A : B] = \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Applying $R_2 \to R_2 + R_1$, $R_4 \to R_4 - 20\ R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \to 3R_3 - R_4$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95 & 190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to \frac{1}{5}\ R_2$, $R_3 \to \frac{1}{19}\ R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 6 & -4 & 16 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \begin{bmatrix} 0 \\ 16 \\ 10 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} i_1 - i_2 + i_3 &= 0 \\ 6i_2 - 4i_3 &= 16 \\ 5i_3 &= 10 \end{aligned}$$

On solving these equations

$i_1 = 2$ amperes

$i_2 = 4$ amperes

$i_3 = 2$ amperes.

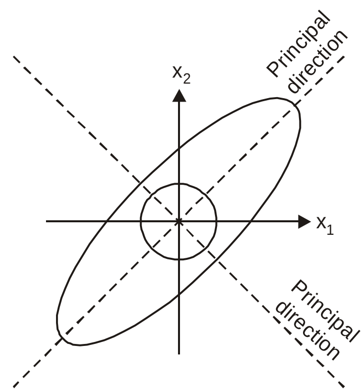


Fig. 3.5

Example 2. An elastic membrane in the x_1, x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$, is stretched 80 that a point $P : (x_1, x_2)$ goes over into the point $Q : (y_1, y_2)$ given by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

in components $y_1 = 5x_1 + 3x_2;\ y_2 = 3x_1 + 5x_2$.

Find the principal directions, that is, the directions of the position vector x of P for which the direction of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Sol. We are looking for vectors x such that $y = \lambda x$.

Since $y = AX \Rightarrow AX = \lambda X$

$\therefore$ The above form is

$$\begin{aligned} 5x_1 + 3x_2 &= \lambda x_1 \\ 3x_1 + 5x_2 &= \lambda x_2 \end{aligned} \quad or \quad \begin{aligned} (5 - \lambda)\, x_1 + 3x_2 &= 0 \\ 3x_1 + (5 - \lambda)\, x_2 &= 0 \end{aligned}$$

$\therefore$ The characteristic equation is $\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5 - \lambda)^2 - 9 = 0$

$$\Rightarrow \quad \lambda_1 = 8,\ \lambda_2 = 2$$

for $\lambda_1 = 8$, $\quad \begin{aligned} -3x_1 + 3x_2 &= 0 \\ 3x_1 - 3x_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 - x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$

Sol. $x_2 = x_1$, x_1 arbitrary.

Let $\quad x_1 = x_2 = 1$

For $\lambda_2 = 2$ our system becomes

$$\begin{aligned} 3x_1 + 3x_2 &= 0 \\ 3x_1 + 3x_2 &= 0 \end{aligned}$$

$\Rightarrow \quad x_2 = -x_1$, x_1 arbitrary

Let $\quad x_1 = 1,\ x_2 = -1$

Thus the eigen vector, for instance are

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These vectors make 45° and 135° angles with the positive x_1-direction. They give the principal directions.

The eigen values show that in the principal directions the membrane is stretched by factors 8 and 2, respectively. Accordingly if we choose the principal directions as directions of a new cartesian u_1, u_2-coordinats system, say with the positive u_1-semiaxis in the first quadrant and the positive u_2-semiaxis in the second quadrant of the x_1 x_2-system and if we get

$$u_1 = r\cos\phi,\ u_2 = r\sin\phi$$

then a boundary point of the unstretched circular membrane has coordinates cos ϕ, sin ϕ. Hence after the stretch we have

$$Z_1 = 8\cos\phi,\ Z_2 = 2\sin\phi$$

Since $\cos^2\phi + \sin^2\phi = 1$, this shows that the deformed boundary is an ellipse.

$$\frac{Z_1^2}{8^2} + \frac{Z_2^2}{2^2} = 1$$

with principal semiaxes 8 and 2 in the principal directions.

EXERCISE 3.8

1. A square matrix A is defined by $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Find the modal matrix P and the resulting diagonal matrix D of A. (*U.P.T.U., 2000*)

$$\left[\textbf{Ans. } P = \begin{bmatrix} -2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right]$$

2. Diagonalise the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and hence find A^4.

$$\left[\textbf{Ans. } A^4 = \begin{bmatrix} 251 & -405 & 235 \\ -405 & 81 & -405 \\ 235 & -405 & 251 \end{bmatrix}\right]$$

3. Diagonalise the matrix A.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}. \qquad \left[\textbf{Ans. } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right]$$

4. Verify whether the following matrices are diagonalisable or not.

(*i*) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, (*ii*) $\begin{bmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{bmatrix}$. [**Ans.** (*i*) Not diagonalisable (*ii*) diagonalisable]

5. Diagonalise $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

$$\left[\textbf{Ans. } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}\right]$$

6. Diagonalise the following matrices:

(*i*) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, (*ii*) $\begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$, (*iii*) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\left[\textbf{Ans. } (i)\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, (ii) \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}, (iii) \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right]$

7. Diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. $\left[\textbf{Ans. } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right]$

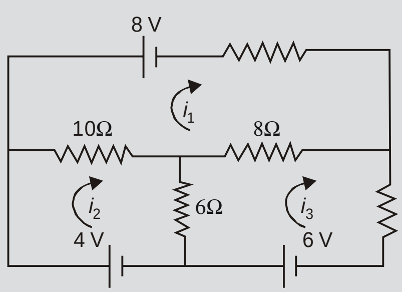
8. Diagonalise the matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ hence find A^5. $\left[\textbf{Ans. } D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, A^5 = \begin{bmatrix} 2344 & 781 \\ 2343 & 782 \end{bmatrix}\right]$

9. Verify the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalisable.

[**Ans.** Not diagonalisable since only one eigen vector $\begin{bmatrix} k \\ 0 \end{bmatrix}$ exists]

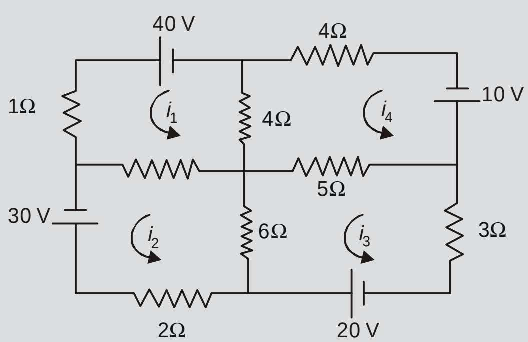
10. Using matrix equation determine the loop current in the following circuits:

(*i*)



$[\textbf{Ans. } i_1 = 1.0826,\ i_2 = 1.4004,\ i_3 = 1.2775]$

(*ii*)



$[\textbf{Ans. } i_1 = 11.43,\ i_2 = 10.55,\ i_3 = 8.04,\ i_4 = 5.84]$

OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer of the choices given below:

1. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then the value of A^4 is

(*i*) $\begin{bmatrix} 1 & a^4 \\ 0 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 4 & 4a \\ 0 & 4 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 4 & a^4 \\ 0 & 4 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$

2. If the matrix $\begin{bmatrix} 1 & b & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ is not invertible, then the value of b is

(*i*) 2 (*ii*) 0

(*iii*) 1 (*iv*) –1

3. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to

(*i*) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (*ii*) $\frac{1}{27}\begin{bmatrix} -1 & 26 \\ 0 & 27 \end{bmatrix}$

(*iii*) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (*iv*) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

4. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then A^{-1} is

(*i*) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (*iv*) None of these

5. If A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$, then $|3AB| =$

(*i*) – 9 (*ii*) – 27

(*iii*) – 81 (*iv*) 81

6. The rank of the unit matrix of order n is

(*i*) 0 (*ii*) 1

(*iii*) 2 (*iv*) 3

7. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{bmatrix}$, then rank ($A$) is

(*i*) 0 (*ii*) 1

(*iii*) 2 (*iv*) 3

8. If a matrix A has a non-zero minor of order r and all minors of higher orders zero, then

(*i*) $\rho(A) < r$ (*ii*) $\rho(A) \leq r$

(*iii*) $\rho(A) > r$ (*iv*) $\rho(A) = r$

9. The system of equations $3x + 2y + z = 0$, $x + 4y + z = 0$, $2x + y + 4z = 0$.

(*i*) is consistent (*ii*) has infinite solutions

(*iii*) has only a trivial solution (*iv*) None of these

10. The system of equations $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ is consistent, if

(*i*) $a + b - c = 0$ (*ii*) $a - b + c = 0$

(*iii*) $a + b + c \neq 0$ (*iv*) $a + b + c = 0$

11. If $X_1 = (2, 3, 1, -1)$; $X_2 = (2, 3, 1, -2)$, $X_3 = (4, 6, 2, -3)$ then

(*i*) $X_1 + X_3 = X_2$ (*ii*) $X_2 + X_3 = X_1$

(*iii*) $X_1 + X_2 = 2X_3$ (*iv*) $X_1 + X_2 = X_3$

B. Fill in the blanks:

1. The vectors (1, –1, 1), (2, 1, 1), (3, a, 2) are linearly dependent if the value of a = ...

2. If A is non-singular matrix of order 3 then $\rho(A)$ = ...

3. If λ is a non-zero characteristic root of a non-singular matrix A, then the characteristic roots of A^{-1} are ....

4. If $\lambda_1, \lambda_2, \lambda_3$ are the characteristic roots of a non-singular matrix A, then the characteristic roots of A^{-1} are ....

5. The characteristic of a real skew symmetric are either ..... or .....

6. The product of the characteristic roots of a square matrix is ....

7. A system of n linear homogeneous equations in n unknowns has a non-trivial solution if the coefficient matrix is ....

8. A system of n linear homogeneous equations in n unknowns is always ....

9. For the set of vectors $(X_1, X_2, \ldots X_n)$ to be linearly independent it is necessary that no X_i is ....

10. The eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are ....

C. Indicate True or False for the following statements:

1. (*i*) If $|A| = 0$, then at least one eigen value is zero.

(*ii*) A^{-1} exists iff 0 is an eigen value of A.

(*iii*) If $|A| \neq 0$ then A is known as singular matrix.

(*iv*) Two vectors X and Y are said to be orthogonal if, $X^TY = Y^TX \neq 0$.

2. (*i*) The eigen values of a real skew symmetric matrix are all real.

(*ii*) If A is a square matrix, then latent roots of A and A' are identical.

(*iii*) If A is a unit matrix, then $|A| = 0$.

(*iv*) If $|A| \neq 0$, then $|A. \text{adj } A| = |A|^{n-1}$, where $A = (a_{ij})_{n \times n}$.

D. Match the following:

1. (*i*) If A is an invertible matrix then	(*a*) $|A| = \pm 1$
(*ii*) If A is orthogonal then	(*b*) $|A| = 0$
(*iii*) $(A - \lambda I)$ is a matrix	(*c*) is 1
(*iv*) The rank of unit matrix of order n	(*d*) Characteristic
2. (*i*) For non-Trivial	(*a*) A^{-1} is a diagonal
(*ii*) A is diagonal	(*b*) A
(*iii*) $(A^{-1})^{-1}$	(*c*) $|A| = 0$
(*iv*) For trivial solution	(*d*) $\rho(A) < n$
3. (*i*) Rank of skew symmetric matrix	(*a*) Have same rank
(*ii*) Two equivalent matrices	(*b*) – A
(*iii*) The rank of I_4 is	(*c*) 1
(*iv*) A^*	(*d*) 4

ANSWERS TO OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer:

1. (*iv*)	**2.** (*iii*)	**3.** (*i*)
4. (*iii*)	**5.** (*iii*)	**6.** (*iv*)
7. (*iii*)	**8.** (*iv*)	**9.** (*iii*)
10. (*iv*)		

B. Fill in the blanks:

1. 0	**2.** 3	**3.** *adj A*
4. $\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}$	**5.** All zero; pure imaginary]	**6.** $|A|$
7. Singular	**8.** Consistent	
9. Independent	**10.** 1, 1, 5	

C. True or False:

1. (*i*) *T*	(*ii*) *F*	(*iii*) *F*	(*iv*) *F*
2. (*i*) *F*	(*ii*) *T*	(*iii*) *T*	(*iv*) *F*

D. Match the following:

1. (*i*) → (*b*), (*ii*) → (*a*), (*iii*) → (*d*), (*iv*) → (*c*)

2. (*i*) → (*d*) (*ii*) → (*a*) (*iii*) → (*b*) (*iv*) → (*c*)

3. (*i*) → (*c*), (*ii*) → (*a*), (*iii*) → (*d*), (*iv*) → (*b*)

❑❑❑

UNIT IV

Multiple Integrals

4.1 MULTIPLE INTEGRALS

The process of integration for one variable can be extended to the functions of more than one variable. The generalization of definite integrals is known as **"multiple integral".**

4.2 DOUBLE INTEGRALS

Consider the region R in the x, y plane we assume that R is a closed*, bounded** region in the x , y plane, by the curve $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$. Let us lay down a rectangular grid on R consisting of a finite number of lines parallel to the coordinate axes. The N rectangles lying entirely within R (the shaded ones in Fig. 4.1). Let (x_r, y_r) be an arbitrarily selected point in the rth partition rectangle for each $r = 1, 2, ..., N$. Then denoting the area $\delta x_r \cdot \delta y_r = \delta S_r$

Thus, the total sum of areas

$$S_N = \sum_{r=1}^{N} f(x_r, y_r)\ \delta S_r$$

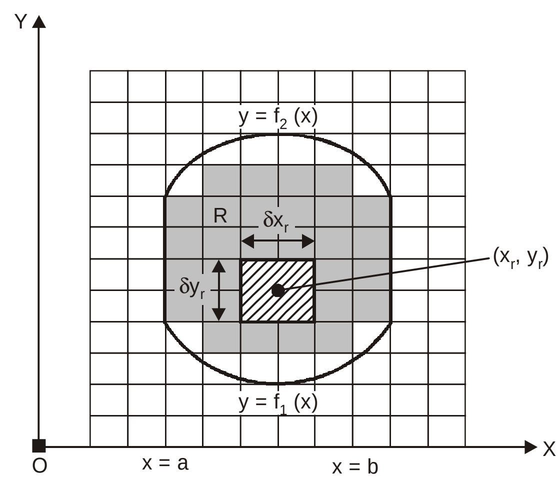


Fig. 4.1

Let the maximum linear dimensions of each portion of areas approach zero, and n increases indefinitely then the sum S_N will approach a limit, "namely the double integral $\iint_R f(x, y)\, dS$ and the value of this limit is given by

* Boundary included *i.e.,* part of the region.

** Can be enclosed within a sufficiently large circle.

$$\iint_R f(x,y)\,dS = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right] dx$$

Similarly if the region is bounded by $y = c$, $y = d$ and by the curves $x = f_1(y)$, $x = f_2(y)$, then

$$\iint_R f(x,y)\,dS = \int_c^d \left[\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx \right] dy$$

4.3 WORKING RULE

(*a*) We integrate with respect to y, x is to be regarded as constant and evaluate the result between the limits $y = f_1(x)$ and $y = f_2(x)$.

(*b*) Then we integrate the result of (*a*) with respect to x between the limits $x = a$ and $x = b$.

4.4 DOUBLE INTEGRATION FOR POLAR CURVES

Let OP and OQ are two radii vectors of the curve $r = f(\theta)$ and the coordinates of P and Q be (r, θ) and $(r + \delta r, \theta + \delta\theta)$.

The area of portion

$$K_1 L_1 K_2 L_2 = \frac{1}{2}(r + \delta r)^2 \delta\theta - \frac{1}{2} r^2 \delta\theta$$

$$= r\delta r\delta\theta + \frac{1}{2}\delta r^2 \delta\theta$$

$$= r\delta r\delta\theta \quad | \text{As } \delta r^2\delta\theta \text{ is very small}$$

Therefore, the area of OPQ

$$= \lim_{\delta r \to 0} [\Sigma r\delta r\delta\theta]$$

$$= \lim_{\delta r \to 0} [\Sigma r\delta r]\,\delta\theta$$

$$= \left[\int_0^{f(\theta)} r dr \right] \delta\theta$$

Hence the area OAB

$$= \lim_{\delta\theta \to 0} \sum \left[\int_0^{f(\theta)} r dr \right] \delta\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \left[\int_0^{f(\theta)} r dr \right] d\theta$$

where α and β are the vectorial angles of A and B.

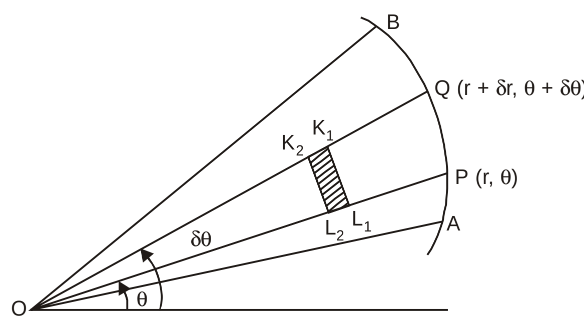


Fig. 4.2

Example 1. Evaluate $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\, dx\, dy.$

Sol. We have

$$I = \int_1^{\log 8}\left[\int_0^{\log y}\left(e^x dx\right)\right] e^y\, dy$$

$$= \int_1^{\log 8}\left[e^x\right]_0^{\log y} e^y\, dy$$

$$= \int_1^{\log 8} e^y\left[e^{\log y} - e^0\right] dy$$

$$= \left[e^y(y-1)\right]_1^{\log 8} - \left[e^y\right]_1^{\log 8} \qquad (\text{As } e^{\log_e y} = y)$$

$$= e^{\log 8}\cdot(\log 8 - 1) - 0 - (e^{\log 8} - e)$$

$$= 8 \log 8 - 8 - 8 + e$$

$$= 8 \log 8 - 16 + e.$$

Example 2. Evaluate $\int_0^{\pi} \int_0^{x} \sin y\, dy\, dx.$

Sol. We have

$$I = \int_0^{\pi}\left[\int_0^{x} \sin y\, dy\right] dx$$

$$= \int_0^{\pi}\left[-\cos y\right]_0^x dx, \text{ treating } x \text{ as constant}$$

$$= -\int_0^{\pi} (\cos x - \cos 0)\, dx$$

$$= -\int_0^{\pi} \cos x\, dx + \int_0^{\pi} dx \qquad (\text{As } \cos 0 = 1)$$

$$= -\left[\sin x\right]_0^{\pi} + \left[x\right]_0^{\pi} = \pi.$$

Example 3. Evaluate $\int_0^{\sqrt{2}} \int_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} y\, dx\, dy.$

Sol. We have

$$I = \int_0^{\sqrt{2}}\left[\int_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} dx\right] y\, dy$$

$$= \int_0^{\sqrt{2}} \left[x\right]_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} y\, dy$$

$$= \int_0^{\sqrt{2}} 2\sqrt{(4-2y^2)}\, y\, dy$$

$$= -\frac{1}{2}\int_4^0 \sqrt{t}\, dt \qquad \Big| \text{ by putting } 4 - 2y^2 = t,\ -4y\, dy = dt$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left[t^{\frac{3}{2}}\right]_4^0$$

$$= \frac{1}{3}\cdot 8 = \frac{8}{3}.$$

Example 4. Evaluate $\iint (x+y)^2 \; dx\, dy$ over the area bounded by the ellipse

[*U.P.T.U. (C.O.), 2004*]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

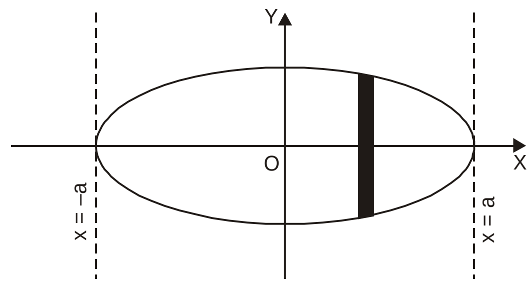


Fig. 4.3

Sol. We have $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \qquad \frac{y}{b} = \pm \sqrt{1-\frac{x^2}{a^2}}$$

$$\Rightarrow \qquad y = \pm \frac{b}{a} \sqrt{a^2-x^2}$$

$$\therefore \quad \iint (x+y)^2 \; dx\, dy = \iint \left(x^2+y^2+2xy\right) dx\, dy$$

$$= \int_{-a}^{a} \int_{\left(\frac{-b}{a}\right)\sqrt{a^2-x^2}}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} \left(x^2+y^2+2xy\right) dx\, dy$$

$$= \int_{-a}^{a} \int_{\left(\frac{-b}{a}\right)\sqrt{a^2-x^2}}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} \left(x^2+y^2\right) dx\, dy + 2\int_{-a}^{a} \int_{\left(-\frac{b}{a}\right)\sqrt{a^2-x^2}}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= \int_{-a}^{a} 2\int_{0}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} \left(x^2+y^2\right) dy\, dx + 0$$

$$\left| \text{As} \int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx \left[\text{when } f(x) \text{ even}\right] = 0 \left[\text{when } f(x) \text{ odd}\right]\right.$$

$$= \int_{-a}^{a} \left[2\left(x^2 y + \frac{y^3}{3}\right)\right]_{0}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x^2 \times \frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\frac{b^3}{a^3}\left(a^2-x^2\right)^{\frac{3}{2}}\right] dx$$

$$= 4\int_{0}^{a} \left[\frac{b}{a}x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}\left(a^2-x^2\right)^{\frac{3}{2}}\right] dx \text{ (Again by definite integral)}$$

[On putting $x = a \sin \theta$ and $dx = a \cos \theta \; d\theta$]

$$= 4\int_{0}^{\frac{\pi}{2}} \left(\frac{b}{a} \cdot a^2 \sin^2 \theta . a \cos\theta + \frac{b^3}{3a^3} a^3 \cos^3 \theta\right) \times a \cos \theta \; d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left(a^3b\sin^2\theta\cos^2\theta\, d\theta + \frac{ab^3}{3}\cos^4\theta\right) d\theta$$

$$= 4a^3b\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\, d\theta + \frac{4ab^3}{3}\int_0^{\frac{\pi}{2}}\sin^0\theta\cos^4\theta\, d\theta$$

$$= 4a^3b\cdot\frac{\left|\frac{3}{2}\right.\left|\frac{3}{2}\right.}{2\left|\frac{2+2+2}{2}\right.} + \frac{4ab^3}{3}\cdot\frac{\left|\frac{1}{2}\right.\left|\frac{5}{2}\right.}{2\left|\frac{0+4+2}{2}\right.} \qquad \left| \text{As} \int_0^{\frac{\pi}{2}}\sin^m\theta\ \cos^n\theta\ d\theta = \frac{\left|\frac{m+1}{2}\right.\left|\frac{n+1}{2}\right.}{2\left|\frac{m+n+2}{2}\right.}\right.$$

$$= 4a^3b.\frac{\frac{1}{2}\left|\frac{1}{2}\right..\frac{1}{2}\left|\frac{1}{2}\right.}{2\left|3\right.} + \frac{4ab^3}{3}.\frac{\frac{1}{2}.\frac{3}{2}.\frac{1}{2}.\left|\frac{1}{2}\right.}{2\left|3\right.}$$

$$= 4a^3b\cdot\frac{\pi}{16} + \frac{4ab^3}{3}\cdot\frac{3\pi}{16} \qquad \left|\text{As } \left|n\right. = (n-1)\left|n-1\right. \text{ and } \left|\frac{1}{2}\right. = \sqrt{\pi}\right.$$

$$\Rightarrow \quad = \frac{\pi a^3 b}{4} + \frac{\pi ab^3}{4} = \frac{\pi}{4}\cdot ab\left(a^2+b^2\right).$$

Example 5. Evaluate $\iint xy\ (x+y)\, dx\, dy$ over the area between $y = x^2$ and $y = x$.

Sol. The area is bounded by the curves $y = f_1(x) = x^2$, $y = f_2(x) = x$.

When $\quad f_1(x) = f_2(x)$,

$x^2 = x$, *i.e.*, $x(x-1) = 0$

or $\quad x = 0,\ x = 1$

i.e., the area of integration is bounded by

$y = x^2,\ y = x,\ x = 0,\ x = 1$

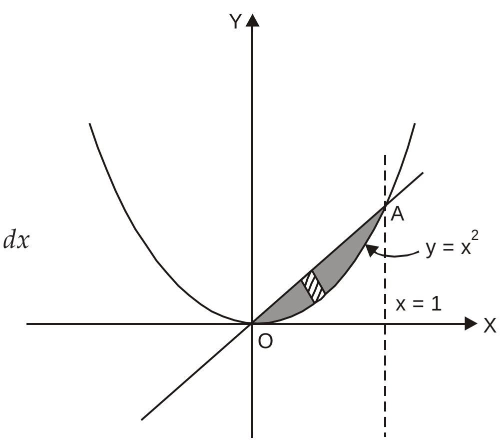


Fig. 4.4

$$\therefore \quad = \iint_A xy\,(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{y=x} xy\,(x+y)\,dy\right] dx$$

$$= \int_0^1\left[\int_{x^2}^{x}\left(x^2y + xy^2\right)dy\right] dx$$

$$= \int_0^1\left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1\left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1 = \left[\frac{1}{6} - \frac{1}{14} - \frac{1}{24}\right] = \frac{3}{56}.$$

Example 6. Find $\iint_D x^2 dx\, dy$ where D is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$. (*U.P.T.U., 2002*)

Sol. We have

$$xy = 16 \qquad ...(i)$$
$$y = x \qquad ...(ii)$$
$$y = 0 \qquad ...(iii)$$
$$x = 8 \qquad ...(iv)$$

From eqns. (*i*) and (*ii*), we get $x = 4,\ y = 4$ *i.e.*, intersection point of curve and the line

$$y = x = (4, 4).$$

Similarly intersection point of (*i*) and (*iv*) = (8, 2)

To evaluate the given integral, we divide the area *OABEO* into two parts by *AG* as shown in the Figure 4.5.

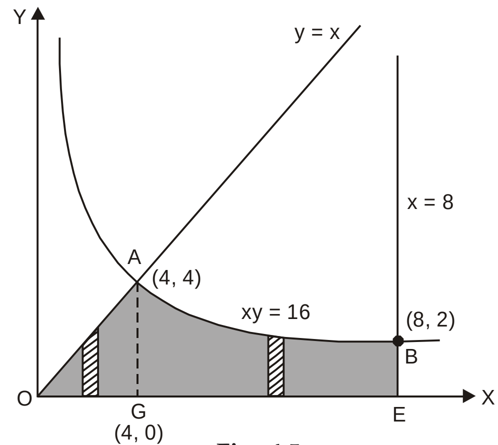


Fig. 4.5

Then, $$\iint_D x^2 dx\, dy = \int_{x=0}^{x=4}\int_{y=0}^{y=x} x^2 dx\, dy + \int_{x=4}^{x=8}\int_{y=0}^{y=\frac{16}{x}} x^2 dx\, dy$$

$$= \int_0^4 x^2 dx \int_0^x dy + \int_4^8 x^2 dx \int_0^{\frac{16}{x}} dy = \int_0^4 x^2 (y)_0^x dx + \int_4^8 x^2 (y)_0^{\frac{16}{x}} dx$$

$$= \int_0^4 x^3 dx + \int_4^8 16x\, dx = \left[\frac{x^4}{4}\right]_0^4 + \left[8x^2\right]_4^8$$

$$= 64 + 8(64 - 16) = 64 + 384 = 448.$$

Example 7. Evaluate $\iint_A xy\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

Sol. Here the region of integration is positive quadrant of circle $x^2 + y^2 = a^2$, where x varies from 0 to a and y varies from 0 to $\sqrt{(a^2 - x^2)}$.

Here, $$\iint_A xy\, dx\, dy = \int_0^a \int_0^{\sqrt{(a^2-x^2)}} xy\, dx\, dy$$

$$= \int_0^a \left[\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} x\, dx \right] dx$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} x\, dx$$

$$= \frac{1}{2}\int_0^a x\left(a^2 - x^2\right) dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{1}{8}a^4.$$

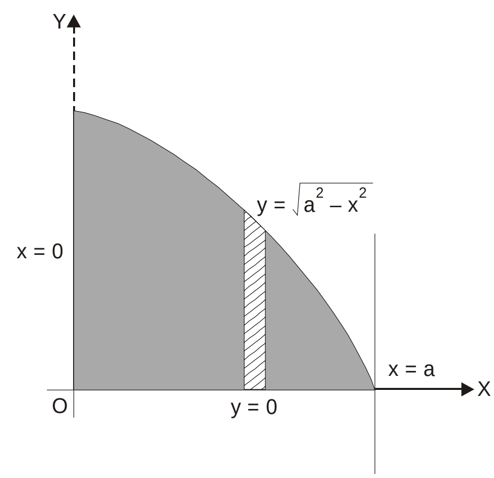


Fig. 4.6

Example 8. Find $\iint_D \left(x^2 + y^2\right) dx\, dy$ where D is bounded by $y = x$ and $y^2 = 4x$.

Sol. $$\iint_D \left(x^2 + y^2\right) dx\, dy = \int_0^4 \int_x^{2\sqrt{x}} \left(x^2 + y^2\right) dy\, dx$$

$$= \int_0^4 \left[x^2 y + \frac{y^3}{3}\right]_x^{2\sqrt{x}} dx$$

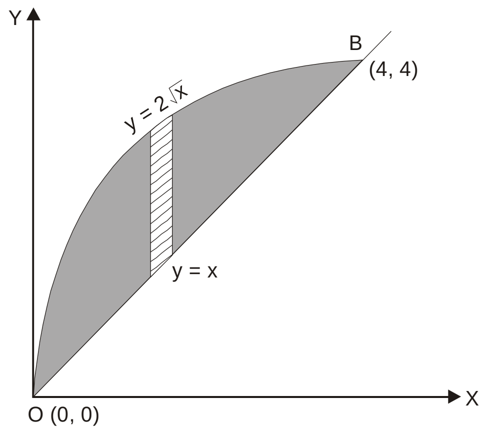


Fig. 4.7

$$= \int_0^4 \left(2x^{5/2} + \frac{8}{3}x^{3/2} - \frac{4}{3}x^3\right)dx$$

$$= \frac{768}{35}.$$

Example 9. Find $\iint_D x^3 y\, dx\, dy$ where D is the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

Sol.

$$\iint_D x^3 y\, dx\, dy = \int_0^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} x^3 y\, dy\, dx$$

$$= \int_0^a \left[\frac{x^3 y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}$$

$$= \frac{b^2}{2a^2} \int_0^a \left(a^2x^3 - x^5\right) dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^4}{4} - \frac{x^6}{6}\right]_0^a = \frac{b^2a^4}{24}.$$

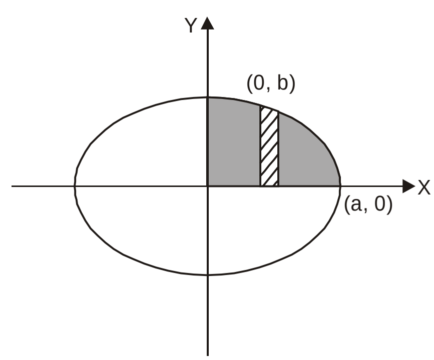


Fig. 4.8

Example 10. Evaluate $I = \int_0^{2\pi} \int_{a\sin\theta}^{a} r\,dr\,d\theta$.

Sol.

$$I = \int_0^{2\pi} \int_{r=a\sin\theta}^{a} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{a\sin\theta}^{a} d\theta = \frac{1}{2}\int_0^{2\pi}\left(a^2 - a^2\sin^2\theta\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{a^2}{2}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$I = \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{\pi a^2}{2}.$$

Example 11. Evaluate $\iint_A r^2 \sin\theta \, d\theta \, dr$ over the area of cordioid $r = a(1 + \cos\theta)$ above the initial line.

Sol. The region of integration A can be covered by radial strips whose ends are at $r = 0$, $r = a(1 + \cos\theta)$.

The strips lie between $\theta = 0$ and $\theta = \pi$

$$\text{Thus } \iint_A r^2 \sin\theta \, d\theta \, dr = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \sin\theta \, d\theta \, dr$$

$$= \int_0^{\pi} \sin\theta \left[\int_0^{a(1+\cos\theta)} r^2 dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} (1 + \cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{16a^3}{3} \int_0^{\pi} \cos^7 \frac{\theta}{2} \sin \frac{\theta}{2} d\theta$$

$$= \frac{16}{3} a^3 \int_0^{\frac{\pi}{2}} \sin\phi \cos^7 \phi \cdot 2d\phi \qquad \left| \text{ Put } \frac{\theta}{2} = \phi \right.$$

$$= 2 \times \frac{16}{3} a^3 \left[\frac{-\cos^8 \phi}{8} \right]_0^{\frac{\pi}{2}} = \frac{4}{3} a^3.$$

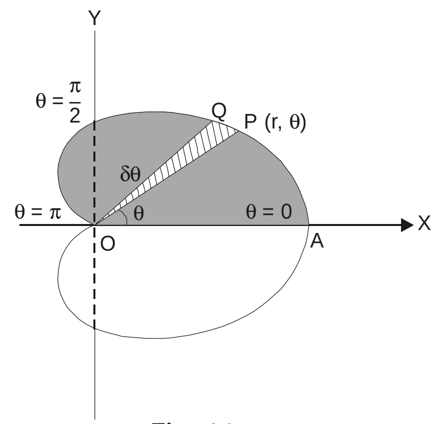


Fig. 4.9

Example 12. Evaluate $\int_0^{\frac{\pi}{2}} \int_{a(1-\cos\theta)}^{a} r^2 dr d\theta$.

$$\textbf{Sol. } \int_0^{\frac{\pi}{2}} d\theta \int_{a(1-\cos\theta)}^{a} r^2 dr = \int_0^{\frac{\pi}{2}} d\theta \left[\frac{r^3}{3} \right]_{a(1-\cos\theta)}^{a}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \left(\frac{a^3}{3} - \frac{a^3 (1 - \cos\theta)^3}{3} \right)$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \left[1 - (1 - \cos\theta)^3 \right] d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \left[1 - \left(1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta \right) \right] d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \left(3\cos\theta - 3\cos^2\theta + \cos^3\theta \right) d\theta$$

$$= \frac{a^3}{3} \left[[3\sin\theta]_0^{\frac{\pi}{2}} - 3 \frac{1}{2} \frac{\pi}{2} + \frac{2}{3 \cdot 1} \right]$$

$$= \frac{a^3}{3} \left[3 - \frac{3\pi}{4} + \frac{2}{3} \right]$$

$$= \frac{a^3}{36} [44 - 9\pi].$$

Example 13. Evaluate $\int_0^{\pi} \int_0^{a\theta} r^3 d\theta\, dr$

Sol. We have

$$I = \int_0^{\pi} \int_0^{a\theta} r^3 d\theta\, dr$$

$$= \int_0^{\pi} \left[\int_0^{a\theta} r^3 dr\right] d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4}\right]_0^{a\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi} a^4 \theta^4 d\theta$$

$$= \frac{a^4}{4} \left[\frac{\theta^5}{5}\right]_0^{\pi} = \frac{a^4 \pi^5}{20}.$$

Example 14. Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^3 \sin\theta \cos\theta\, d\theta\, dr$.

Sol. We have

$$I = \int_0^{\pi} \sin\theta \cos\theta \left[\int_0^{a(1+\cos\theta)} r^3 dr\right] d\theta$$

$$\int_0^{\pi} \sin\theta \cos\theta \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi} (1+\cos\theta)^4 \sin\theta \cos\theta\, d\theta$$

Put $1 + \cos\theta = t$ and $-\sin\theta\, d\theta = dt$

$$= \frac{a^4}{4} \int_2^0 t^4 (t-1)(-dt)$$

$$= \frac{a^4}{4} \int_0^2 (t^5 - t^4)\, dt = \frac{16}{15} a^4.$$

EXERCISE 4.1

1. Evaluate $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\, dx\, dy.$ **[Ans.** $8 \log 8 - 16 + e$**]**

2. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}.$ **[Ans.** $\frac{\pi}{4} \log(\sqrt{2}+1)$**]**

3. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(x+y)\, dy\, dx.$ **[Ans.** -2**]**

4. $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy.$ **[Ans.** $\frac{\pi a^3}{6}$**]**

5. $\int_0^1 \int_0^{x^2} e^{\frac{x}{y}} dx\, dy$. $\left[\textbf{Ans. } \frac{1}{2}\right]$

6. Evaluate $\iint_S \sqrt{xy - y^2}\; dy\, dx$ where S is the triangle with vertices (0, 0), (10, 1) and (1, 1). $[\textbf{Ans. } 6]$

7. Evaluate $\iint_S \left(x^2 + y^2\right) dx\, dy$, where S is the area enclosed by the curves, $y = 4x$, $x + y = 3$, $y = 0$ and $y = 2$. $\left[\textbf{Ans. } \frac{463}{48}\right]$

8. Evaluate $\iint x^2 y^2\; dx\, dy$ over the region $x^2 + y^2 \le 1$. $\left[\textbf{Ans. } \frac{\pi}{24}\right]$

9. Evaluate $\iint \left(x^2 + y^2\right) dx\, dy$ over the region bounded by $x = 0$, $y = 0$, $x + y = 1$. $\left[\textbf{Ans. } \frac{1}{6}\right]$

10. Evaluate $\iint_A \frac{xy}{\sqrt{1 - y^2}}\; dx\, dy$, where the region of integration is the positive quadrant of the circle $x^2 + y^2 = 1$. $\left[\textbf{Ans. } \frac{1}{6}\right]$

11. Evaluate $\iint xy\; dx\, dy$ over the region in the positive quadrant for which $x + y \le 1$. $\left[\textbf{Ans. } \frac{1}{24}\right]$

12. Evaluate $\int_{-1/2}^{1} \int_{-x}^{1+x} \left(x^2 + y\right) dy\, dx$. $\left[\textbf{Ans. } \frac{63}{32}\right]$

13. $\iint_D \left(4xy - y^2\right) dx\, dy$, where D is the rectangle bounded by $x = 1$, $x = 2$, $y = 0$, $y = 3$. $[\textbf{Ans. } 18]$

14. $\iint_A (1 + x + y)\; dx\, dy$, A is the region bounded by the lines $y = -x$, $x = \sqrt{y}$, $y = 2$, $y = 0$.

$\left[\textbf{Hint: } \text{Limits } x : -y \text{ to } \sqrt{y};\; y : 0 \text{ to } 2\right]$. $\left[\textbf{Ans. } \frac{44}{15}\sqrt{2} + \frac{13}{3}\right]$

15. $\int_0^1 \int_0^{\sqrt{1+x^2}} \left(1 + x^2 + y^2\right)^{-1}\; dx\, dy$. $\left[\textbf{Ans. } \frac{\pi}{4} \log\left(1 + \sqrt{2}\right)\right]$

16. Evaluate $\iint_A y\, dx\, dy$, where A is bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$. $\left[\textbf{Ans. } \frac{48}{5}\right]$

17. Show that $\int_0^1 \left[\int_0^1 \frac{x - y}{(x + y)^2} dy\right] dx \ne \int_0^1 \left[\int_0^1 \frac{x - y}{(x + y)^2} dx\right] dy$.

18. Evaluate $\int_0^1 \int_{y^2}^{y} \left(1+xy^2\right)\, dx\, dy.$ $\left[\textbf{Ans. } \frac{41}{240}\right]$

19. $\iint_A \left(x^2+y^2\right) dx\, dy$, where A is bounded by $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $b > a$.

$\left[\textbf{Ans. } \frac{\pi}{2}\left(b^4 - a^4\right)\right]$

20. $\int_3^4 \int_1^2 \frac{dy\, dx}{(x+y)^2}.$ $\left[\textbf{Ans. } \log\left(\frac{25}{24}\right)\right]$

Using polar coordinates evaluate the following double integral.

21. $\int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta.$ $\left[\textbf{Ans. } \frac{45\pi}{2}\right]$

22. $\int_0^{\pi} \int_0^{\cos\theta} \rho \sin\theta\, d\rho\, d\theta.$ $\left[\textbf{Ans. } \frac{1}{3}\right]$

23. $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta.$ $[\textbf{Ans. } 0]$

24. $\iint r^3\, dr\, d\theta$, over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

$\left[\textbf{Ans. } \frac{45}{2}\pi\right]$

25. Show that $\iint_R r^2 \sin\theta\, dr\, d\theta = \frac{2a^3}{3}$, where r is the region bounded by the semicircle $r = 2a\cos\theta$.

26. $\int_0^{\frac{\pi}{2}} \int_0^{a} r^n \sin^n\theta \cos\theta\, d\theta\, dr$, for $n + 1 > 0$. $\left[\textbf{Ans. } \frac{a^{n+1}}{(n+1)^2}\right]$

4.5 CHANGE OF THE ORDER OF INTEGRATION

If the limits of x and y are constant then $\iint F(x,y)\, dx\, dy$ can be integrated in either order, but if the limits of y are functions of x, then the new limits of x in functions of y are to be determined. The best method is by geometrical consideration.

Thus in several problems, the evaluation of double integral becomes easier with the change of order of integration, which of course, changes the limits of integration also.

Remark. If y depends on x and we want that x depend on y *i.e.*, $x = f(y)$ then construct a strip parallel to x-axis.

Example 1. Evaluate the integral $\int_0^{\infty}\int_0^{x} x \exp\left(-\frac{x^2}{y}\right) dx\, dy$ by changing the order of integration.

(*U.P.T.U., 2005*)

Sol. Here $y = 0$ and $y = x$

$x = 0$ and $x = \infty$

Here x start from $x = y$ and goes to $x \to \infty$ and y varies from $y = 0$ to $y \to \infty$

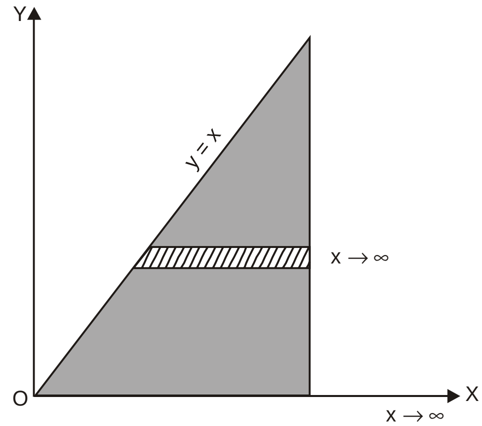


Fig. 4.10

$$\therefore \int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx\, dy$$

$$= \int_{y=0}^\infty \int_{x=y}^\infty x e^{-\frac{x^2}{y}} \cdot dy\, dx$$

$$= \int_0^\infty \int_y^\infty -\frac{y}{2}\left(-\frac{2x}{y} e^{-\frac{x^2}{y}}\right) dy\, dx$$

Put $e^{-\frac{x^2}{y}} = t \Rightarrow -\frac{2x}{y} e^{-\frac{x^2}{y}} dx = dt$

$$= \int_0^\infty \left[-\frac{y}{2} e^{-\frac{x^2}{y}}\right]_y^\infty dy$$

$$= \int_0^\infty \left[0 + \frac{y}{2} e^{-\frac{y^2}{y}}\right] dy = \int_0^\infty \frac{y}{2} e^{-y} dy$$

$$= \left[\frac{y}{2}\left(-e^{-y}\right) - \frac{1}{2}\left(e^{-y}\right)\right]_0^\infty \qquad \text{(Integration by parts)}$$

$$= (0 - 0) + \left(0 + \frac{1}{2}\right) = \frac{1}{2}.$$

Example 2. Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x, y)\, dx\, dy.$ (*U.P.T.U., 2007*)

Sol. Here the limits are

$x^2 = ay$ and $y = 2a - x$

i.e., $x^2 = ay$ and $x + y = 2a$

also $x = 0$ and $x = a$

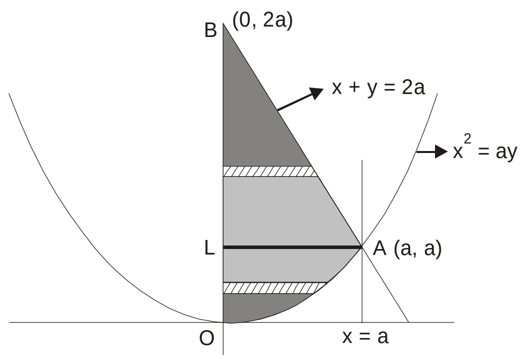


Fig. 4.11

Now $$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x, y)\, dx\, dy$$

$$= \int_0^a \int_0^{\sqrt{ay}} f(x, y)\, dy\, dx$$

$$+ \int_a^{2a} \int_0^{2a-y} f(x, y)\, dy\, dx.$$

Example 3. Evaluate $\int_0^2 \int_1^{e^x} dx\, dy$ changing the order of integration. (*U.P.T.U., 2003*)

Sol. The given limits are $x = 0$, $x = 2$, $y = 1$ and $y = e^x$.

Here $$\int_0^2 \int_1^{e^x} dx\, dy = \int_1^{e^2} \int_{x=\log y}^2 dy\, dx$$ As x varies from $x = \log y$ to $x = 2$

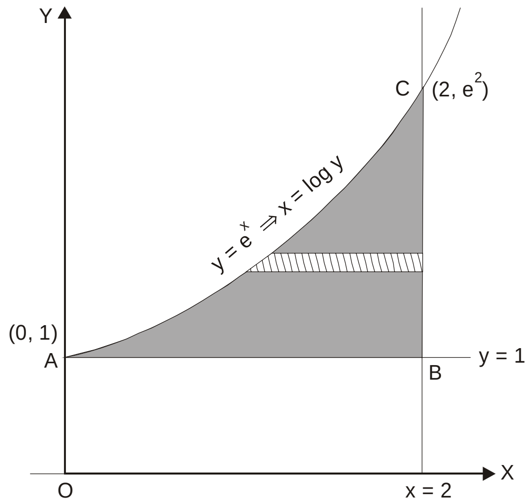


Fig. 4.12

$$= \int_1^{e^2} (2 - \log y)\ dy$$

$$= (2y - y\log y + y)_1^{e^2} = (3y - y\log y)_1^{e^2}$$

$$= (3e^2 - 2e^2) - 3$$

$$= e^2 - 3.$$

Example 4. $\int_{-2}^{1}\int_{x^2+4x}^{3x+2} dy\, dx$.

Sol. Here the curves $y = x^2 + 4x$, and the straight lines $y = 3x + 2$, $x = -2$ and $x = 1$ as shown shaded in Figure 4.13. Here $A(-2, -4)$ $B(0, 0)$, $E(1, 5)$, $F(0, 2)$, $G\left(-\frac{2}{3}, 0\right)$.

Considering horizontal strip, x varies from the upper curve $x = \sqrt{4+y} - 2$ the lower curve $x = \left(\frac{y-2}{3}\right)$ and then y varies from -4 to 5. Changing the order of integration to first x and later to y, the double integral becomes

$$= \int_{-2}^{1} \int_{y=x^2+4x}^{3x+2} dy\, dx$$

$$= \int_{-4}^{5} \int_{x=\frac{y-2}{3}}^{\sqrt{y+4}-2} dx\, dy$$

$$= \int_{-4}^{5} \left[\sqrt{y+4} - 2 - \left(\frac{y-2}{3}\right)\right] dy$$

$$= \left[\frac{2}{3}(y+4)^{\frac{3}{2}} - \frac{y^2}{6} - \frac{4}{3}y\right]_{-4}^{5} = \frac{9}{2}.$$

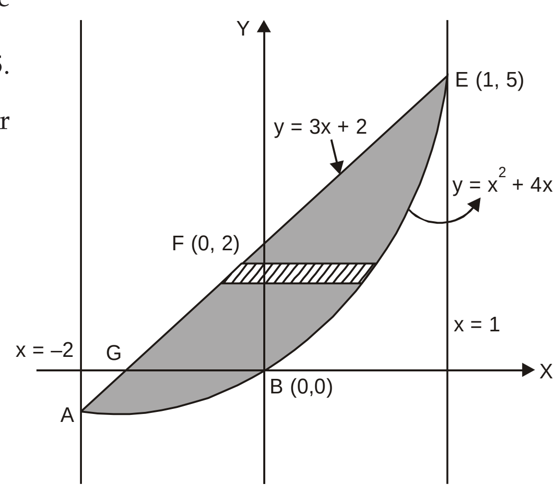


Fig. 4.13

Example 5. Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate the same. (*U.P.T.U., 2004*)

Sol. The limits are $y = x^2$ and $y = 2 - x$

$x = 0$ and $x = 1$

The point of intersection of the parabola $y = x^2$ and the line $y = 2 - x$ is B (1, 1).

We have taken a strip parallel to x axis in the area OBC and second strip in the area ABC. The limits of x in the area OBC are 0 and $\sqrt{y}$ and the limits of x in the area ABC are 0 and $2 - y$.

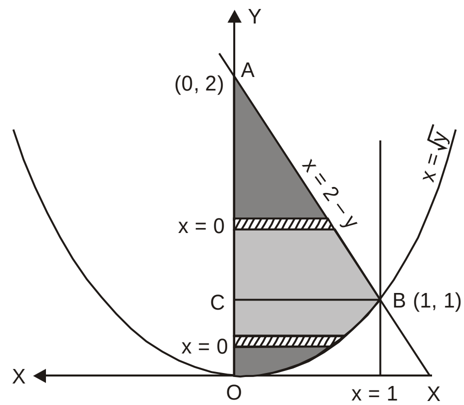


Fig. 4.14

$$\therefore \int_0^1 \int_{x^2}^{2-x} xydxdy = \int_0^1 y\, dy \int_0^{\sqrt{y}} xdx + \int_1^2 y\, dy \int_0^{2-y} x\, dx$$

$$= \int_0^1 y\, dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_1^2 ydy \left[\frac{x^2}{2}\right]_0^{2-y}$$

$$= \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 y(2-y)^2 \, dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2} \int_1^2 \left(4y - 4y^2 + y^3\right) dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2 = \frac{1}{6} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{96 - 128 + 48 - 24 + 16 - 3}{12}\right]$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}.$$

Example 6. Change the order of integration in $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} \phi(x, y)\, dx\, dy$.

Sol. Here the limits are $y = \sqrt{a^2 - x^2}$ and $y = x + 2a$ and $x = 0$, $x = a$.

Now, $y = \sqrt{x^2 - a^2}$ *i.e.*, $x^2 + y^2 = a^2$ is a circle and the straight line $y = x + 2a$ and the limits of x are given by the straight lines $x = 0$ and $x = a$.

The integral extends to all points in the space bounded by the axis of y the circle with centre O, and the straight line $x = a$.

We draw BN, KP perpendiculars to AM.

Now the order of integration is to be changed. For this, we consider parallel strips.

The integral is broken into three parts.

Ist part is BAN, bounded by lines $x = 0$, $x = a$ and the circle.

2nd part is *KPNB*, bounded by lines $y = a$, $y = 2a$ and $x = 0$, $x = a$.

3rd part is triangle *KPM* bounded by the lines $y = 2a$, $y = x + 2a$, $x = a$.

Hence $\int_0^a \int_{\sqrt{(a^2-x^2)}}^{x+2a} \phi(x, y)\, dx\, dy$

$$= \int_0^a \int_{\sqrt{a^2-y^2}}^{a} \phi(x, y)\, dy\, dx + \int_a^{2a} \int_0^a \phi(x, y)\, dy\, dx$$

$$= + \int_{2a}^{3a} \int_{y-2a}^{a} \phi(x, y)\, dy\, dx.$$

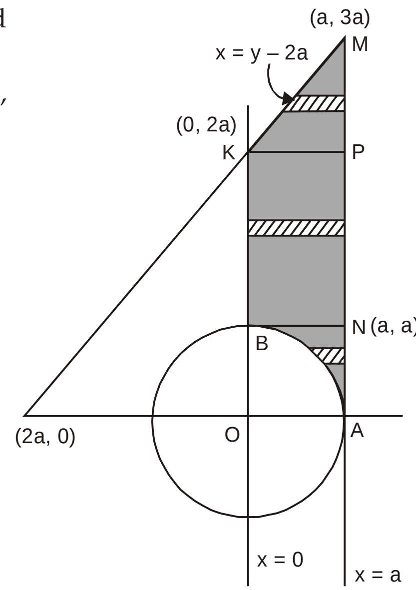


Fig. 4.15

Example 7. Change the order of integration and hence evaluate

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dx\, dy}{\sqrt{y^4 - a^2x^2}}.$$

Sol. The limits are $y^2 = ax$, $y = a$ and $x = 0$, $x = a$, $y = 0$ to $y = a$ and x varies from $x = 0$ to $x = \dfrac{y^2}{a}$.

By changing the order of integration. Hence, the given integral,

$$\int_{x=0}^a \int_{y=\sqrt{ax}}^a \frac{y^2\, dx\, dy}{\sqrt{y^4 - a^2x^2}} = \int_{y=0}^a \int_{x=0}^{y^2/a} \frac{y^2\, dy\, dx}{\sqrt{y^4 - a^2x^2}}$$

$$= \frac{1}{a}\int_0^a \int_0^{\frac{y^2}{a}} \frac{y^2\, dy\, dx}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}$$

$$= \frac{1}{a}\int_0^a y^2 \left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{\frac{y^2}{a}} dy$$

$$= \frac{1}{a}\int_0^a y^2 \;[\sin^{-1}(1) - \sin^{-1}(0)]\; dy$$

$$= \frac{\pi}{2a}\int_0^a y^2 dy = \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi}{6a}(a^3) = \frac{\pi a^2}{6}.$$

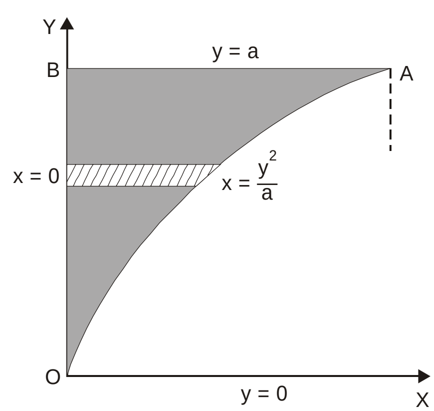


Fig. 4.16

Example 8. By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px\, dx\, dy$ show that

$$\int_0^\infty \frac{\sin px}{x}\, dx = \frac{\pi}{2}.$$

(*U.P.T.U., 2004, 2008*)

Sol. We have

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \sin px \left\{ \int_0^\infty e^{-xy} dy \right\} dx \qquad \text{|As the limits are constants.}$$

$$= \int_0^\infty \sin px \left[\frac{e^{-xy}}{-x} \right]_0^\infty dx$$

$$= \int_0^\infty \frac{\sin px}{x} \, dx \qquad \text{...}(i)$$

Again $\displaystyle\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \left[\int_0^\infty e^{-xy} \sin px \, dx \right] dy$

$$= \int_0^\infty \left[\frac{-e^{-xy}}{p^2 + y^2} (p \cos px + y \sin px) \right]_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2 + y^2} \, dy = \left[\tan^{-1}\left(\frac{y}{p} \right) \right]_0^\infty = \frac{\pi}{2} \qquad \text{...}(ii)$$

Hence from (*i*) and (*ii*)

$$\int_0^\infty \frac{\sin px}{x} \, dx = \frac{\pi}{2}. \quad \textbf{Hence proved.}$$

Example 9. Change the order of integration in the following integral and evaluate:

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy dx \cdot$$

Sol. The limits are $y = \dfrac{x^2}{4a}$ and $y = 2\sqrt{ax}$ and $x = 0, \ x = 4a$.

From the parabola $y^2 = 4ax$ *i.e.*, $x = \dfrac{y^2}{4a}$ and from parabola $x^2 = 4ay$ *i.e.*, $x = 2\sqrt{ay}$ *i.e.*, x varies from $\dfrac{y^2}{4a}$ to $2\sqrt{ay}$.

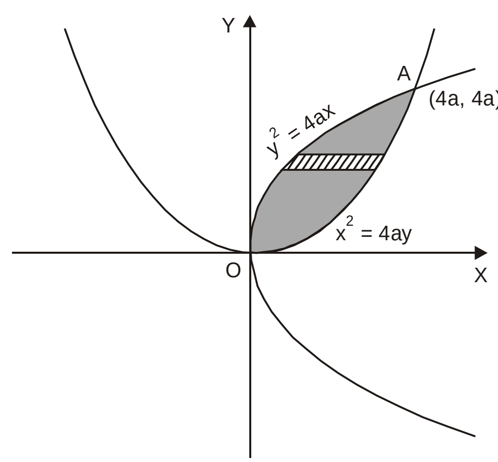


Fig. 4.17

$$\therefore \quad \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy \, dx = \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy \, dx$$

$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a} \right) dy$$

$$= \left[2\sqrt{a} . \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{12a} \right]_0^{4a}$$

$$= \frac{4}{3} \sqrt{a} \cdot (4a)^{\frac{3}{2}} - \frac{64a^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}.$$

Example 10. Change the order of integration evaluate $\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dx\,dy$.

Sol. Here the limits are $x = \frac{y^2}{a}$ *i.e.*, $y^2 = ax$, $x = y$ and $y = 0$, $y = a$

intersection point of parabola and line $y = x$ is $A(a, a)$.

Here x varies from 0 to a and y varies from x to $\sqrt{ax}$

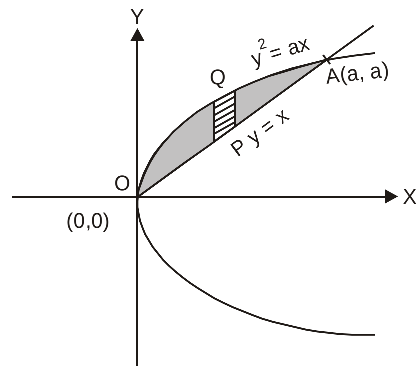


Fig. 4.18

$$\therefore \int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dx\,dy$$

$$= \int_0^a \int_x^{\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dy\,dx$$

$$= \int_0^a \frac{1}{(a-x)}\left[-\left(ax-y^2\right)^{1/2}\right]_x^{\sqrt{ax}} dx \qquad \left| \begin{aligned} ax - y^2 &= t \\ -2y\,dy &= dt \\ \int \frac{y}{\sqrt{ax-y^2}}\,dy &= -\left(ax-y^2\right)^{1/2} \end{aligned} \right.$$

$$= \int_0^a \frac{1}{a-x}\left[0 + \left(ax - x^2\right)^{1/2}\right] dx$$

$$= \int_0^a \frac{\sqrt{ax-x^2}}{(a-x)}\,dx = \int_0^a \frac{\sqrt{x}\,\sqrt{a-x}}{(a-x)}\,dx = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}}\,dx$$

Let $x = a\sin^2\theta$ *i.e.*, $dx = 2a \sin\theta \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{\sqrt{a}\,.\sin\theta}{\sqrt{a}\,.\cos\theta} \cdot 2a \sin\theta \cos\theta\, d\theta$$

$$= 2a\int_0^{\pi/2} \sin^2\theta\, d\theta = 2a\int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) = a\int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta$$

$$= a\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = a\left[\frac{\pi}{2} - 0\right] = \frac{a\pi}{2}.$$

4.6 CHANGE OF VARIABLES IN A MULTIPLE INTEGRAL

Sometimes it becomes easier to evaluate definite integrals by changing one system of variables to another system of variables, such as cartesian coordinates system to polar coordinates system.

Let us consider the transformation of

$$= \iint_S F(x,y)\, dx\, dy \qquad ...(i)$$

when the variables are changed from x and y to u and v by the relations

$$x = \phi\,(u, v),\ y = \psi\,(u, v) \qquad ...(ii)$$

Let these relations transform the function $F(x, y)$ to $F(u, v)$ to express $dx\, dy$ in terms of new variables $u, v,$ we proceed as follows:

First solve equations (*ii*) for $u, v,$ to get

$$u = F_1\,(x, y) \text{ and } v = F_2\,(x, y) \qquad ..(iii)$$

Then u = constant and v = constant form two systems of curves in the xy-plane. Divide the region S into elementary areas by the curve u = constant, $u + \delta u$ = constant, v = constant and $v + \delta v$ = constant.

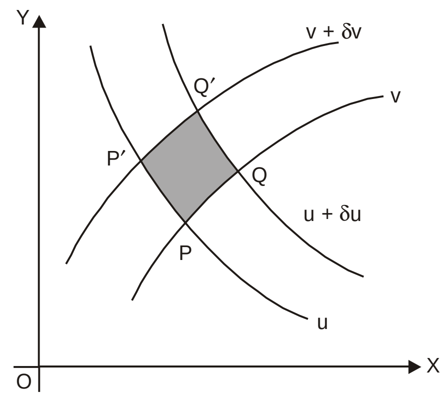


Fig. 4.19

Let P be the intersection of u = constant and v = constant, and Q is the intersection of $u + \delta u$ = constant and v = constant.

Thus, if P is the point (x, y), so that $x = \phi\,(u, v)$, $y = \psi\,(u, v)$

Then Q is the point

$[(\phi(u + \delta u, v), \psi\,(u + \delta u, v)]$

Now using Taylor's theorem to the first order of approximation, we obtain

$$\phi\,(u + \delta u, v) = \phi\,(u, v) + \frac{\delta\phi}{\delta u}\delta u$$

and

$$\psi\,(u + \delta u, v) = \psi\,(u, v) + \frac{\delta\psi}{\delta u}\delta u$$

Therefore, the coordinates of Q are $\left(x + \frac{\delta x}{\delta u}\delta u, y + \frac{\delta y}{\delta u}\delta u\right)$ $\quad\Big|$ As $x = \phi\,(u, v)$, $y = \psi\,(u, v)$

Similarly, P' is the point $\left(x + \frac{\delta x}{\delta v}\delta v, y + \frac{\delta y}{\delta v}\delta v\right)$

and Q' is the point $\left(x + \frac{\delta x}{\delta u}\delta u + \frac{\delta x}{\delta v}\delta v, y + \frac{\delta y}{\delta u}\delta u + \frac{\delta y}{\delta v}\delta v\right)$

Therefore, to the first order of approximation $PQQ'\,P'$ will be a parallelogram and its area would be double that of the triangle PQP'.

Hence the elementary area $PQP'\,Q' = [2\ \Delta\ PQP']$

$$= 2 \times \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x + \frac{\delta x}{\delta u}\delta u & y + \frac{\delta y}{\delta u}\delta u & 1 \\ x + \frac{\delta x}{\delta v}\delta v & y + \frac{\delta y}{\delta v}\delta v & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ \frac{\delta x}{\delta u}\delta u & \frac{\delta y}{\delta u}\delta u & 0 \\ \frac{\delta x}{\delta v}\delta v & \frac{\delta y}{\delta v}\delta v & 0 \end{vmatrix} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta y}{\delta u} \\ \frac{\delta x}{\delta v} & \frac{\delta y}{\delta v} \end{vmatrix}.\delta u \delta v$$

$$= \frac{\delta(x,y)}{\delta(u,v)} .\ \delta u\ \delta v = J.\delta u\ \delta v \quad ..(iv)$$

Thus, if the whole region S be divided into elementary areas by the system of curves

$$F_1 = \text{constant and } F_2 = \text{constant then, we have}$$

$$\lim \Sigma\ F\ (x, y)\ \delta S = \lim\ \Sigma F\ (u, v)|\ J\ |\ .\ \delta u\ \delta v$$

i.e.,

$$\boxed{\iint_S F(x,y)\,dx\,dy = \iint_S F(u,v)\,|\ J\ |\,du\,dv} \quad ...(v)$$

$$J = \frac{\delta(x,y)}{\delta(u,v)} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta y}{\delta u} \\ \frac{\delta x}{\delta v} & \frac{\delta y}{\delta v} \end{vmatrix} \text{ or } \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix}$$

Hence,

$$dx\ dy = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta y}{\delta u} \\ \frac{\delta x}{\delta v} & \frac{\delta y}{\delta v} \end{vmatrix} du\ dv.$$

Example 11. Transform to polar coordinates and integrates

$$\iint \sqrt{\left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)}\ dx\ dy$$

the integral being extended over all positive values of x and y subject to $x^2 + y^2 \leq 1$.

Sol. Put $x = r\cos\theta,\ y = r\sin\theta$

then

$$\frac{\delta(x,y)}{\delta(r,\theta)} = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Therefore, $dx\ dy = rd\theta\ dr$ and limits of r are from 0 to 1 and those of θ from 0 to $\frac{\pi}{2}$,

Hence

$$I = \iint \sqrt{\left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)}\ dx\ dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\left(\frac{1-r^2}{1+r^2}\right)}\ r\ d\theta\ dr$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{\left(\frac{1-r^2}{1+r^2}\right)} r\ dr$$

Now, suppose $r^2 = \cos\phi,\ 2rdr = -\sin\phi\ d\phi$

$$= \frac{\pi}{2} \int_{\frac{\pi}{2}}^0 \frac{1}{2} \sqrt{\left(\frac{1-\cos\phi}{1+\cos\phi}\right)}\ (-\sin\phi)\ d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}(1-\cos\phi)\,d\phi \qquad \left| \text{As } \int_0^a F(x)dx = -\int_a^0 F(x)dx \right.$$

$$= \frac{\pi}{4}\left[\phi - \sin\phi\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}\left(\frac{\pi}{2}-1\right).$$

Example 12. Evaluate $\iint_R (x+y)^2\ dx\ dy$, where R is region bounded by the parallelogram $x + y = 0$, $x + y = 2$, $3x - 2y = 0$, $3x - 2y = 3$. (*U.P.T.U., 2006*)

Sol. By changing the variables x, y to the new variables u, v, by the substitution $x + y = u$, $3x - 2y = v$, the given parallelogram R reduces to a rectangle R^* as shown in the Figure 4.20.

$$\frac{\delta(v,v)}{\delta(x,y)} = \begin{vmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -5$$

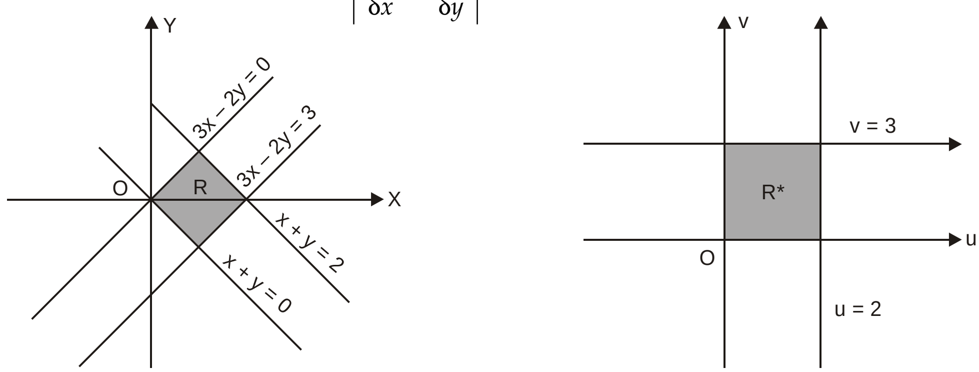


Fig. 4.20

So required Jacobian $\quad J = \frac{\delta(x,y)}{\delta(u,v)} = -\frac{1}{5}$

Since $u = x + y = 0$ and $u = x + y = 2$, u varies from 0 to 2, while v varies from 0 to 3 since $3x - 2y = v = 0$, $3x - 2y = v = 3$.

Thus, the given integral in terms of the new variables u, v is

$$\iint_R (x+y)^2\ dx\ dy = \iint_{R^*} u^2 \left|\frac{1}{-5}\right|\ du\ dv$$

$$= \frac{1}{5}\int_0^3\int_0^2 u^2 du\ dv$$

$$= \frac{1}{5}\int_0^3\left[\frac{u^3}{3}\right]_0^2 dv = \frac{8}{15}\cdot[v]_0^3 = \frac{8}{5}.$$

Example 13. $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\ dy$ by changing to polar coordinates hence, show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. (*U.P.T.U., 2002*)

Sol. Let $\quad x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = r^2$

Here r varies from 0 to ∞

and $\theta = \tan^{-1}\frac{y}{x}$

Now, $\theta = \tan^{-1} 0 = 0,\ \theta = \tan^{-1} \infty = \dfrac{\pi}{2}$

$\Rightarrow$ θ varies from 0 to $\dfrac{\pi}{2}$

Hence, $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\, dr\, d\theta$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^\infty (-2r) e^{-r^2}\, dr\, d\theta = -\frac{1}{2} \int_0^{\frac{\pi}{2}} \left[e^{-r^2} \right]_0^\infty d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (0-1)\, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} 1\, d\theta = \frac{\pi}{4}.$$

Let $\displaystyle I = \int_0^\infty e^{-x^2}\, dx$ ...(*i*)

$\displaystyle I = \int_0^\infty e^{-y^2}\, dy$ ...(*ii*)

[Property of definite integrals]

Multiplying (*i*) and (*ii*), we get

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \frac{\pi}{4}.$$

[As obtained above]

$$I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$ **Proved.**

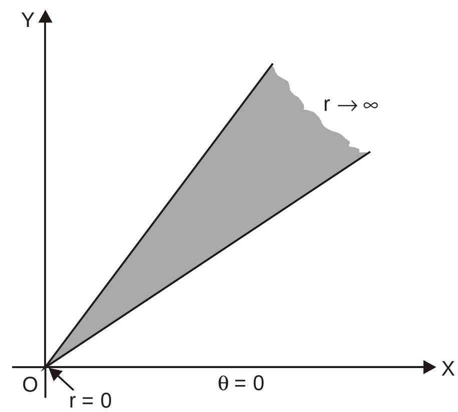


Fig. 4.21

Example 14. Evaluate $\displaystyle\int_0^1 \int_0^x \sqrt{x^2+y^2}\, dx\, dy$, the transformation is $x = u,\ y = uv$.

Sol. Region of integration R is the triangle bounded by $y = 0,\ x = 1$ and $y = x$

Put $x = u,\ y = uv$

$$J = \text{Jacobian} = \frac{\delta(x, y)}{\delta(u, v)} = \begin{vmatrix} \dfrac{\delta x}{\delta u} & \dfrac{\delta x}{\delta v} \\ \dfrac{\delta y}{\delta u} & \dfrac{\delta y}{\delta v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$$

In the given region R, x: varies from 0 to 1 while y varies from 0 to x. Since $u = x$, so u varies from 0 to 1.

Similarly, since $0 \le y = uv \le x = u$, so v varies from 0 to 1. Thus

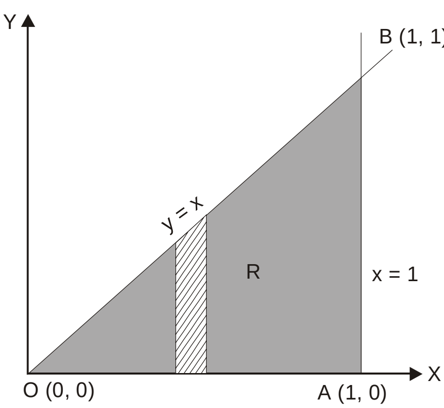


Fig. 4.22

$$= \int_0^1 \int_0^x \sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_0^1 \int_0^1 u\sqrt{1+v^2}\, u\,du\, dv = \frac{1}{3} \int_0^1 \sqrt{1+v^2}\, dv$$

$$\frac{1}{3}\left[\frac{v\sqrt{(1+v^2)}}{2}+\frac{1}{2}\sin h^{-1}v\right]_0^1$$

$$= \frac{1}{3}\left[\frac{\sqrt{2}}{2}+\frac{1}{2}\sin h^{-1}1\right].$$

Example 15. Evaluate $\iint_R (x+y)^2 \; dx\,dy$, where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$ and $v = x - 2y$.

(*U.P.T.U., 2003*)

Sol. Since $u = x + y$ and $v = x - 2y$

$\therefore\ u = 1 + 0 = 1,\ v = 1 - 0 = 1$

Similarly, $u = 4\ v = 1,\ u = 4,\ v = -2$

and $u = 1,\ v = -2$

and $$J = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$\therefore\ dx\,dy = |J|\,du\,dv = \frac{1}{3}\,du\,dv$$

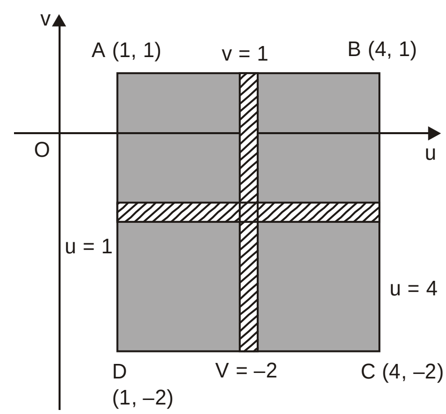


Fig. 4.23

Thus, $$\iint_R (x+y)^2\; dx\,dy = \int_{-2}^{1}\int_1^4 u^2.\frac{1}{3}\,du\,dv$$

$$= \int_{-2}^{1}\frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv = \int_{-2}^{1} 7dv = 21.$$

Example 16. Evaluate the following by changing into polar coordinates

$$\int_0^a\int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\;dy\,dx.$$ (*U.P.T.U., 2007*)

Sol. Here $y = 0,\ y = a$ and $x = 0,\ x = \sqrt{a^2-y^2}$

Let $x = r\cos\theta,\ \ y = r\sin\theta\ \ \therefore\ \ x^2 + y^2 = r^2$ or $r = a$

Also, when $x = 0,\ r\cos\theta = 0 \Rightarrow r = 0$, or $\theta = \frac{\pi}{2}$

when $y = 0,\ r\sin\theta = 0 \Rightarrow r = 0$ or $\theta = 0$

when $y = a,\ r\sin\theta = a \Rightarrow a\sin\theta = a \Rightarrow \theta = \frac{\pi}{2}$

Hence, we have

$$\int_0^a\int_0^{\sqrt{a^2-y^2}} y^2.\sqrt{x^2+y^2}\;dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2\sin^2\theta\, r.\,r\,dr\,d\theta = \int_0^{\pi/2}\sin^2\theta\,d\theta.\int_0^a r^4.dr$$

$$= \frac{a^5}{5}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{a^5}{10}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi a^5}{20}.$$

Example 17. Transform $\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{\frac{\sin\phi}{\sin\theta}}\,d\phi\,d\theta$ by the transformation $x = \sin\phi\cos\theta$, $y = \sin\phi.\sin\theta$ and show that its value is π.

Sol. We have $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$

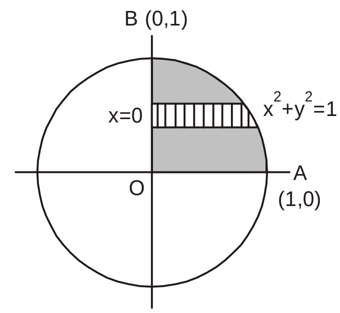


Fig. 4.24

$\therefore \quad x^2 + y^2 = \sin^2\phi$ and $\theta = \tan^{-1}\frac{y}{x}$

when $\theta = 0, \tan^{-1}\frac{y}{x} = 0 \Rightarrow y = 0$

when $\theta = \frac{\pi}{2}, \tan^{-1}\frac{y}{x} = \frac{\pi}{2} \Rightarrow x = 0$

when $\phi = 0, x^2 + y^2 = 0 \Rightarrow x = 0$ or $y = 0$

when $\phi = \frac{\pi}{2}, x^2 + y^2 = 1$

Now $$J = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{vmatrix} = \begin{vmatrix} -\sin\phi.\cos\theta & \cos\phi\ \cos\theta \\ \sin\phi\cos\theta & \cos\phi\ \sin\theta \end{vmatrix} = -\sin\phi\cos\phi$$

$$\therefore \quad d\phi\,d\theta = \frac{1}{|J|}dxdy = \frac{1}{\sin\phi\cos\phi}.dxdy$$

Thus $$\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{\frac{\sin\phi}{\sin\theta}}\,d\phi\,d\theta = \int_0^1\int_0^{\sqrt{1-y^2}}\sqrt{\frac{\sin\phi}{\sin\theta}}\cdot\frac{1}{\sin\phi\cos\phi}\cdot dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}}\frac{1}{\cos\phi\sqrt{\sin\phi\sin\phi}}\cdot dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}}\frac{1}{\left(\sqrt{1-\sin^2\phi}\right)\sqrt{y}}dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}}\frac{dy\,dx}{\sqrt{1-\left(x^2+y^2\right)}\cdot\sqrt{y}} = \int_0^1\int_0^{\sqrt{1-y^2}}\frac{dy\,dx}{\sqrt{\left(1-y^2\right)-x^2}\ \sqrt{y}}$$

$$= \int_0^1\left[\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}}\cdot\frac{dy}{\sqrt{y}} = \frac{\pi}{2}\int_0^1 y^{-1/2}dy$$

$$= \frac{\pi}{2}\left[2y^{1/2}\right]_0^1 = \pi.$$ **Hence proved.**

EXERCISE 4.2

Change the order of integration and then evaluate the following double integrals:

1. $\int_0^2 \int_{y^3}^{4\sqrt{2y}} y^2 \, dx \, dy.$ $\left[\textbf{Ans. } \frac{160}{21}\right]$

2. $\int_1^2 \int_3^4 (x+y) \, dx \, dy.$ $[\textbf{Ans. } 5]$

3. $\int_0^1 \int_{y^2}^{y^{1/3}} f(x, y) \, dx \, dy.$ $\left[\textbf{Ans. } \int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y) \, dy \, dx\right]$

4. $\int_{-1}^2 \int_{-x}^{2-x^2} f(x, y) \, dy \, dx.$ $\left[\textbf{Ans. } \int_{-2}^1 \int_{-y}^{\sqrt{2-y}} f(x, y) dx \, dy + \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x, y) dx \, dy\right]$

5. $\int_0^2 \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y \, dx \, dy.$ $\left[\textbf{Ans. } \frac{8}{3}\right]$

6. $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2) \, dy \, dx.$ $\left[\textbf{Ans. } \frac{a^3}{28} + \frac{a}{20}\right]$

7. $\int_0^{2a} \int_0^{\sqrt{2ay - y^2}} dx \, dy.$ $\left[\textbf{Ans. } \frac{\pi a^2}{2}\right]$

8. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2} \, dy \, dx.$ $\left[\textbf{Ans. } \frac{\pi a^3}{a}\right]$

Change the following integrals in to polar coordinates and show that

9. $\int_0^a \int_y^a \frac{1}{x^2 + y^2} \, dy \, dx = \frac{\pi a}{4}.$

10. $\int_0^1 \int_x^{\sqrt{(2x - x^2)}} (x^2 + y^2) \, dx \, dy = \frac{3\pi}{8} - 1.$

11. $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} y\sqrt{(x^2 + y^2)} \, dx \, dy = \frac{\pi a^5}{20}.$

Evaluate the following by changing the order of integration:

12. $\int_0^a \int_0^{bx/a} x \, dy \, dx.$ $\left[\textbf{Ans. } \frac{1}{3} a^2 b\right]$

13. $\int_0^b \int_0^{\frac{a}{b}\sqrt{(b^2 - y^2)}} xy \, dx \, dy.$ $\left[\textbf{Ans. } \frac{a^2 b^2}{8}\right]$

14. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy.$ $\left[\textbf{Ans.} = \frac{2}{3}a^4\right]$

15. $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx.$ $\left[\textbf{Ans.} = \frac{16a^2}{3}\right]$

16. $\int_0^1\int_0^1 (x^2+y^2)\,dx\,dy + \int_1^2\int_0^{2-y}(x^2+y^2)\,dx\,dy.$ $\left[\textbf{Ans.} = \frac{5}{3}\right]$

17. $\int_0^1 \int_{e^x}^{e} \frac{1}{\log y}\cdot dx\,dy$ $\left[\textbf{Ans.} = e-1\right]$

18. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy.$ $\left[\textbf{Ans.} = 1\right]$

Change the order of integration in the following integrals:

19. $\int_0^a \int_{mx}^{lx} V(x,y)\,dx\,dy.$ $\left[\textbf{Ans.} \int_0^{ma}\int_{y/l}^{y/m} V(x,y)dy\,dx + \int_{ma}^{la}\int_{\frac{y}{l}}^{a} V(x,y)dy\,dx.\right]$

20. $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} V dx\,dy.$ $\left[\textbf{Ans.} \int_0^a\int_{\frac{y2}{2a}}^{a-\sqrt{a^2-y^2}} V dy\,dx + \int_0^a \int_{a+\sqrt{(a^2-y^2)}}^{2a} V dy\,dx + \int_a^{2a}\int_{\frac{y2}{2a}}^{2a} V dy\,dx\right]$

21. $\int_a^{a\cos a} \int_{x\tan a}^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$ $\left[\textbf{Ans.} \int_0^{a\sin a}\int_0^{y\cot a} f(x,y)dy\,dx + \int_{a\sin a}^{a}\int_0^{\sqrt{(a^2-x^2)}} f(x,y)dy\,dx\right]$

22. $\int_1^2 \int_x^{x^2} (x^2+y^2)\,dx\,dy.$ $\left[\textbf{Ans.}\ 9\frac{61}{105}\right]$

23. $\int_0^1\int_x^{\sqrt{(2-x^2)}} \frac{x\,dx\,dy}{\sqrt{(x^2+y^2)}}.$ $\left[\textbf{Ans.}\ 1-\frac{\sqrt{2}}{2}\right]$

24. Using the transformation $x + y = u$, $y = uv$ show that

$$\int_0^1\int_0^{1-x} e^{y}/(x+y)\,dy\,dx = \frac{1}{2}(e-1).$$

25. $\iint [xy(1-x-y)]^{\frac{1}{2}}\,dx\,dy = \frac{2\pi}{105}$, integration being taken over the area of the triangle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$.

26. $\iint_D (y-x)\ dx\ dy$, D: region in xy-plane bounded by the straight lines $y = x + 4$,

$$y = x - 3,\ y = -\frac{1}{3}x + \frac{7}{3},\ y = -\frac{1}{3}x + 5.$$ **[Ans. 8]**

27. $\iint_R (x+y)^2\ dx\ dy$. R: parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1).

[Ans. 21]

28. $\iint_{D_e} (x-y)/(x+y)\ dx\ dy$, D: triangle bounded by $y = 0$, $x = 1$ and $y = x$. Use $x = u - uv$, $y = uv$ to transfrom the double integrals. $\left[\textbf{Ans. } \frac{(e^2-1)}{4e}\right]$

29. Find $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta$ by transforming it into cartesian coordinate. **[Ans. = 0]**

4.7 BETA AND GAMMA FUNCTIONS *(U.P.T.U., 2007)*

The first and second Eulerian Integrals which are also called "Beta and Gamma functions" respectively are defined as follows:

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$$

and

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$ *[U.P.T.U. (C.O.), 2003]*

$\beta(m, n)$ is read as "Beta m, n" and Γn is read as "Gamma n". Here the quantities m and n are positive numbers which may or may not be integrals.

4.7.1 Properties of Beta and Gamma Functions

(a) The function $\beta(m, n)$ is symmetrical w.r.t. m, n *i.e.,*

$$\beta(m, n) = \beta(n, m)$$

we have $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx$

putting $1 - x = t \Rightarrow dx = -dt$

$$= -\int_1^0 (1-t)^{m-1} \cdot t^{n-1}\ dt = \int_0^1 (1-t)^{m-1} \cdot t^{n-1}\ dt$$

$$= \int_0^1 (1-x)^{m-1} \cdot x^{n-1} \cdot dx = \int_0^1 x^{n-1}(1-x)^{m-1} \quad \left(\text{By } \int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right)$$

$$\Rightarrow \quad \boxed{\beta(m, n) = \beta(n, m)}.$$

(*b*) Evaluation of Beta Function β (*m*, *n*)

$$\boldsymbol{\beta(m, n) = \frac{\lfloor\underline{m-1}\,\lfloor\underline{n-1}}{\lfloor\underline{(m+n-1)}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}} \qquad \text{(U.P.T.U., 2008)}$$

We have $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}.\,dx$

$$= \left[\frac{x^m}{m}(1-x)^{n-1}\right]_0^1 + \frac{(n-1)}{m}\int_0^1 x^m\,(1-x)^{n-2}\cdot dx$$

(Integrate by part)

or
$$\beta(m, n) = \frac{(n-1)}{m}\int_0^1 x^m\,(1-x)^{n-2}.\,dx$$

Again integrate by parts above, we get

$$\beta(m, n) = \frac{(n-1)}{m}.\,\frac{(n-2)}{m+1}\int_0^1 x^{m+1}(1-x)^{n-3}.\,dx$$

Continuing the above process of integrating by parts

$$\beta(m, n) = \frac{(n-1)(n-2)\ldots 2\cdot 1}{m\,(m+1)\ldots(m+n-2)}\int_0^1 x^{m+n-2}\,.\,dx$$

or
$$\boldsymbol{\beta(m, n) = \frac{\lfloor\underline{n-1}}{m\,(m+1)\ldots(m+n-2)(m+n-1)}} \qquad \ldots(i)$$

n is a positive integer

In case *n* is alone a positive integer.

Similarly, if *m* is positive integer, then $\beta(m, n) = \beta(n, m)$

$$\boldsymbol{\beta(n, m) = \frac{\lfloor\underline{m-1}}{n(n+1)\ldots(n+m-1)}} \qquad \ldots(ii)$$

In case both *m* and *n* are positive integer, then multiplying (*i*) numerator and denominator by 1·2·3... (*m* – 1) or (*ii*) by 1·2·3... (*n* – 1), then, we get

$$\beta(m, n) = \frac{1\cdot 2\cdot 3\ldots.(m-1)\lfloor\underline{n-1}}{1\cdot 2\cdot 3\ldots(m-1)\cdot m(m+1)\ldots(m+n-1)}$$

or
$$\boxed{\beta(m, n) = \frac{\lfloor\underline{m-1}\,\lfloor\underline{n-1}}{\lfloor\underline{m+n-1}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}\,.$$

(c) Evaluation of Gamma Function

$$\boldsymbol{\Gamma(n) = (n-1)\,\Gamma(n-1) \text{ or } \Gamma(n+1) = n\,\Gamma(n) = \lfloor\underline{n}}$$

we know that $\Gamma(n) = \int_0^\infty x^{n-1}.e^{-x}\cdot dx$

Integrating by parts keeping x^{n-1} as first function

$$\Gamma(n) = \left[-e^{-x}\cdot x^{n-1}\right]_0^\infty + (n-1)\int_0^\infty x^{n-2}\cdot e^{-x}\,dx$$

$$= \left[-\lim_{x\to\infty} e^{-x}\cdot x^{n-1} + 0\right] + (n-1)\int_0^\infty x^{n-2}.e^{-x}\,dx \quad ...(i)$$

$$\left| \text{As } \lim_{x\to 0} e^{-x}.x^{n-1} = 0 \right.$$

But $$\lim_{x\to\infty} e^{-x}\cdot x^{n-1} = \lim_{x\to\infty}\cdot\frac{x^{n-1}}{e^x} = \lim_{x\to\infty}\frac{x^{n-1}}{1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^n}{\lfloor n}+\ldots}$$

$$= \lim_{x\to\infty}\frac{1}{\frac{1}{x^{n-1}}+\frac{1}{x^{n-2}}+\frac{1}{\lfloor 2\cdot x^{n-3}}\ldots} = \frac{1}{\infty} = 0$$

∴ From (i), we get

$$\Gamma n = 0 + (n-1)\int_0^\infty x^{n-2}\cdot e^{-x}\,dx = (n-1)\int_0^\infty x^{(n-1)-1}\cdot e^{-x}\,dx$$

$$\Rightarrow \quad \boxed{\Gamma n = (n-1)\,\Gamma(n-1)}. \quad ...(ii)$$

Replace n by $n + 1$ in equation (ii) then, we get

$$\boxed{\Gamma(n+1) = n\,\Gamma n}. \quad ...(iii)$$

(d) $$\Gamma(n+1) = \lfloor n$$

from (i) $$\Gamma n = (n-1)(n-2) \ldots 3.\ 2.\ 1$$

$$\Gamma n = \lfloor n-1$$

replace n by $(n + 1)$, then

$$\boxed{\Gamma(n+1) = \lfloor n}.$$

4.8 TRANSFORMATIONS OF GAMMA FUNCTION

We have $$\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx \quad ...(i)$$

(1) Put $x = \lambda y$ or $dx = \lambda\, dy$

Then from (i), we get $$\Gamma n = \int_0^\infty e^{-\lambda y}(\lambda y)^{n-1}\,\lambda\,dy$$

$$= \lambda^n\int_0^\infty e^{-\lambda y}y^{n-1}\,dy$$

or $$\boxed{\int_0^\infty y^{n-1}\,e^{-\lambda y}\,dy = \frac{\Gamma n}{\lambda^n}} \quad ...(ii)$$

(2) Put $x^n = z$ in (i) then $nx^{n-1}\,dx = dz$ and $x = z^{1/n}$

$\therefore$ From (i), we get
$$\Gamma n = \int_0^\infty e^{-z^{1/n}} \left(\frac{1}{n}\right) dz$$

or
$$\boxed{\int_0^\infty e^{-z^{1/n}}\,dz = n\Gamma n = \Gamma(n+1)} \quad ...(iii)$$

(3) Put $e^{-x} = t$ in (i).

Then $-e^{-x}\,dx = dt$ and $e^x = \dfrac{1}{t}$

$\therefore$ From (i), we get
$$\Gamma n = \int_1^0 (-\log t)^{n-1}(-dt)$$
$$= \int_0^1 \left[\log\left(\frac{1}{t}\right)\right]^{n-1} dt$$

$\therefore$
$$\boxed{\int_0^1 \left[\log\left(\frac{1}{t}\right)\right]^{n-1} dt = \Gamma n} \quad ...(iv)$$

(4) Value of $\Gamma\left(\frac{1}{2}\right)$. Putting $n = \frac{1}{2}$ in (iii), we get

$$\frac{1}{2}\Gamma\frac{1}{2} = \int_0^\infty e^{-z^2}\,dz = \frac{1}{2}\sqrt{\pi} \qquad \left|\text{As } \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\right.$$

or
$$\boxed{\Gamma\frac{1}{2} = \sqrt{(\pi)}}\,.$$

4.9 TRANSFORMATIONS OF BETA FUNCTION

We know that
$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ...(i)$$

(1) Putting $x = \dfrac{1}{1+y}$ or $dx = -\dfrac{1}{(1+y)^2}\,dy$

Also
$$(1-x) = 1 - \frac{1}{1+y} = \frac{y}{(1+y)}$$

Also, when $x = 0$ then $y = \infty$ and when $x = 1$ then $y = 0$

$\therefore$ From (i), we get
$$\beta(m, n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \cdot \left(\frac{y}{1+y}\right)^{n-1} \left\{\frac{-1}{(1+y)^2}\right\} dy$$
$$= \int_0^\infty \frac{y^{n-1}dy}{(1+y)^{m-1+n-1+2}}$$

or
$$\boxed{\beta(m, n) = \int_0^\infty \frac{y^{n-1}dy}{(1+y)^{m+n}}} \quad ...(ii)$$

(2) Also, as $\beta(m, n) = \beta(n, m)$,
therefore interchanging m and n in (*ii*), we have

$$\boxed{\beta(n, m) = \int_0^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}}}. \qquad (U.P.T.U., 2008)$$

4.10 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

[*U.P.T.U. (C.O.), 2002, U.P.T.U., 2003*]

We know that $\int_0^\infty y^{n-1} e^{-xy}\, dy = \frac{\Gamma n}{x^n}$ [§ 4.8 eqn. (*ii*)]

$$\text{or} \quad \Gamma n = \int_0^\infty x^n y^{n-1} e^{-xy}\, dy \qquad \text{...}(i)$$

Also $\Gamma m = \int_0^\infty x^{m-1} e^{-x}\, dx$ ...(*ii*)

Multiplying both sides of (*i*) by $x^{m-1} e^{-x}$, we get

$$\Gamma n \cdot x^{m-1} e^{-x} = \int_0^\infty x^{n+m-1} y^{n-1} e^{-(y+1)x} dy$$

Integrating both sides with respect to x within limits $x = 0$ to $x = \infty$, we have

$$\Gamma n \int_0^\infty x^{m-1} e^{-x} dx = \int_0^\infty \left[\int_0^\infty x^{n+m-1} e^{-(y+1)x} dx\, y^{n-1} dy\right]$$

But $\int_0^\infty x^{(n+m)-1} e^{-(y+1)x} dx = \frac{\Gamma(n+m)}{(1+y)^{m+n}}$ ...(*iii*)

[by putting $\lambda = 1 + y$ and 'n' $= m + n$ in 4.8 on (*ii*)]

Using this result in (*ii*), we get

$$\Gamma n\, \Gamma m = \int_0^\infty \Gamma(n+m) \cdot \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \Gamma(m+n) \int_0^\infty \frac{y^{n-1} dy}{(1+y)^{m+n}}$$

$$= \Gamma(m+n) \cdot \beta(m, n)$$

$$\therefore \quad \boxed{\beta(m, n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}}.$$

4.11 SOME IMPORTANT DEDUCTIONS

(1) To prove that $\Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ (*U.P.T.U., 2008*)

We know that $\beta(m, n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$.

Putting $m + n = 1$ or $m = (1 - n)$, we get

$$\frac{\Gamma n \cdot \Gamma(1-n)}{\Gamma 1} = \beta(n, 1-n) \quad ...(i)$$

But
$$\beta(m, n) = \int_0^\infty \frac{y^{n-1} dy}{(1+y)^{m+n}}$$

$\therefore$
$$\beta(n, 1-n) = \int_0^\infty \frac{y^{n-1} dy}{(1+y)} \qquad \left| \text{As} \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi} \right.$$

$$= \frac{\pi}{\sin n\pi}, \ n < 1$$

$\therefore$ From (*i*), we have $\boxed{\Gamma n \, \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}}$.

(2) To prove that $\Gamma(1+n)\,\Gamma(1-n) = \dfrac{n\pi}{\sin n\pi}$.

We have
$$\Gamma n \,.\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Multiplying both sides by n, we get

$$n\Gamma n \, \Gamma(1-n) = \frac{n\pi}{\sin n\pi}$$

or
$$\Gamma(1+n)\,\Gamma(1-n) = \frac{n\pi}{\sin n\pi}. \qquad \left| \text{As } \Gamma(1+n) = n\Gamma n \right.$$

(3) To prove that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$.

We know that
$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx \quad ...(i)$$

$$= \frac{\Gamma q \cdot \Gamma q}{\Gamma(p+q)}.$$

Putting $x = \sin^2\theta$ and $dx = 2 \sin \theta \cos \theta \, d\theta$, then, we get from (*i*)

$$\beta(p, q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1}(1-\sin^2\theta)^{q-1} \,.\, 2 \sin \theta \cos \theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta$$

$\therefore$
$$\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta = \frac{1}{2}\beta(p, q)$$

$$= \frac{\Gamma p \cdot \Gamma q}{2\Gamma(p+q)}$$

Putting $2p - 1 = m$ and $2q - 1 = n$,

or $$p = \left(\frac{m+1}{2}\right) \text{ and } q = \left(\frac{n+1}{2}\right), \text{ we get}$$

$$\boxed{\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}}.$$

4.12 DUPLICATION FORMULA

To prove that $\Gamma m \; \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{(\pi)}}{2^{2m-1}} \cdot \Gamma 2m.$ *(U.P.T.U., 2001)*

Hence show that $\beta\,(m, m) = 2^{1-2m}\,\beta\left(m, \frac{1}{2}\right).$

Since $$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \qquad ...(i)$$

Putting $2n - 1 = 0$ or $n = \frac{1}{2}$, we obtain

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(m+\frac{1}{2}\right)}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \cdot \sqrt{\pi}}{2\,\Gamma\left(m+\frac{1}{2}\right)} \qquad ...(ii)$$

$$\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Again putting $n = m$ in (*i*), we get

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\{\Gamma(m)\}^2}{2\,\Gamma(2m)}$$

i.e., $$\frac{1}{2^{2m-1}} \int_0^{\pi/2} (2 \sin\theta \cos\theta)^{2m-1} d\theta = \frac{\{\Gamma(m)\}^2}{2\,\Gamma(2m)}$$

i.e., $$\frac{1}{2^{2m}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} 2\, d\theta = \frac{\{\Gamma(m)\}^2}{2\,\Gamma(2m)}$$

Putting $2\theta = \phi$ and $2\ d\theta = d\phi$, we get

$$\frac{1}{2^{2m}}\int_0^{\pi}\sin^{2m-1}\phi\ d\phi = \frac{\{\Gamma(m)\}^2}{2\Gamma(2m)}$$

or $$\frac{2}{2^{2m}}\int_0^{\pi/2}\sin^{2m-1}\phi\ d\phi = \frac{\{\Gamma(m)\}^2}{2\Gamma(2m)} \qquad \left|\ \text{As } \int_0^a f(x)\,dx = 2\int_0^{\pi/2} f(x)\,dx\right.$$

or $$\int_0^{\pi/2}\sin^{2m-1}\theta\ d\theta = \frac{2^{2m-1}\{\Gamma(m)\}^2}{2\Gamma(2m)} \qquad ...(iii)$$

$$\left(\text{As } \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$$

Equating two values of $\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta$ from (*ii*) and (*iii*), we get

$$\frac{2^{2m-1}[\Gamma(m)]^2}{2\Gamma(2m)} = \frac{\Gamma(m)\cdot\sqrt{(\pi)}}{2\Gamma\left(m+\frac{1}{2}\right)} \qquad \left|\ \text{As } \Gamma\frac{1}{2} = \sqrt{\pi}\right.$$

Hence, $$\boxed{\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{(\pi)}}{2^{2m-1}}\Gamma(2m)}$$

$\Rightarrow$ Multiplying above by Γm, we get

$$\Gamma m\,\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\,\Gamma 2m\cdot\Gamma m}{2^{2m-1}} \qquad \left|\ \text{As } \sqrt{\pi} = \Gamma\frac{1}{2}\right.$$

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = \frac{2^{1-2m}\,\Gamma m\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\boxed{\beta\,(m, m) = 2^{1-2m}\,\beta\left(m, \frac{1}{2}\right)} \qquad \left|\ \text{As, } \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}.\right.$$

Example 1. Compute: $\Gamma\left(-\frac{1}{2}\right), \Gamma\left(-\frac{3}{2}\right), \Gamma\left(-\frac{5}{2}\right)$.

Sol. We have $\Gamma n\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ ...(*i*)

(1) Putting $n = -\frac{1}{2}$, in (*i*), we get

$$\Gamma\left(-\frac{1}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right) = \frac{\pi}{\sin\left(-\frac{1}{2}\pi\right)} = -\pi$$

or $$\Gamma\left(-\frac{1}{2}\right)=-\frac{\pi}{\Gamma\frac{3}{2}} = -\frac{\pi}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}=-2\sqrt{\pi} \qquad ...(ii)$$

$$\Rightarrow \quad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\,.$$

(2) Putting $n=-\frac{3}{2}$, in (i), we get

$$\Gamma\left(-\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right) = \frac{\pi}{\sin\left(-\frac{3}{2}\pi\right)} = -\frac{\pi}{\left(\sin\frac{3}{2}\pi\right)} = \pi$$

or $$\Gamma\left(-\frac{3}{2}\right) = \frac{\pi}{\Gamma\left(\frac{5}{2}\right)} = \frac{\pi}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}=\frac{4\sqrt{\pi}}{3}. \qquad ...(iii)$$

(3) Putting $n=-\frac{5}{2}$, in (i), we get

$$\Gamma\left(-\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right) = \frac{\pi}{\sin\left(-\frac{5}{2}\pi\right)}=-\frac{\pi}{\left(\sin\frac{5}{2}\pi\right)}=-\pi$$

or $$\Gamma\left(-\frac{5}{2}\right)=\frac{\pi}{\Gamma\left(\frac{7}{2}\right)} = -\frac{\pi}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}=-\frac{8\sqrt{\pi}}{15}. \qquad ...(iv)$$

Example 2. Evaluate $\int_0^\infty \frac{dx}{1+x^4}$.

Sol. Putting $x^4 = y$ or $dx = \frac{dy}{4x^3}$, we have

$$\int_0^\infty \frac{dx}{1+x^4} = \int_0^\infty \frac{\frac{1}{4}y^{-\frac{3}{4}}\,dy}{(1+y)}$$

$$= \frac{1}{4}\int_0^\infty \frac{y^{\left(\frac{1}{4}-1\right)}}{1+y}\,dy\cdot$$

Also, we have $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$, we get

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{4}\cdot\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}=\frac{\pi\sqrt{2}}{4}.$$

Example 3. Show that $\int_0^{\pi/2} \tan^n \theta \, d\theta = \frac{1}{2}\pi \sec\left(\frac{1}{2}n\pi\right)$.

Sol. We have

$$\int_0^{\pi/2} \tan^n \theta \, d\theta = \int_0^{\pi/2} \sin^n \theta \, (\cos\theta)^{-n} \, d\theta$$

$$= \frac{\left\lceil \left\{\frac{1}{2}(n+1)\right\} \cdot \left\lceil \left\{\frac{1}{2}(-n+1)\right\}}{2 \left\lceil \left\{\frac{1}{2}(n-n+2)\right\}}$$

$$= \frac{\left\lceil \frac{(n+1)}{2} \cdot \left\lceil \left(1-\frac{n+1}{2}\right)}{2\lceil 1}$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\frac{1}{2}(n+1)\pi} \qquad \left| \text{As } \lceil n \lceil (1-n) = \frac{\pi}{\sin n\pi} \right.$$

$$= \frac{1}{2}\pi \operatorname{cosec}\left\{\frac{1}{2}(n+1)\pi\right\}$$

$$= \frac{1}{2}\pi \operatorname{cosec}\left(\frac{\pi}{2}+\frac{n\pi}{2}\right) = \frac{1}{2}\pi \sec\left(\frac{n\pi}{2}\right).$$ **Hence proved.**

Example 4. Show that $\int_0^{\pi/2} \sqrt{(\tan\theta)} \, d\theta = \frac{1}{2} \left\lceil \left(\frac{3}{4}\right) \left\lceil \left(\frac{1}{4}\right) = 2\int_0^{\infty} \frac{x^2 dx}{1+x^4}\right.\right.$.

Sol. We have

$$\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} (\sin\theta)^{1/2} (\cos\theta)^{-1/2} \, d\theta$$

$$= \frac{\left\lceil \left\{\frac{1}{2}\left(\frac{1}{2}+1\right)\right\} \left\lceil \left\{\frac{1}{2}\left(-\frac{1}{2}+1\right)\right\}}{2\left\lceil \left\{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}+2\right)\right\}}$$

$$= \frac{\left\lceil \left(\frac{3}{4}\right) \left\lceil \left(\frac{1}{4}\right)}{2\lceil (1)}$$

$$= \frac{1}{2} \left\lceil \left(\frac{3}{4}\right) \left\lceil \left(\frac{1}{4}\right)\right.\right.$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) \quad \because \beta(m, n) = \frac{\lceil m \cdot \lceil n}{\lceil (m+n)}$$

$$= \frac{1}{2}\int_0^\infty \frac{y^{\left(\frac{3}{4}-1\right)}}{(1+y)}\,dy \quad \because \ \beta(m, n) = \int_0^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}}$$

$$= \frac{1}{2}\int_0^\infty \frac{x^{-1}\cdot 4x^3 dx}{\left(1+x^4\right)}. \text{ Putting } y = x^4,\ dy = 4x^3\,dx$$

$$= 2\int_0^\infty \frac{x^2 dx}{1+x^4}\cdot \ \textbf{Hence proved.}$$

Example 5. Show that $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$. (*U.P.T.U., 2008*)

Sol. We have

$$\text{R.H.S.} = \frac{\sqrt{(m+1)}\cdot\sqrt{n}}{\sqrt{(m+1+n)}} + \frac{\sqrt{m}\cdot\sqrt{(n+1)}}{\sqrt{(m+n+1)}}$$

$$= \frac{m\sqrt{m}\cdot\sqrt{n}}{(m+n)\sqrt{(m+n)}} + \frac{\sqrt{m}\cdot n\sqrt{n}}{(m+n)\sqrt{m+n}}$$

$$= \frac{\sqrt{m}\cdot\sqrt{n}}{\sqrt{(m+n)}}\left[\frac{m}{m+n} + \frac{n}{m+n}\right]$$

$$= \frac{\sqrt{m}\cdot\sqrt{n}}{\sqrt{(m+n)}} = \beta\,(m, n). \ \textbf{Hence proved.}$$

Example 6. Show that $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{\sqrt{\pi}\,\sqrt{\left(\frac{1}{n}\right)}}{n\sqrt{\left(\frac{1}{n}+\frac{1}{2}\right)}}$.

Sol. Put $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

$$\therefore \quad dx = \frac{2}{n}\sin^{\left(\frac{2}{n}-1\right)}\theta\ \cos\theta\ d\theta$$

$$\therefore \quad \int \frac{dx}{\sqrt{(1-x^n)}} = \left(\frac{2}{n}\right)\int_0^{\pi/2} \frac{\sin^{\left(\frac{2}{n}-1\right)}\theta\cos\theta\,d\theta}{\sqrt{(1-\sin^2\theta)}}$$

$$= \frac{2}{n}\int_0^{\pi/2}\sin^{\left(\frac{2}{n}-1\right)}\theta\ \cos^0\theta\ d\theta$$

$$= \frac{2}{n}\,\frac{\sqrt{\left\{\frac{1}{2}\left(\frac{2}{n}-1+1\right)\right\}}\sqrt{\left\{\frac{1}{2}(0+1)\right\}}}{2\sqrt{\left\{\frac{1}{2}\left(\frac{2}{n}-1+0+2\right)\right\}}}$$

$$= \frac{1}{n}\cdot\frac{\sqrt{\left(\frac{1}{n}\right)}\cdot\sqrt{\left(\frac{1}{2}\right)}}{\sqrt{\left\{\frac{1}{2}\left(\frac{2}{n}+1\right)\right\}}} = \frac{\sqrt{\left(\frac{1}{n}\right)}\cdot\sqrt{\pi}}{n\sqrt{\left(\frac{1}{n}+\frac{1}{2}\right)}}\cdot \ \textbf{Hence proved.}$$

4.13 EVALUATE THE INTEGRALS [*U.P.T.U. (C.O.), 2004*]

(*i*) $\int_0^\infty e^{-ax} \cos bx \cdot x^{m-1} dx$

(*ii*) $\int_0^\infty e^{-ax} \cdot \sin bx \cdot x^{m-1} dx$ (*U.P.T.U., 2003*)

Proof: Both the above integrals are respectively the real and imaginary parts of

$$\int_0^\infty e^{-ax} e^{ibx} \cdot x^{m-1} dx$$

or
$$\int_0^\infty e^{-(a-ib)x} \cdot x^{m-1} dx$$

Now, we have
$$\int_0^\infty e^{-\lambda x} x^{n-1} dx = \frac{\Gamma n}{\lambda^n}$$

$$\therefore \quad \int_0^\infty e^{-(a-ib)x} x^{m-1}\, dx = \frac{\Gamma m}{(a-ib)^m}$$

$$= \Gamma m \cdot \frac{(a+ib)^m}{(a^2+b^2)^m}$$

Let us put $a = r \cos \theta$, $b = r \sin \theta$ in R.H.S.

$$\therefore \quad \int_0^\infty e^{-ax}(\cos\ bx + i \sin\ bx) x^{m-1} dx = \Gamma m \cdot \frac{r^m (\cos \theta + i \sin \theta)^m}{r^{2m}} \quad | \; r^2 = x^2 + y^2$$

or
$$\int_0^\infty \left[e^{-ax} \cos bx \cdot x^{m-1} + i e^{-ax} \sin bx \cdot x^{m-1} dx\right]$$

$$= \frac{\Gamma m}{r^m}(\cos m\theta + i \sin m\theta)$$

Equating real and imaginary parts, we get

$$\boxed{\int_0^\infty e^{-ax} \cos bx \cdot x^{m-1}\, dx = \frac{\Gamma m}{r^m} \cos\ m\theta}$$

and
$$\boxed{\int_0^\infty e^{-ax} \sin bx \cdot x^{m-1}\, dx = \frac{\Gamma m}{r^m} \sin m\theta}$$

where $r = \sqrt{(a^2+b^2)}$ and $\theta = \tan^{-1} \frac{b}{a}$.

Example 7. Prove that $\int_0^\infty xe^{-ax}\cos bx\cdot dx = \dfrac{a^2-b^2}{(a^2+b^2)^2}$, where $a > 0$.

Sol. We have

$$\int_0^\infty e^{-ax}\cos bx\cdot x^{m-1}\,dx = \frac{\Gamma m}{r^m}\cos m\theta$$

Put $m - 1 = 1$, *i.e.*, $m = 2$

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{\Gamma 2}{r^2}\cos 2\theta = \frac{1}{(a^2+b^2)}\cdot\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1}{a^2+b^2}\cdot\frac{\left(1-\dfrac{b^2}{a^2}\right)}{\left(1+\dfrac{b^2}{a^2}\right)}$$

$$= \frac{a^2-b^2}{(a^2+b^2)^2}.$$ **Hence proved.**

Example 8. Find the value of $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\ \Gamma\left(\frac{n-1}{n}\right)$, where n is a positive integer.

Sol. Let

$$P = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\ \Gamma\left(\frac{n-2}{n}\right)\Gamma\left(\frac{n-1}{n}\right) \qquad \ldots(i)$$

or

$$P = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\ \Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{1}{n}\right)$$

Writing in the reverse order,

$$P = \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\ \Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{1}{n}\right) \qquad \ldots(ii)$$

Multiplying (*i*) and (*ii*), we get

$$P^2 = \left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right]\ldots$$
$$\left[\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(1-\frac{n-2}{n}\right)\right]\left[\Gamma\left(\frac{n-1}{n}\right)\cdot\Gamma\left(1-\frac{n-1}{n}\right)\right]$$

Now, we know that

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

or

$$\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\dfrac{\pi}{n}} \qquad \ldots(iii)$$

$\therefore$

$$P^2 = \frac{\pi}{\sin\dfrac{\pi}{n}}\cdot\frac{\pi}{\sin\dfrac{2\pi}{n}}\ldots\frac{\pi}{\sin\dfrac{n-2}{n}\pi}\ \frac{\pi}{\sin\dfrac{n-1}{n}\pi} \qquad \ldots(iv)$$

From Trigonometry, we know that

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1} \sin\left(\theta+\frac{\pi}{n}\right) \sin\left(\theta+\frac{2\pi}{n}\right) \ldots \sin\left(\theta+\frac{n-2}{n}\pi\right) \sin\left(\theta+\frac{n-1}{n}\pi\right)$$

Putting $\theta = 0$

$$\lim_{\theta\to 0} \frac{\sin n\theta}{\sin\theta} = \lim_{\theta\to 0}\left(n\cdot\frac{\sin n\theta}{n\theta}\cdot\frac{\theta}{\sin\theta}\right)$$

$$= n \times 1 \times 1 = n$$

$$\therefore \quad n = 2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \ldots \sin\frac{n-2}{n}\pi \sin\frac{n-1}{n}\pi$$

$$\therefore \quad P^2 = \pi^{n-1} \frac{2^{n-1}}{n}$$

$$\therefore \quad P = \frac{(2\pi)^{n-1/2}}{\sqrt{n}}$$

Hence $\quad \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1/2)}}{\sqrt{n}}.$

Example 9. Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$.

Sol. We have

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Putting $n = \frac{1}{2}$, we get

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$ **Hence proved.**

Example 10. Show that $\iint_D x^{l-1}y^{m-1}\, dx\, dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} a^{l+m}$

where D is the domain $x \geq 0$, $y \geq 0$ and $x + y \leq a$.

Sol. Putting $x = aX$ and $y = aY$ in the given integral then, we get

$$I = \iint_{D'} (aX)^{l-1}(aY)^{m-1} a^2 dX dY$$

Where D' is the domain $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$

$$I = a^{l+m} \int_0^1 \int_0^{1-X} X^{l-1} Y^{m-1} dY\, dX$$

$$= a^{l+m} \int_0^1 X^{l-1} \left[\frac{Y^m}{m}\right]_0^{1-X} dX = \frac{a^{l+m}}{m} \int_0^1 X^{l-1} (1-X)^m dX$$

$$= \frac{a^{l+m}}{m} \beta(l, m+1) = \frac{a^{l+m}}{m} \cdot \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= a^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}.$$ **Hence proved.**

Example 11. Evaluate $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.$

Sol. The given integral

$$I = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

or $I = I_1 + I_2$ (say) ...(*i*)

Putting $x = \frac{1}{z}$ in I_2, we have

$$I_2 = \int_\infty^1 \frac{\left(\frac{1}{z}\right)^{n-1}\left(-\frac{1}{z^2}\right)dz}{\left[1+\left(\frac{1}{z}\right)\right]^{m+n}}$$

$$= \int_1^\infty \frac{z^{m-1}dz}{(1+z)^{m+n}}$$

$$= \int_1^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}$$ (By definite integral)

∴ From equation (*i*)

$$I = \int_0^1 \frac{x^{m-1}dx}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}$$

$$= \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = \beta(m, n)$$

$$= \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}.$$

Example 12. Using Beta and Gamma functions, evaluate

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx.$$ (*U.P.T.U., 2006*)

Sol. Let $x^3 = \sin^2\theta \Rightarrow 3x^2dx = 2\sin\theta\cos\theta\, d\theta$

$$\therefore \quad \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \times 2\frac{\sin\theta\cos\theta}{3\cdot\sin^{4/3}\theta}\,d\theta = \frac{2}{3}\int_0^{\pi/2} \sin^{2-4/3}\theta\; d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \sin^{2/3}\theta\cos^0\theta\,d\theta = \frac{2}{3}\,\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{2/3+0+2}{2}\right)} = \frac{2}{3}\,\frac{\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\cdot\sqrt{\pi}}{2\,\Gamma\left(\frac{4}{3}\right)}$$

$$= \frac{2}{9}\cdot\frac{\Gamma\left(\frac{2}{3}\right)\sqrt{\pi}}{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)} = \frac{2}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{3}\right)}.$$

Example 13. Prove that: $\beta(l, m)\,\beta(l+m, n)\,\beta(l+m+n, p) = \dfrac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)}$

Sol. $\beta(l, m)\cdot\beta(l+m, n)\cdot\beta(l+m+n, p)$

$$= \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}\cdot\frac{\Gamma(l+m)\,\Gamma n}{\Gamma(l+m+n)}\cdot\frac{\Gamma(l+m+n)\,\Gamma p}{\Gamma(l+m+n+p)} = \frac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)}.$$ **Proved.**

Example 14. Find $\displaystyle\int_0^\infty x^{1/2}\cdot e^{-x^{\frac{1}{3}}}\,dx$.

Sol. Let $x^{1/3} = t$ or $x = t^3 \Rightarrow dx = 3t^2\,dt$

$$\therefore \quad \int_0^\infty x^{1/2}\cdot e^{-x^{\frac{1}{3}}}dx = \int_0^\infty t^{3/2}\cdot e^{-t}\cdot 3t^2dt = 3\int_0^\infty t^{7/2}e^{-t}dt$$

$$= 3\int_0^\infty t^{\frac{9}{2}-1}\cdot e^{-t}\cdot dt = 3\,\Gamma\left(\frac{9}{2}\right) = \frac{315}{16}\sqrt{\pi}.$$

Example 15. Evaluate—

(*a*) $\displaystyle\int_0^\infty \cos x^2dx$ (*b*) $\displaystyle\int_{-\infty}^\infty \cos\frac{\pi}{2}x^2dx$

Sol. (*a*) We know that

$$\int_0^\infty e^{-ax}\cdot x^{n-1}\cos bx\,dx = \frac{\Gamma(n)\cos n\theta}{(a^2+b^2)^{n/2}}, \text{ where } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Put $a = 0$, $\displaystyle\int_0^\infty x^{n-1}\cos\,bx\,dx = \frac{\Gamma(n)}{b^n}\cos\frac{n\pi}{2}$ $\quad\left|\; \theta = \tan^{-1}\left(\frac{b}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}\right.$

Put $x^n = z$ so that $x^{n-1}\,dx = \dfrac{dz}{n}$ and $x = z^{1/n}$

Then $$\int_0^\infty \cos bz^{1/n}\, dz = \frac{n\,\Gamma(n)}{b^n} \cos\frac{n\pi}{2}$$

$$\int_0^\infty \cos (bx^{1/n})\, dx = \frac{\Gamma(n+1)}{b^n} \cos\frac{n\pi}{2} \quad \text{(By definite integral)} \quad ...(i)$$

Putting $b = 1,\ n = \frac{1}{2}$

$$\therefore \quad \int_0^\infty \cos x^2 dx = \Gamma\left(\frac{3}{2}\right)\cos\frac{\pi}{4} = \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,.$$

(*b*) $$I = \int_{-\infty}^\infty \cos\frac{\pi x^2}{2}\, dx = 2\int_0^\infty \cos\frac{\pi x^2}{2}\, dx \quad ...(ii) \quad |\text{By definite integral}$$

Putting $b = \frac{\pi}{2}$ and $n = \frac{1}{2}$ in equation (*i*), we get

$$\int_0^\infty \cos\left(\frac{\pi}{2}x^2\right)dx = \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}}\cos\frac{\pi}{4}$$

$\therefore$ From (*ii*) $$\int_{-\infty}^\infty \cos\frac{\pi x^2}{2}dx = 2\,\frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}}\cos\frac{\pi}{4}$$

$$= 2\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{\sqrt{2}}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{2}} = 1.$$

EXERCISE 4.3

1. Prove that $\displaystyle\int_0^1 x^4(1-x)^3 dx = \frac{1}{280}$.

2. Prove that $\displaystyle\int_0^2 x(8-x^3)^{1/3} dx = \frac{16\pi}{9\sqrt{3}}$.

3. Prove that $\displaystyle\int_0^2 \frac{x^2}{\sqrt{(2-x)}}dx = \frac{64\sqrt{2}}{15}$.

4. Show that $\displaystyle\int_0^\infty \frac{e^{-st}}{\sqrt{t}}dt = \frac{\sqrt{\pi}}{s}$, s > 0.

5. Show that $\displaystyle\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \Gamma(n), n > 0$.

6. Prove that $\displaystyle\int_0^1 (1-x^n)^{1/n} dx = \frac{1}{n}\cdot\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{2\,\Gamma\left(\frac{2}{n}\right)}$.

7. Show that $\displaystyle\int_0^1 x^2(1-x)^3\, dx = 60$.

8. Show that $\displaystyle\int_0^2 (4-x^2)^{3/2} dx = 3\pi$.

9. Show that $\int_0^3 \frac{dx}{\sqrt{(3x-x)^2}} = \pi.$

10. Show that $\int_0^1 \frac{x^2dx}{\sqrt{(1-x^3)}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{3\,\Gamma\left(\frac{5}{6}\right)}.$

11. Prove that $\int_0^\infty \frac{x^2dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}.$

12. Prove that $\int_0^1 \frac{x^2dx}{\sqrt{(1-x^3)}} = \frac{2}{3}.$

13. Prove that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\,.\,d\theta = \pi.$

14. Show that, if $n > -1$,

$$\int_0^\infty x^n e^{-k^2x^2}\,dx = \frac{1}{2k^{n+1}}\Gamma\left(\frac{n+1}{2}\right).$$

Hence or otherwise evaluate $\int_{-\infty}^0 e^{-k^2x^2}\,dx.$

15. Show that $\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)} = \sqrt{(\pi)}\,2^{1/3}.$

16. Show that $\frac{\left[\Gamma\left(\frac{1}{3}\right)\right]^2}{\Gamma\left(\frac{1}{6}\right)} = \frac{\sqrt{(\pi)}\cdot 2^{1/3}}{3^{1/2}}.$

17. Prove that $\int_0^\infty e^{-4x}\,x^{3/2}\,dx = \frac{3}{128}\sqrt{(\pi)}.$

18. Prove that $\int_0^\infty x^2\,e^{-x^2}\,dx = \frac{\sqrt{(\pi)}}{4}.$

19. Evaluate $\int_0^\infty x^{m-1}\cos bx\,dx$ and $\int_0^\infty x^{m-1}\sin bx\,dx.$ $\left[\textbf{Ans. } \frac{\Gamma m}{b^m}\cos\frac{m\pi}{2};\ \frac{\Gamma m}{b^m}\sin\frac{m\pi}{2}\right]$

20. Prove that $\int_0^\infty x^{m-1}\,e^{-ax^2}\,dx = \frac{\Gamma n}{2a^n}.$

4.14 APPLICATION TO AREA (DOUBLE INTEGRALS)

4.14.1 Area in Cartesian Coordinates

(a) Let the area $ABCD$ is enclosed by $y = f(x)$, $y = 0$ and $x = a$, $x = b$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve AD whose equation is $y = f(x)$.

Then the area of the element is $\delta x\ \delta y$.

Consequently the area of the strip $PNMQ$.

$\int_{y=0}^{f(x)} dx\,dy$, where $y = f(x)$ is the equation of AD.

∴ The required area

$$ABCD = \int_{x=a}^{b} \int_{y=0}^{f(x)} dx\, dy \quad .$$

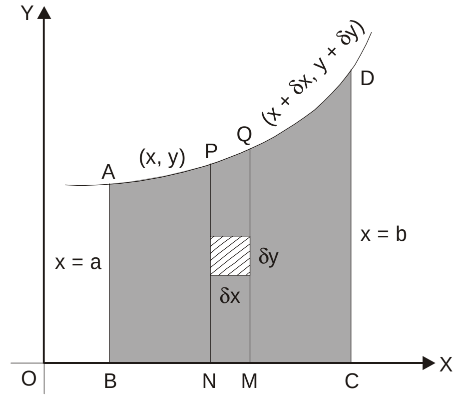


Fig. 4.25

(b) In a similar way we can prove that the area bounded by the curve $x = f(y)$, the y axis and the abscissae at $y = a$ and $y = b$ is given by

$$\int_{y=a}^{b} \int_{x=0}^{f(y)} dy\, dx$$

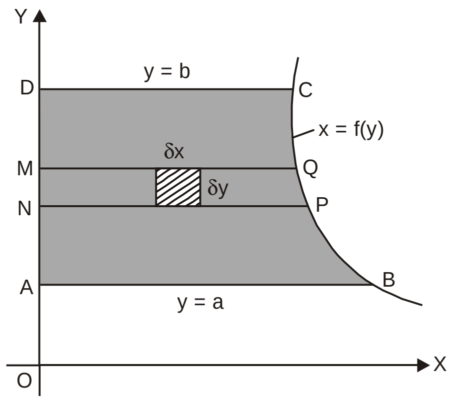


Fig. 4.26

(c) If we are to find the area bounded by the two curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$ and $x = b$ *i.e.*, the area $ABCD$ in the Figure 4.27 then the required area

$$\int_{x=a}^{b} \int_{y=f_2(x)}^{f_1(x)} dx\, dy$$

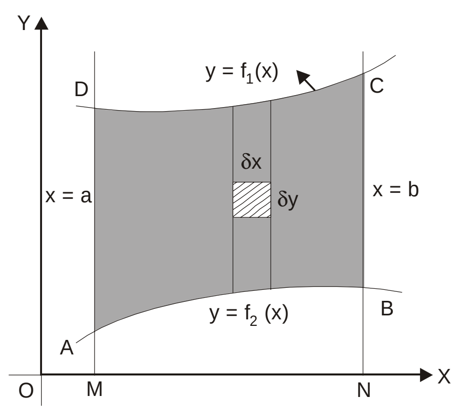


Fig. 4.27

Example 1. Find the area lying between the parabola $y = 4x - x^2$ and line $y = x$ by the method of double integration. (*U.P.T.U., 2007*)

Sol. OA is the line $y = x$ ...(*i*)

and $OBAD$ is the parabola

$$y = 4x - x^2 \quad \text{...}(ii)$$

Solving $y = 0$ and Eqn. (*ii*), we find that the parabola Eqn. (*ii*) meets the x-axis at $O\,(0, 0)$ and $D\,(4, 0)$.

Solving Eqns. (*i*) and (*ii*), we find $x = 4x - x^2$ or $x(x - 3) = 0$

i.e., $x = 0, 3$.

Also $x = 0$ gives $y = 0$ and $x = 3$ gives $y = 3$.

∴ The line (*i*) meets the parabola (*ii*) in $O\,(0, 0)$ and $A\,(3, 3)$ we are to find the area $OBAO$.

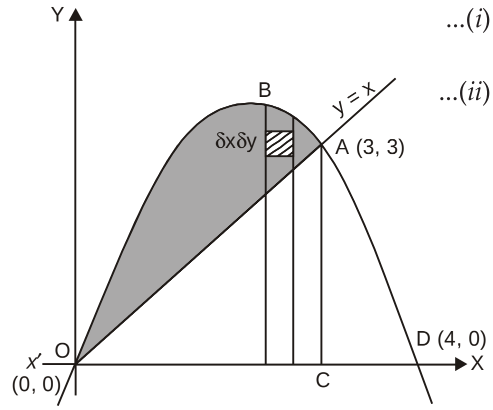


Fig. 4.28

∴ Required area $OBAO$ = area $OCABO$ – area of $\Delta\, OCA$.

$$= \int_{x=0}^{3}\int_{y=0}^{4x-x^2} dx\,dy - \left(\frac{1}{2}\times OC\times CA\right)$$

$$= \int_0^3 (y)_0^{4x-x^2}\,dx - \left(\frac{1}{2}\times 3\times 3\right)$$

$$= \int_0^3 (4x - x^2)dx - \frac{1}{2}(9) = \left(2x^2 - \frac{1}{3}x^3\right)_0^3 - \frac{9}{2}$$

$$= 2(3)^2 - \frac{1}{3}(3)^3 - \frac{9}{2} = 18 - 9 - \frac{9}{2} = \frac{9}{2}.$$

Example 2. $a^2y^2 = a^2x^2 - x^4$ find the whole area within the curve.

Sol. First of all trace the curve as shown in the Figure 4.29.

Whole area = 4 × area OAC

$$= 4\int_{x=0}^{a}\int_{y=0}^{f(x)} dx\,dy$$

where $y = f(x)$ *i.e.*,

$$y = \frac{\sqrt{(a^2x^2 - x^4)}}{a}$$

is the equation of the curve

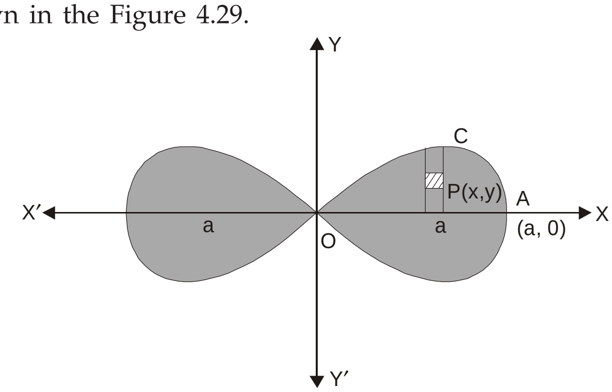


Fig. 4.29

$$= 4\int_{x=0}^{a}[y]_0^{f(x)}\,dx$$

$$= 4\int_0^a f(x)dx = \frac{4}{a}\int_0^a \sqrt{(a^2x^2 - x^4)}dx$$

$$= \frac{4}{a}\int_0^a x\sqrt{(a^2 - x^2)}dx = \frac{4}{a}\int_0^{\pi/2} a\sin\theta\, a\cos\theta\, a\cos\theta\, d\theta$$

Putting $x = a\sin\theta$

$$= 4a^2\int_0^{\pi/2}\sin\theta\cos^2\theta\, d\theta = \frac{4a^2\,\overline{|1}\,\overline{|(3/2)}}{2\,\overline{|(5/2)}}$$

$$= \frac{4a^2\, 1\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4a^2}{3}.$$

Example 3. Determine the area of region bounded by the curves

$$xy = 2,\ 4y = x^2,\ y = 4.$$

[*U.P.T.U. (C.O.), 2003, 2008*]

Sol. Required area of shaded region

$$= \int_{y=1}^{4}\int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\,dy$$

$$= \int_{1}^{4}\left(2\sqrt{y}-\frac{2}{y}\right)dy$$

$$= 2\left(\frac{2}{3}y^{3/2}-\log y\right)_1^4$$

$$= 2\left[\left(\frac{16}{3}-2\log 2\right)-\frac{2}{3}\right]$$

$$= \frac{28}{3}-4\log 2.$$

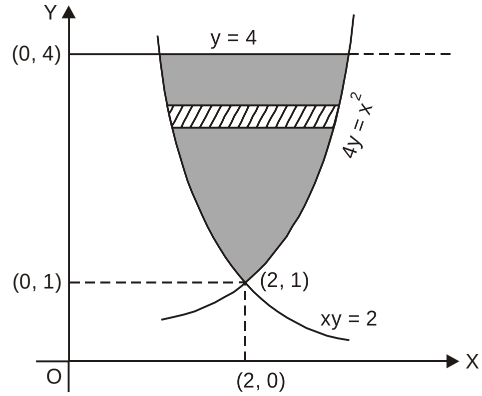


Fig. 4.30

Example 4. By double integration, find the whole area of the curve

$$a^2x^2 = y^3(2a - y). \qquad \textit{(U.P.T.U., 2001)}$$

Sol. Required area = 2 (area *OAB*)

$$= 2\int_{y=0}^{2a}\int_{x=0}^{f(y)} dy\,dx \qquad ...(i)$$

where $x = f(y) = \dfrac{y^{3/2}\sqrt{2a-y}}{a}$ is the equation of the given curve.

From eqn. (*i*), required area

$$= 2\int_0^{2a} f(y)dy = \frac{2}{a}\int_0^{2a} y^{3/2}\sqrt{2a-y}\ dy$$

$$= \frac{2}{a}\int_0^{\pi/2}(2a\sin^2\theta)^{3/2}\sqrt{2a-2a\sin^2\theta}\cdot 4\,a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta \qquad (\text{Put } y = 2a\sin^2\theta)$$

$$= 32a^2\cdot\frac{\left|\!\overline{\left(\frac{5}{2}\right)}\right.\left|\!\overline{\left(\frac{3}{2}\right)}\right.}{2\left|\!\overline{(4)}\right.} = 32a^2\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot 3\cdot 2\cdot 1} = \pi a^2.$$

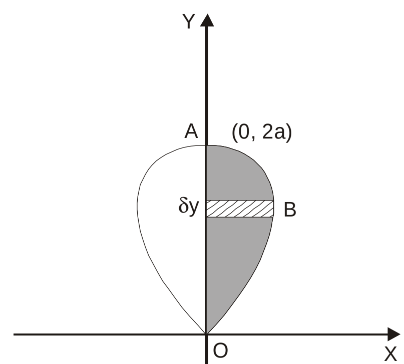


Fig. 4.31

Example 5. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is

$$\frac{16}{3}a^2(8\pi-\sqrt{3}).$$

Sol. $y^2 = 12ax$ is a parabola, whose vertex is (0, 0) latus rectum = $12a$, $x^2 + y^2 = 64a^2$ is a circle, whose centre is (0, 0) and radius = $8a$.

Solving $x^2 + y^2 = 64a^2$ and $y^2 = 12ax$, we get

$$x^2 + 12ax - 64a^2 = 0 \text{ or } x = 4a.$$

$\therefore$ The x-coordinate of P the point of intersection of the two curves is $4a$. Also the circle $x^2 + y^2 = 64a^2$ meets x axis at $x^2 = 64a^2$ (Putting $y = 0$) *i.e.*, at $x = \pm 8a$. Hence the coordinates of A and A' are $(8a, 0)$ and $(-8a, 0)$.

Also for the parabola $y = \sqrt{(12ax)}$ and for the circle we have

$$y = \sqrt{(64a^2 - x^2)}$$

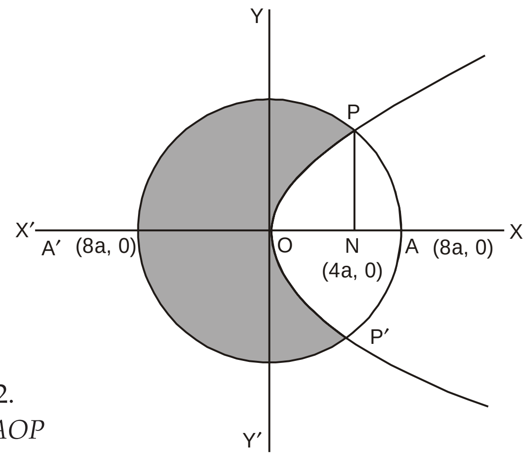


Fig. 4.32

Required area = shaded area in the Figure 4.32.

$= $ area of the circle – area $OP'\ AOP$

$= \pi(8a)^2 - 2$ [area OAP]

$= 64\ \pi a^2 - 2$ [area OPN + area PAN]

$$= 64\pi a^2 - 2\left[\int_0^{4a}\sqrt{(12ax)}dx + \int_{4a}^{8a}\sqrt{(64a^2 - x^2)}dx\right]$$

$$= 64\pi a^2 - 2\left[2\sqrt{(3a)}\int_0^{4a}\sqrt{x}dx + \left\{\frac{1}{2}x\sqrt{(64a^2 - x^2)} + \frac{1}{2}\cdot 64a^2\sin^{-1}\left(\frac{x}{8a}\right)\right\}_{4a}^{8a}\right]$$

$$= 64\pi a^2 - 4\sqrt{(3a)}\left\{\frac{2}{3}x^{3/2}\right\}_0^{4a} - 2\left\{32a^2\sin^{-1}(1) - \frac{1}{2}\cdot 4a\cdot\sqrt{48a^2} - 32a^2\sin^{-1}\left(\frac{1}{2}\right)\right\}$$

$$= 64\pi a^2 - \frac{64a^2}{3}\sqrt{3} - 32\pi a^2 + 16a^2\sqrt{3} + \frac{32}{3}\pi a^2$$

$$= \frac{128\pi a^2}{3} - \frac{16a^2}{3}\sqrt{3}$$

$$= \frac{16}{3}a^2(8\pi - \sqrt{3}).$$

4.14.2 Area of Curves in Polar Coordinate

The area bounded by the curve $r = f(\theta)$, where $f(\theta)$ is single valued function of θ in the domain (α, β) and redii vectors $\theta = \alpha$ and $\theta = \beta$ is $\int_{\theta=\alpha}^{\beta}\int_{r=0}^{f(\theta)} r\,d\theta\,dr$ $(\alpha < \beta)$.

Let O be the pole, OX the initial line and AB be the portion of the arc the curve $r = f(\theta)$ included between the radii vectors OA and OB *i.e.*, $\theta = \alpha$ and $\theta = \beta$.

The area of element $CDEF$ = area of sector OCD–area of sector OFE

$$= \frac{1}{2}(r + \delta r)^2\delta\theta - \frac{1}{2}r^2\delta\theta$$

$= r\ \delta\theta\ \delta r$, neglecting higher powers of δr and $\delta\theta$.

Hence the element of area in polar coordinates is $r\ \delta\theta\ \delta r$.

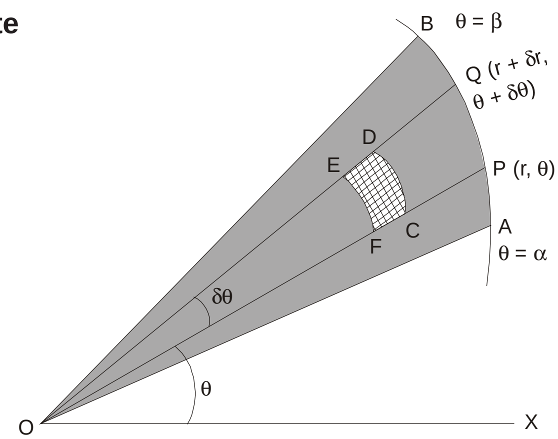


Fig. 4.33

$\therefore$ The area of the figure *OPQO*

$= \int_{\theta=\alpha}^{\beta} \int_{r=0}^{f(\theta)} r \, d\theta \, dr$, where $r = f(\theta)$ is the curve *AB*.

$\therefore$ The required area $= \int_{\theta=\alpha}^{\beta} \int_{r=0}^{f(\theta)} r \, d\theta \, dr$.

Example 6. Find the area of the region enclosed by $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and $x + y = a$.

Sol. We have $\sqrt{x} + \sqrt{y} = \sqrt{a}$ ...(*i*)

$x + y = a$ ...(*ii*)

From (*i*) $\sqrt{x} = \sqrt{a} - \sqrt{y} \Rightarrow x = \left(\sqrt{a} - \sqrt{y}\right)^2$

From (*ii*) $\left(\sqrt{a} - \sqrt{y}\right)^2 + y = a$

$$a + y - 2\sqrt{ay} + y = a$$

$$2y - 2\sqrt{ay} = 0 \Rightarrow y = \sqrt{ay}$$

Squaring on both sides, we get

$y^2 = ay \Rightarrow y(y - a) = 0 \Rightarrow y = 0, a$

Using the value of y in equation (*ii*), we get, $x = a, 0$.

So, the intersection point of given curve are $(a, 0)$ and $(0, a)$.

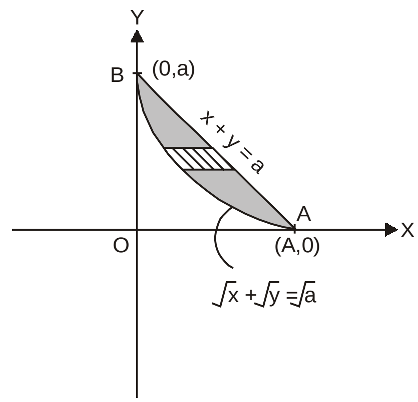


Fig. 4.34

Required area of the shaded region.

$$= \int_0^a \int_{x=\left(\sqrt{a}-\sqrt{y}\right)^2}^{(a-y)} dx \, dy = \int_0^a [x]_{\left(\sqrt{a}-\sqrt{y}\right)^2}^{(a-y)} . dy$$

$$= \int_0^a \left\{a - y - \left(\sqrt{a} - \sqrt{y}\right)^2\right\} dy = \int_0^a \left(2\sqrt{ay} - 2y\right) dy$$

$$= \left[\frac{2\sqrt{a}.y^{3/2}}{3/2} - \frac{2y^2}{2}\right]_0^a = \frac{4}{3}a^2 - a^2 = \frac{a^2}{3}. \quad \textbf{Ans.}$$

Example 7. Find the area of the region enclosed by the curves $y = \dfrac{3x}{x^2+2}$ and $4y = x^2$.

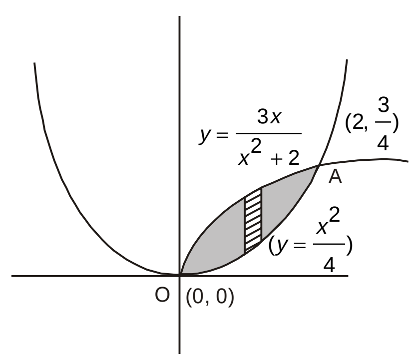


Fig. 4.35

Sol. We have $y = \dfrac{3x}{x^2+2}$ ...(*i*)

$4y = x^2$ ...(*ii*)

From (*i*) and (*ii*) $\dfrac{x^2}{4} = \dfrac{3x}{x^2+2}$

$\Rightarrow$ $x^4 + 2x^2 - 12x = 0$

$$x(x^3 + 2x - 12) = 0$$

$$x(x - 2)(x^2 + 2x + 6) = 0$$

$$\Rightarrow \quad x = 0, 2 \text{ (real values only)}$$

Using the values of x in (i), we get

$$y = 0, \frac{3}{4}$$

So the intersection points of curves are (0, 0) and $\left(2, \frac{3}{4}\right)$.

Hence, the required area of the shaded region.

$$= \int_{x=0}^{2} \int_{y=x^2/4}^{3x/(x^2+2)} dy\,dx = \int_0^2 [y]_{x^2/4}^{3x/(x^2+2)} dx = \int_0^2 \left(\frac{3x}{x^2+2} - \frac{x^2}{4}\right) dx$$

$$= \left[\frac{3}{2}\log\left(x^2+2\right) - \frac{x^3}{12}\right]_0^2 = \frac{3}{2}(\log 6 - \log 2) - \frac{8}{12}$$

$$= \frac{3}{2}\log\frac{6}{2} - \frac{2}{3}$$

$$= \frac{3}{2}\log 3 - \frac{2}{3}.$$

Example 8. Calculate the area which is inside the cardioid $r = 2(1 + \cos\theta)$ and outside the circle $r = 2$.

Sol. $r = 2$ is a circle centred at origin and of radius 2. The shaded area in Figure 4.36 is the region R which is outside the given circle and inside the cardioid. So r varies from the circle $r = 2$ the cardioid $r = 2(1 + \cos\theta)$. While θ varies from $-\pi/2$ to $\pi/2$. Since R is symmetric about x–axis, the required area A of the region R is given by

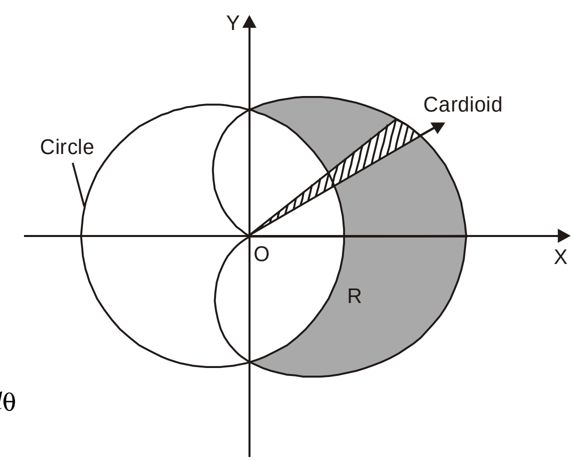


Fig. 4.36

$$A = \int\int_R dA = \int_{-\pi/2}^{\pi/2} \int_{r=2}^{2(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \int_2^{2(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\pi/2} \left.\frac{r^2}{2}\right|_2^{2(1+\cos\theta)} d\theta$$

$$= 4\int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 4\left[2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \pi + 8.$$

Example 9. Find, by double integration, the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.

Sol. Required area

= area $ABCDA$ = 2(area $ABDA$)

$$= 2\int_0^{\pi/2} \int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta$$

$$= a^2 \int_0^{\pi/2}\left[(1+\cos\theta)^2 - 1\right]d\theta$$

$$= a^2 \int_0^{\pi/2}\left(\cos^2\theta + 2\cos\theta\right)d\theta$$

$$= a^2 \left(\frac{1}{2}\cdot\frac{\pi}{2} + 2\right) = \frac{a^2}{4}(\pi + 8).$$

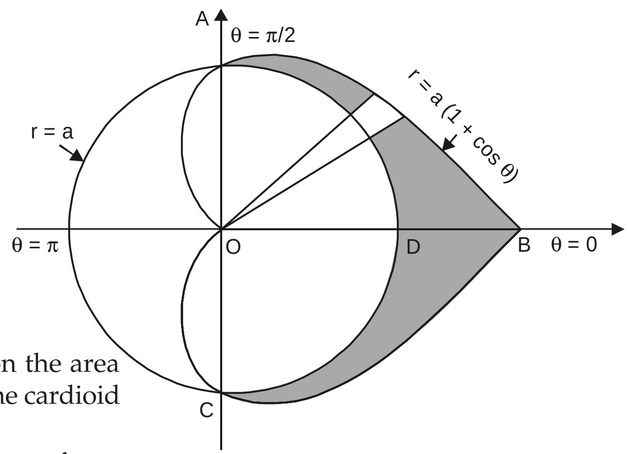


Fig. 4.37

Example 10. Find, by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$.

Sol. Eliminating r between the equations of two curves,
$\sin\theta = 1 - \cos\theta$ or $\sin\theta + \cos\theta = 1$
Squaring $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$

$$\therefore \qquad 2\theta = 0 \text{ or } \pi \Rightarrow \theta = 0 \text{ or } \frac{\pi}{2}$$

For the required area, r varies from $a(1 - \cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

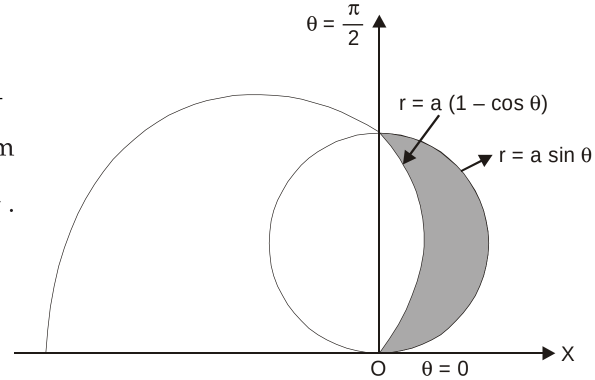


Fig. 4.38

$$\therefore \text{ Required area } = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2\left[\sin^2\theta - (1-\cos\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(-2\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left(1 - \frac{\pi}{4}\right).$$

Example 11. Find by double integration the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola

$$r(1 + \cos\theta) = 1.$$

Sol. The required area = area $CAFBEDC$
= 2 (area $DAFBED$)

$$= 2\int_{\theta=0}^{\pi/2}\int_{r \text{ for parabola}}^{r \text{ for cardioid}} r\,d\theta\,dr$$

$$= 2\int_0^{\pi/2}\left(\frac{1}{2}r^2\right)_{1/(1+\cos\theta)}^{1+\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos\theta)^2 - \{1/(1+\cos\theta)\}^2\right]d\theta$$

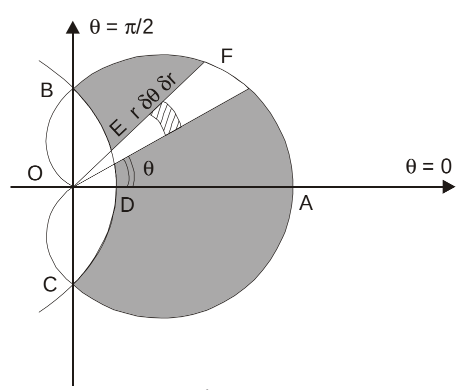


Fig. 4.39

$$= \int_0^{\pi/2} (1+\cos\theta)^2 d\theta - \int_0^{\pi/2} \frac{1}{(1+\cos\theta)^2} d\theta \quad ...(i)$$

Now, $\int_0^{\pi/2} (1+\cos\theta)^2 d\theta = \int_0^{\pi/2} (1+2\cos\theta+\cos^2\theta)\, d\theta$

$$= \int_0^{\pi/2} [(1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)]\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\left(\frac{1}{2}\pi\right) + 2\sin\left(\frac{1}{2}\pi\right) + \frac{1}{4}\sin\pi - 0$$

$$= \frac{3}{4}\pi + 2 = \frac{1}{4}(3\pi+8) \quad ...(ii)$$

and $\int_0^{\frac{\pi}{2}} \left[1/(1+\cos\theta)^2\right] d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sec^4\left(\frac{\theta}{2}\right) d\theta$ | As $1+\cos\theta = 2\cos^2\left(\frac{1}{2}\theta\right)$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^4\phi\, d\phi, \text{ Putting } \frac{1}{2}\theta = \phi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1+\tan^2\phi\right)\sec^2\phi\, d\phi, \quad [\because \sec^2\phi = 1+\tan^2\phi]$$

$$= \frac{1}{2}\int_0^1 \left(1+t^2\right) dt, \text{ where } t = \tan\phi$$

$$= \frac{1}{2}\left(t+\frac{1}{3}t^3\right)_0^1 = \frac{1}{2}\left(1+\frac{1}{3}\right) = \frac{2}{3} \quad ...(iii)$$

$\therefore$ From (*i*) with the help of (*ii*) and (*iii*), we get the required area

$$= \frac{1}{4}(3\pi+8) - \frac{2}{3}$$

$$= \frac{3}{4}\pi + 2 - \frac{2}{3} = \frac{3}{4}\pi + \frac{4}{3}.$$

Example 12. Find the area of the curve $r^2 = a^2 \cos 2\theta$.

Sol. Since the curve is symmetric about origin so the required area of shaded region.

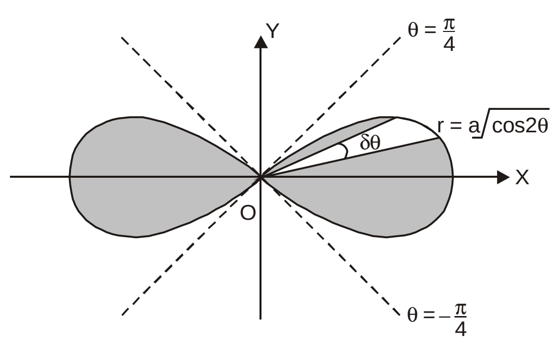


Fig. 4.40

$$= 2\int_{-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,d\theta\,dr$$

$$= 2\int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} .\,d\theta = \int_{-\pi/4}^{\pi/4} a^2\cos 2\theta\,d\theta$$

$$= 2a^2\int_0^{\pi/4}\cos 2\theta\,d\theta = \frac{2a^2}{2}\left[\sin 2\theta\right]_0^{\pi/4} = a^2\left[\sin\frac{\pi}{2} - \sin 0\right] = a^2.$$

Example 13. Find the area included between the curves $r = a(\sec\theta + \cos\theta)$ and its asymptote $r = a\sec\theta$.

Sol. The curve is symmetric about the line $\theta = 0$, so the required area of shaded region.

$$= \int_{-\pi/2}^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,d\theta\,dr$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta = \frac{a^2}{2}\int_{-\pi/2}^{\pi/2}\left\{(\sec\theta+\cos\theta)^2 - \sec^2\theta\right\}d\theta$$

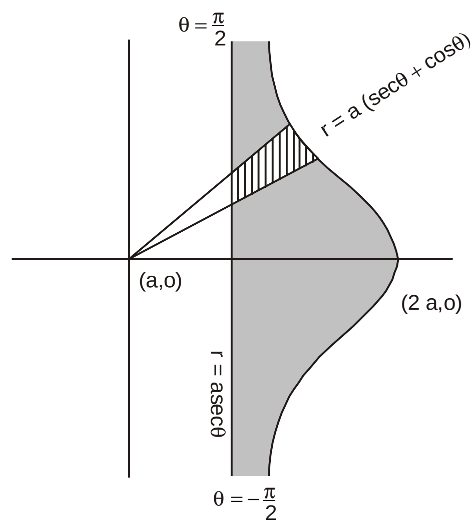


Fig. 4.41

$$= a^2\int_0^{\pi/2}\left(\sec^2\theta + 2\sec\theta\cos\theta + \cos^2\theta - \sec^2\theta\right)d\theta$$

$$= a^2\int_0^{\pi/2}\left(2+\cos^2\theta\right)d\theta = a^2\int_0^{\pi/2}\left(2+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(5+\cos 2\theta)d\theta = \frac{a^2}{2}\left[5\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{5\pi}{2} - 0\right] = \frac{5\pi a^2}{4}.$$

Example 14. Find the area included between the curve

$$x^2y^2 = a^2(y^2 - x^2) \text{ and its asymptotes.}$$

Sol. Required area $= 4 \times \text{area } OAB = 4\int_0^a y\,dx = 4\int_0^a \frac{ax}{\sqrt{(a^2 - x^2)}}\,dx$

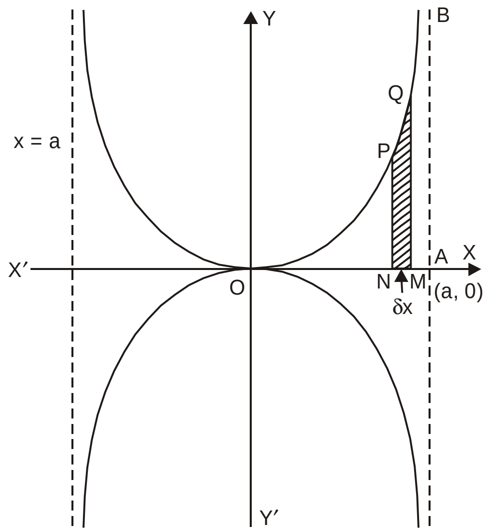


Fig. 4.42

$$= 4a\int_0^{\pi/2} \frac{a\sin\theta \cdot a\cos\theta\,d\theta}{a\cos\theta}, \quad \text{Putting } x = a\sin\theta$$

$$= 4a^2\int_0^{\pi/2} \sin\theta\,d\theta = 4a^2[-\cos\theta]_0^{\pi/2} = 4a^2.$$

Example 15. Show that the entire area between the curve

$$y^2(2a - x) = x^3 \text{ and its asymptote is } 3\pi a^2.$$

Sol. Required area = 2 × area OAB

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{(2a - x)}}\,dx$$

$$= 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}\,4a\sin\theta\cos\theta}{\sqrt{2a}\cdot\cos\theta}\,d\theta$$

$$= 16a^2\int_0^{\pi/2} \sin^4\theta\,d\theta = 16a^2\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$= 3\pi a^2$. **Hence proved.**

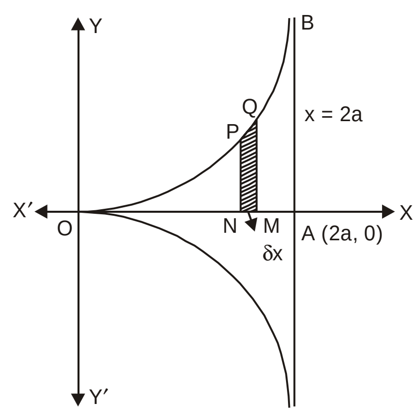


Fig. 4.43

EXERCISE 4.4

1. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum. $\left[\textbf{Ans. } \dfrac{8a^2}{3}\right]$

2. Find by double integration the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ $[\textbf{Ans. } \pi ab]$

3. Show by double integration that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\dfrac{16}{3}a^2$.

4. Find the area of the region bounded by quadrant of $x^2 + y^2 = a^2$ and $x + y = a$.

$\left[\textbf{Ans. } \dfrac{1}{4}(\pi - 2)a^2\right]$

5. Find by the double integration the area of the region bounded by $y^2 = x^3$ and $y = x$.

$\left[\textbf{Ans. } \dfrac{1}{10}\right]$

6. Find the area of the curve $3ay^2 = x(x - a)^2$. $\left[\textbf{Ans. } \dfrac{8a^2}{15\sqrt{3}}\right]$

7. Find the area included between the curve and its asymptotes. $\left[\textbf{Ans. } \pi a^2\right]$

8. Find the area between the curve $y^2(2a - x) = x^3$ and its asymptote. $\left[\textbf{Ans. } 3\pi a^2\right]$

9. Find the area of the portion bounded by the curve $x(x^2 + y^2) = a(x^2 - y^2)$ and its asymptote.

$\left[\textbf{Ans. } 2a^2\left(\dfrac{\pi}{4} - 1\right)\right]$

10. Find the area included between the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$. $\left[\textbf{Ans. } \dfrac{8\sqrt{ab}}{3}\right]$

11. Find the whole area of the curve $r = a\cos 2\theta$. $\left[\textbf{Ans. } \dfrac{1}{2}\pi a^2\right]$

12. Find the area of a loop of the curve $r = a\sin 2\theta$. $\left[\textbf{Ans. } \dfrac{1}{8}\pi a^2\right]$

13. Find the area enclosed by the curve $r = 3 + 2\cos\theta$. $[\textbf{Ans. } 11\pi]$

14. Show by double integration that the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$ is $\dfrac{1}{4}a^2(\pi + 8)$.

15. Find the area outside the circle $r = 2a\cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$.

$\left[\textbf{Ans. } \dfrac{\pi a^2}{2}\right]$

16. Find the area included between the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and its base.

$\left[\textbf{Ans. } 3\pi a^2\right]$

17. Find the area bounded by the curve $xy = 4$, y-axis and the lines $y = 1$ and $y = 4$.

[**Ans.** 8 log 2]

18. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$\left[\textbf{Ans. } \frac{\pi}{8}\right]$

4.15 TRIPLE INTEGRALS

Triple integral is a generalization of a double integral. Let a function $f(x, y, z)$ defined at every point of a three dimensional region V; Divide the region V into n elementary volumes δV_1, δV_2, ..., δV_n and let (x_r, y_r, z_r) be any point inside the rth sub-division δV_r.

Find the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$.

Then
$$\iiint_V f(x, y, z)dV = \lim_{\substack{n \to \infty \\ \delta V_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r .$$

To extend definition of repeated integrals for triple integrals, consider a function $F(x, y, z)$ and keep x and y constant and integrate with respect to z between limits in general depending upon x and y. This would reduce $F(x, y, z)$ to a function of x and y only. Thus let

$$\phi(x, y) = \int_{z_1(x,y)}^{z_2(x,y)} F(x, y, z)\, dz$$

Then in $\phi(x, y)$ we can keep x constant and integrate with respect to y between limits in general depending upon x this leads to a function of x alone say

$$\psi(x) = \int_{y_1(x)}^{y_2(x)} \phi(x, y)\, dy$$

Finally $\psi(x)$ is integrated with respect to x assuming that the limits for x are from a to b. Thus

$$\iiint_V F(x, y, z)\, dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} F(x, y, z)\, dx\, dy\, dz$$

$$\boxed{\iiint_V F(x, y, z)\, dV = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{\int_{z_1(x,y)}^{z_2(x,y)} F(x, y, z) dz\right\} dy\right] dx} .$$

Example 1. Evaluate the integral $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(x+y+z+1)^3}$.

Sol. Let
$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2}\frac{1}{(x+y+z+1)^2}\right]_0^{1-x-y} dx\, dy$$

$$\Rightarrow \qquad = -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right] dx\, dy$$

$$\Rightarrow \qquad = -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-\frac{1}{(x+y+1)^2}\right]dx\,dy$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1}{4}y+\frac{1}{x+y+1}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1}{4}(1-x)+\frac{1}{2}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\int_0^1\left(\frac{3}{4}-\frac{1}{4}x-\frac{1}{x+1}\right)dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}x-\frac{1}{8}x^2-\log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{3}{4}-\frac{1}{8}-\log 2\right]=\frac{1}{2}\left[\log 2-\frac{5}{8}\right].$$

Example 2. Evaluate $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{(4z-x^2)}}dz\,dx\,dy$.

Sol. We have

$$I = \int_0^4\int_0^{2\sqrt{z}}[z]_0^{\sqrt{(4z-x^2)}}dz\,dx$$

$$= \int_0^4\int_0^{2\sqrt{z}}\sqrt{(4z-x^2)}\,dz\,dx$$

$$= \int_0^4 \frac{1}{2}x\sqrt{(4z-x^2)}+\frac{1}{2}\cdot 4z\sin^{-1}\left(\frac{x}{\sqrt{4z}}\right)\Bigg]_0^{2\sqrt{z}}dz$$

$$= \int_0^4\left[\frac{1}{2}\cdot 2\sqrt{z}\sqrt{(4z-4z)}+\frac{1}{2}\cdot 4z\sin^{-1}(1)\right]dz$$

$$= \int_0^4 \pi z\,dz = \pi\left[\frac{1}{2}z^2\right]_0^4 = 8\pi.$$

Example 3. Evaluate $\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$ where V is the volume of the cube bounded by the coordinate planes and the planes $x = y = z = a$.

Sol. Here a column parallel to z-axis is bounded by the planes $z = 0$ and $z = a$.

Here the region S above which the volume V stands is the region in the xy-plane bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$.

Hence, the given integral $= \int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz$

$$= \int_0^a\int_0^a\left(x^2z+y^2z+\frac{z^3}{3}\right)_0^a dx\,dy$$

$$= \int_0^a\int_0^a\left(x^2a+y^2a+\frac{1}{3}a^3\right)dx\,dy$$

$$= \int_0^a\left[x^2ay+\frac{1}{3}y^3a+\frac{1}{3}a^3y\right]_0^a dx$$

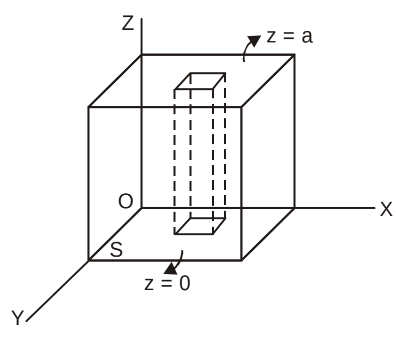


Fig. 4.44

$$= \int_0^a \left(x^2 a^2 + \frac{1}{3} a^4 + \frac{1}{3} a^4 \right) dx$$

$$= \left[\frac{1}{3} x^3 a^2 + \frac{1}{3} a^4 x + \frac{1}{3} a^4 x \right]_0^a = a^5.$$

Example 4. Evaluate $\iiint_V (2x + y)\, dx\, dy\, dz$, where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$.

Sol. Here a column parallel to z-axis is bounded by the plane $z = 0$ and the surface $z = 4 - x^2$ of the clinder. This cylinder $z = 4 - x^2$ meets the z-axis $x = 0$, $y = 0$ at (0, 0, 4) and the x-axis $y = 0$, $z = 0$ at (2, 0, 0) in the given region.

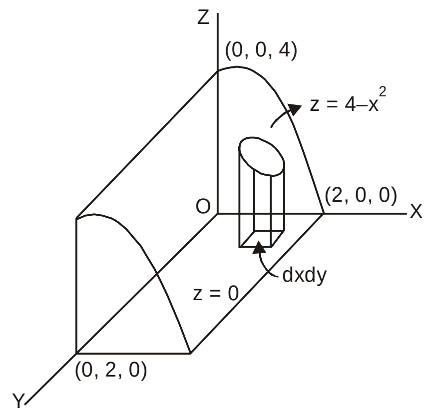


Fig. 4.45

Therefore, it is evident that the limits of integration for z are from 0 to $4 - x^2$, for y from 0 to 2 and for x from 0 to 2.

Hence, the given integral

$$= \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{4-x^2} (2x + y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x + y) [z]_0^{4-x^2}\, dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x + y)(4 - x^2)\, dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} \left[8x - 2x^3 + (4 - x^2) y \right] dx\, dy$$

$$= \int_{x=0}^{2} \left[8xy - 2x^3 y + \frac{1}{2} (4 - x^2) y^2 \right]_0^2 dx$$

$$= \int_0^2 \left[16x - 4x^3 + 2(4 - x^2) \right] dx$$

$$= \left[8x^2 - x^4 + 8x - \frac{2}{3} x^3 \right]_0^2$$

$$= \left(32 - 16 + 16 - \frac{16}{3} \right) = \frac{80}{3}.$$

EXERCISE 4.5

1. Evaluate the integral $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz.$ $\left[\textbf{Ans. } \frac{8}{3} \log 2 - \frac{19}{9} \right]$

2. Evaluate $\iiint_S xyz\, dx\, dy\, dz$, where

   $S = [(x, y, z) : x^2 + y^2 + z^2 \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0]$. $\left[\textbf{Ans. } \frac{1}{48} \right]$

3. Evaluate $\iiint_S \sqrt{x^2+y^2}\,dx\,dy\,dz,$ where S is the solid bounded by the surfaces $x^2 + y^2 = z^2$, $z = 0,\ z = 1.$ $\left[\textbf{Ans. } \dfrac{\pi}{6}\right]$

4. Evaluate $\int_{x=0}^{1}\int_{y=0}^{y=\sqrt{1-x^2}}\int_{z=0}^{z=\sqrt{(1-x^2-y^2)}} xyz\,dz\,dy\,dx.$ $\left[\textbf{Ans. } \dfrac{-5}{48}\right]$

5. Evaluate $\iiint_V xy^2\,dx\,dy\,dz,$ where V is the region bounded between the xy-plane and the sphere $x^2 + y^2 + z^2 = 1$. $\left[\textbf{Ans. } \dfrac{\pi}{24}\right]$

6. Evaluate $\iiint_V dx\,dy\,dz$ over the region V enclosed by the cylinder $x^2 + z^2 = 9$ and the planes $x = 0,\ y = 0,\ z = 0,\ y = 8$. **[Ans.** 18π**]**

7. Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$ and state precisely what is the region of integration.

[**Ans.** $\dfrac{1}{8}\left(e^{4a}-6e^{2a}+8e^a-3\right)$. Region of integration is the volume enclosed by the planes $x = a,\ y = 0,\ y = x,\ z = 0$ and $z = x + y$]

8. Evaluate $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz.$ $\left[\textbf{Ans. } \dfrac{3}{8}\log 3-\dfrac{19}{9}\right]$

9. Evaluate $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^a r^2\sin\theta\,dr\,d\theta\,d\theta.$ $\left[\textbf{Ans. } \dfrac{2\pi a^3}{3\sqrt{2}}\left(\sqrt{2}-1\right)\right]$

10. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,d\theta\,dr\,dz\,.$ $\left[\textbf{Ans. } \dfrac{5a^3\pi}{64}\right]$

4.16 APPLICATION TO VOLUME (TRIPLE INTEGRALS)

Volume of a solid contained in the domain V is given by the triple integral with $f(x, y, z) = 1$

Hence $$V = \iiint dx\,dy\,dz$$

and Volume in cylindrical coordinates, $V = \iiint r\,dr\,d\theta\,dz$

Volume in spherical polar coordinates, $V = \iiint r^2\sin\theta\,dr\,d\theta\,d\phi$

Example 1. Find the volume bounded by the elliptic paraboloids

$z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$. (*U.P.T.U., 2006*)

Sol. We have
$$z = x^2 + 9y^2 \quad ...(i)$$
$$z = 18 - x^2 - 9y^2 \quad ...(ii)$$

From eqns. (*i*) and (*ii*), we get $x^2 + 9y^2 = 9 \Rightarrow \dfrac{x^2}{3^2} + \dfrac{y^2}{1} = 1$ ...(*iii*)

The projection of this volume on to *xy* plane is the plane region *D* enclosed by ellipse (*iii*) as shown in the Figure 4.46.

Here

z varies from $z_1(x, y) = x^2 + 9y^2$ to $z_2(x, y) = 18 - x^2 - 9y^2$

y varies from $y_1(x) = -\sqrt{\dfrac{9-x^2}{9}}$ to $y_2(x) = \sqrt{\dfrac{9-x^2}{9}}$

x varies from -3 to 3.

Thus, the volume V bounded by the elliptic paraboloids

$$V = \int_{-3}^{3}\int_{y_1(x)}^{y_2(x)}\int_{z_1(x,y)}^{z_2(x,y)} dz\, dy\, dx$$

$$= \int_{-3}^{3}\int_{y_1(x)}^{y_2(x)}\left[\left(18 - x^2 - 9y^2\right) - \left(x^2 + 9y^2\right)\right] dy\, dx$$

$$= 2\int_{-3}^{3}\int_{y_1(x)}^{y_2(x)}\left(9 - x^2 - 9y^2\right) dy\, dx$$

$$= 2\int_{-3}^{3}\left[\left(9y - x^2y - 3y^3\right)\right]_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}} dx$$

$$= \frac{8}{9}\int_{-3}^{3}\left(9 - x^2\right)^{\frac{3}{2}} dx$$

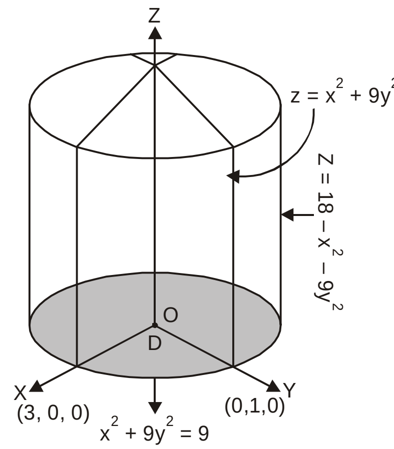


Fig. 4.46

Putting $x = 3\cos\theta$ so $dx = -3\sin\theta\, d\theta$

$$= 72\int_0^{\pi}\sin^4\theta\, d\theta$$

$$= 144\int_0^{\pi/2}\sin^4\theta\cos^0\theta\, d\theta$$

$$= 144\cdot\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}$$

$$= 144\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\pi}{2\cdot 2}$$

$$= 27\pi.$$

Example 2. Calculate the volume of the solid bounded by the surface $x = 0$, $y = 0$, $x + y + z = 1$ and $z = 0$. (*U.P.T.U., 2004*)

Sol. Here $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$

and z varies from 0 to $1 - x - y$

y varies from 0 to $1 - x$

x varies from 0 to 1

$$
\begin{aligned}
V &= \iiint dx\, dy\, dz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \\
&= \int_0^1 dx \int_0^{1-x} dy\, [z]_0^{1-x-y} = \int_0^1 dx \int_0^{1-x} dy\, (1-x-y) \\
&= \int_0^1 dx \left(y - xy - \frac{y^2}{2} \right)_0^{1-x} \\
&= \int_0^1 dx \left[1 - x - x(1-x) - \frac{1}{2}(1-x)^2 \right] \\
&= \int_0^1 \left(1 - x - x + x^2 - \frac{1}{2} + x - \frac{x^2}{2} \right) dx = \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2} \right) dx \\
&= \left[\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.
\end{aligned}
$$

Example 3. Find the volume of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.

Sol. We have

$$x^2 + y^2 + z^2 = 4a^2 \qquad \text{...}(i)$$

$$x^2 + y^2 - 2ay = 0 \qquad \text{...}(ii)$$

Changing it in polar coordinates

$$x = r\cos\theta,\ y = r\sin\theta$$

From (i) $\quad z^2 = 4a^2 - r^2$

From (ii) $\quad r^2 = 2ar\sin\theta = 0 \Rightarrow r = 2a\sin\theta$

Hence, r varies from $r = 0$ to $r = 2a\sin\theta$

z varies from $z = 0$ to $z = \sqrt{4a^2 - r^2}$

and θ varies from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

$$\therefore \quad V = \iiint dx\, dy\, dz = 4\int_0^{\frac{\pi}{2}} d\theta \int_{r=0}^{2a\sin\theta} r dr \int_{z=0}^{\sqrt{4a^2-r^2}} dz \qquad \Big| \text{Volume in 4 quadrant}$$

$$
\begin{aligned}
&= 4\int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\sin\theta} r\, dr [z]_0^{\sqrt{4a^2-r^2}} = 4\int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\sin\theta} r\, dr \sqrt{4a^2 - r^2} \\
&= 4\int_0^{\frac{\pi}{2}} d\theta \left[-\frac{1}{3}(4a^2 - r^2)^{\frac{3}{2}} \right]_0^{2a\sin\theta} = \frac{4}{3}\int_0^{\frac{\pi}{2}} \left[-(4a^2 - 4a^2\sin^2\theta)^{\frac{3}{2}} + 8a^3 \right] d\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}} (-8a^3\cos^3\theta + 8a^3)\, d\theta = \frac{8 \times 4a^3}{3}\int_0^{\frac{\pi}{2}} (1 - \cos^3\theta)\, d\theta \\
&= \frac{32a^3}{3}\int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{4}\cos 3\theta - \frac{3}{4}\cos\theta \right) d\theta \\
&= \frac{32a^3}{3}\left[\theta - \frac{1}{12}\sin 3\theta - \frac{3}{4}\sin\theta \right]_0^{\frac{\pi}{2}} = \frac{32a^3}{3}\left[\frac{\pi}{2} + \frac{1}{12} - \frac{3}{4} \right] \\
&= \frac{32a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3} \right].
\end{aligned}
$$

Example 4. Find the volume of the cylindrical column standing on the area common to the parabolas $x = y^2$, $y = x^2$ as base and cut off by the surface $z = 12 + y - x^2$. (*U.P.T.U., 2001*)

Sol. Volume $= \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{12+y-x^2} dz$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (12 + y - x^2)\, dy$$

$$= \int_0^1 dx \left(12y + \frac{y^2}{2} - x^2 y\right)_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 \left(12\sqrt{x} + \frac{x}{2} - x^{\frac{5}{2}} - 12x^2 - \frac{x^4}{2} + x^4\right) dx$$

$$= \left(8x^{\frac{3}{2}} + \frac{x^2}{4} - \frac{2}{7}x^{\frac{7}{2}} - 4x^3 - \frac{x^5}{10} + \frac{x^5}{5}\right)_0^1$$

$$= 8 + \frac{1}{4} - \frac{2}{7} - 4 - \frac{1}{10} + \frac{1}{5} = \frac{569}{140}.$$

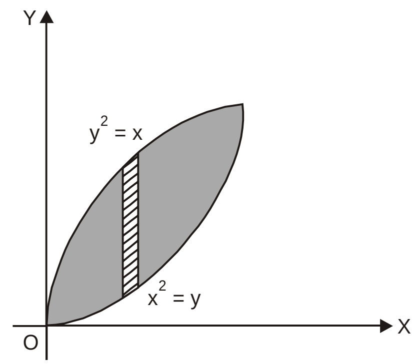


Fig. 4.47

Example 5. Find the volume enclosed between the cylinders $x^2 + y^2 = ax$ and $z^2 = ax$.

Sol. We have $x^2 + y^2 = ax$

$$z^2 = ax$$

$$V = \iiint dx\, dy\, dz$$

$$= \int_0^a dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy \int_{-\sqrt{ax}}^{\sqrt{ax}} dz$$

$$= \int_0^a dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy (z)_{-\sqrt{ax}}^{\sqrt{ax}}$$

$$= 2\int_0^a dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy\, \sqrt{ax}$$

$$= 2\int_0^a \sqrt{ax}\, dx [y]_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}}$$

$$= 2\int_0^a \sqrt{ax}\, dx \left(2\sqrt{ax - x^2}\right) = 4\sqrt{a} \int_0^a x\sqrt{a-x}\, dx$$

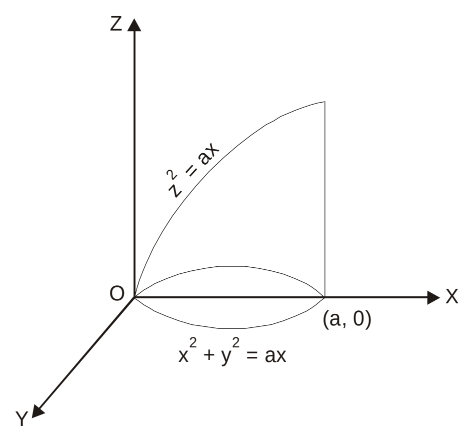


Fig. 4.48

Put $x = a \sin^2 \theta$ so that $dx = 2a \sin \theta \cos \theta\, d\theta$

$$= 4\sqrt{a} \int_0^{\frac{\pi}{2}} a \sin^2 \theta \sqrt{a - a\sin^2 \theta} \cdot 2a \sin \theta \cos \theta\, d\theta$$

$$= 8a^3 \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^2 \theta\, d\theta$$

$$= 8a^3 \frac{\Gamma 2\, \Gamma\frac{3}{2}}{2\Gamma\frac{7}{2}} = 4a^3 \frac{\Gamma\frac{3}{2}}{\frac{5}{2} \cdot \frac{3}{2} \Gamma\frac{3}{2}} = \frac{16a^3}{15}.$$

Example 6. Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $x^2 + y^2 = z^2$.

Sol. We have $x^2 + y^2 + z^2 = a^2$ ...(*i*)

$x^2 + y^2 = z^2$ ...(*ii*)

Let $x = r\cos\theta,\ y = r\sin\theta$

From eqns. (*i*) and (*ii*) $z^2 = a^2 - r^2$

and $z^2 = r^2$

Here z varies from r to $\sqrt{a^2 - r^2}$

r varies from 0 to $\dfrac{a}{\sqrt{2}}$ $\quad\left|\ r^2 = a^2 - r^2 \Rightarrow r = \dfrac{a}{\sqrt{2}}\right.$

θ varies from 0 to 2π

$$\therefore \quad \text{Volume } V = \int_0^{2\pi}\int_0^{\frac{a}{\sqrt{2}}}\int_r^{\sqrt{a^2-r^2}} dz\cdot(r\,dr\,d\theta)$$

$$= \int_0^{2\pi}\int_0^{a/\sqrt{2}} (z)_r^{\sqrt{a^2-r^2}}\ r\,dr\,d\theta = \int_0^{2\pi}\int_0^{a/\sqrt{2}} \left(\sqrt{a^2-r^2} - r\right)\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{(a^2-r^2)^{\frac{3}{2}}}{-2\cdot\left(\frac{3}{2}\right)} - \frac{1}{3}r^3\right]_0^{a/\sqrt{2}} d\theta = \frac{1}{3}\int_0^{2\pi}\left[-\left(\frac{a^2}{2}\right)^{\frac{3}{2}} + a^3 - \frac{a^3}{2\sqrt{2}}\right]d\theta$$

$$= \frac{1}{3}a^3\left(1 - \frac{1}{\sqrt{2}}\right)(\theta)_0^{2\pi} = \frac{1}{3}a^3\left(1 - \frac{1}{\sqrt{2}}\right)\cdot 2\pi$$

$$= \left(2 - \sqrt{2}\right)\frac{\pi a^3}{3}.$$

Example 7. Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $x^2 + y^2 = z^2$.

Sol. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$ ...(*i*)

and that of the cone is $x^2 + y^2 = z^2$ ...(*ii*)

In spherical coordinates $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

From eqn. (*i*) $(r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2 = a^2$

$\Rightarrow r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = a^2$

$\Rightarrow r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta = a^2$

$\Rightarrow r^2\sin^2\theta + r^2\cos^2\theta = a^2$

$\Rightarrow r^2(\sin^2\theta + \cos^2\theta) = a^2$

$\Rightarrow r^2 = a^2$

$\Rightarrow r = a$

From (*ii*) $(r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 = (r\cos\theta)^2$

$\Rightarrow r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\cos^2\theta$

$\Rightarrow r^2\sin^2\theta = r^2\cos^2\theta$

$\Rightarrow \tan^2\theta = 1$

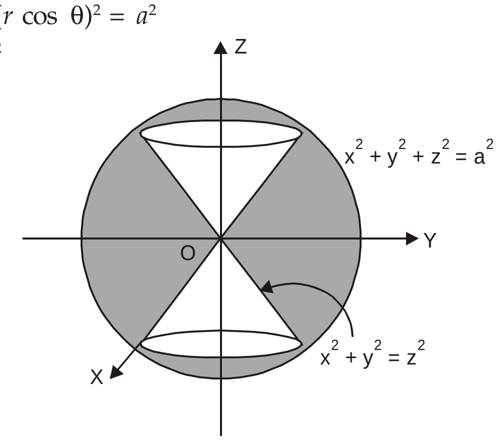


Fig. 4.49

$\Rightarrow \qquad \tan\,\theta = 1 \Rightarrow \theta = \pm\,\dfrac{\pi}{4}$

Thus $\qquad r = a$ and $\theta = \pm\,\dfrac{\pi}{4}$

The volume in the first octant is one-fourth only. Limits in the first octant r varies 0 to a, θ from 0 to $\dfrac{\pi}{4}$ and ϕ from 0 to $\dfrac{\pi}{2}$

$$\therefore \qquad V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi = 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^a$$

$$= 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \cdot \frac{a^3}{3} = \frac{4a^3}{3}\int_0^{\frac{\pi}{2}} d\phi \left[-\cos\theta\right]_0^{\frac{\pi}{4}} = \frac{4a^3}{3}(\phi)_0^{\frac{\pi}{2}}\left[-\frac{1}{\sqrt{2}}+1\right]$$

$$= \frac{4a^3}{3\sqrt{2}}\left(\sqrt{2}-1\right)\frac{\pi}{2} = \frac{2}{3}\pi a^3\left(1-\frac{1}{\sqrt{2}}\right).$$

Spherical coordinates:

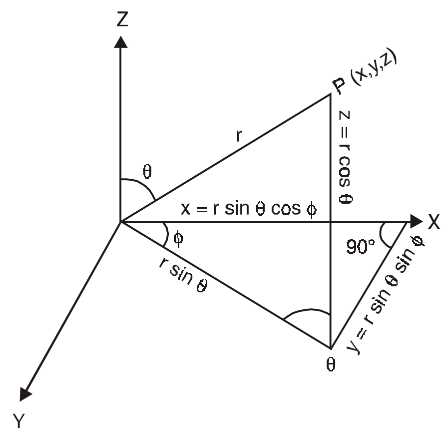


Fig. 4.50

Example 8. Find by triple integration, the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut off by the plane $z = 4$. (*U.P.T.U., 2005*)

Sol. By symmetry, the required volume is 4 times the volume in the positive octant.

The section of the paraboloid by the plane $z = 4$ is the circle $x^2 + y^2 = 16$, $z = 4$ and its projection on the xy-plane is the circle $x^2 + y^2 = 16$, $z = 0$.

The volume in the positive octant is bounded by $z = \dfrac{x^2+y^2}{4}$, $z = 4$, $y = 0$, $y = \sqrt{16-x^2}$ and $x = 0$, $x = 4$.

$$\text{Volume } V = 4\int_0^4\int_0^{\sqrt{16-x^2}}\int_{\frac{(x^2+y^2)}{4}}^{4} dz\, dy\, dx = 4\int_0^4\int_0^{\sqrt{16-x^2}} \left[z\right]_{\frac{(x^2+y^2)}{4}}^{4} dy\, dx$$

$$= 4\int_0^4\int_0^{\sqrt{16-x^2}} \left(4-\frac{x^2+y^2}{4}\right) dy\, dx = 4\int_0^4 \left[\left(4-\frac{x^2}{4}\right)y - \frac{1}{4}\cdot\frac{y^3}{3}\right]_0^{\sqrt{16-x^2}} dx$$

$$= 4\int_0^4 \left[\left(4-\frac{x^2}{4}\right)\sqrt{16-x^2} - \frac{1}{12}\left(16-x^2\right)^{\frac{3}{2}}\right] dx$$

$$= 4\int_0^4 \left[\frac{1}{4}(16-x^2)\sqrt{16-x^2} - \frac{1}{12}(16-x^2)^{\frac{3}{2}}\right] dx$$

$$= 4\int_0^4 \frac{1}{6}(16-x^2)^{\frac{3}{2}} dx = \frac{2}{3}\int_0^4 (16-x^2)^{3/2}\, dx$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} (16)^{3/2} \cdot \cos^3\theta \cdot 4\cos\theta\, d\theta, \text{ where } x = 4\sin\theta$$

$$= \frac{512}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= \frac{512}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= 32\pi.$$

Example 9. Find the volume of the solid under the surface $az = x^2 + y^2$ and whose base R is the circle $x^2 + y^2 = a^2$. (*U.P.T.U., 2008*)

Sol. We have $az = x^2 + y^2$, $x^2 + y^2 = a^2$

Here x varies from 0 to a

y varies from 0 to $\sqrt{a^2 - x^2}$

and z varies from 0 to $\dfrac{x^2 + y^2}{a}$.

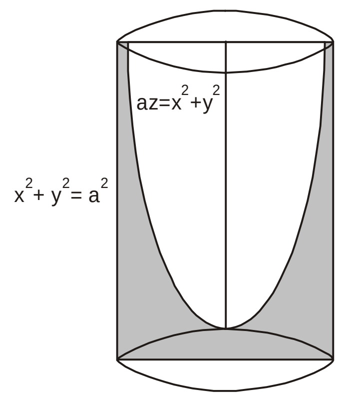


Fig. 4.51

By symmetry, the required volume is 4 times the volume in positive octant.

$$V = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\frac{x^2+y^2}{a}} dx\,dy\,dz = \frac{4}{a}\int_0^a\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy$$

$$= \frac{4}{a}\int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{4}{a}\int_0^a \left[x^2\sqrt{a^2-x^2} + \frac{(a^2-x^2)^{3/2}}{3}\right] dx$$

Putting $x = a\sin\theta$ *i.e.*, $dx = a\cos\theta\, d\theta$

$$= \frac{4}{a}\left[a^4\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta + \frac{a^4}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta\right]$$

$$= \frac{4}{a}\left[\frac{a^4.\,\Gamma(3/2)\,\Gamma(3/2)}{2\,\Gamma(3)} + \frac{a^4.\,\Gamma(5/2)\,\Gamma(1/2)}{3.2\,\Gamma(3)}\right] = \frac{4a^3}{4}\left[\frac{1}{2}\cdot\frac{1}{2}\pi + \frac{3.\,\pi}{3.\,2.\,2}\right]$$

$$= a^3\left[\frac{2\pi}{4}\right] = \frac{\pi a^3}{2}.$$

Example 10. A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the plane $z = 0$ and surface $z = c + xy$.

[*U.P.T.U. (C.O.), 2003*]

Sol. Here x varies from 0 to a

y varies from 0 to $\dfrac{bx}{a}$

z varies from 0 to $c + xy$

Hence, the volume is

$$V = \int_0^a \int_0^{bx/a} \int_0^{c+xy} dx\, dy\, dz = \int_0^a \int_0^{bx/a} (c+xy)dx\, dy$$

$$= \int_0^a \left[cy + \frac{xy^2}{2}\right]_0^{bx/a} dx = \int_0^a \left(\frac{bcx}{a} + \frac{b^2x^3}{2a^2}\right)dx$$

$$= \left[\frac{bcx^2}{2a} + \frac{b^2x^4}{8a^2}\right]_0^a = \frac{bca^2}{2a} + \frac{b^2a^4}{8a^2} = \frac{ab}{8}(4c+ab).$$

EXERCISE 4.6

1. Find $\iiint_V x^2yz\, dx\, dy\, dz$, where V is the volume bounded by the surface $x^2 + y^2 = 9$, $z = 0$, $z = 2$. $\left[\textbf{Ans. } \frac{648}{5}\right]$

2. Compute the volume of the solid enclosed between the two surfaces elliptic paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + 3y^2$.

   [**Hint:** Projection of the volume on to xy - plane is the ellipse

   $x^2 + 2y^2 = 4$, so limits are z; $x^2 + 3y^2$ to $8 - x^2 - y^2$; $y: \pm\sqrt{\frac{4-x^2}{2}}$, $x: \pm 2$].

   $\left[\textbf{Ans. } 8\pi\sqrt{2}\right]$

3. Compute the volume of the solid bounded by the plane $2x + 3y + 4z = 12$, xy-plane and the cylinder $x^2 + y^2 = 1$.

   $\left[\textbf{Hint}: \text{Limits } z: 0 \text{ to } \frac{1}{4}(12-2x-3y), x: -1 \text{ to } 1, y: \pm\sqrt{1-x^2}\right]$. $[\textbf{Ans. } 3\pi]$

4. Find the volume of the solid common to two cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$.

   $\left[\textbf{Hint: } z: \pm\sqrt{a^2-x^2},\ \underset{x:\pm a}{y}: \pm\sqrt{a^2-x^2}\right]$. $\left[\textbf{Ans. } \frac{16a^3}{3}\right]$

5. Find the volume of the cylindrical column standing on the area common to the parabolas $y^2 = x$, $x^2 = y$ and cut off by the surface $z = 12 + y - x^2$. (*U.P.T.U., 2001*)

   $\left[\textbf{Ans. } \frac{566}{140}\right]$

6. A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the planes $z = 0$ and surface $z = c + xy$. [*U.P.T.U. (C.O.), 2003*]

   $\left[\textbf{Ans. } \frac{ab}{8}(4c+ab)\right]$

7. Compute the volume bounded by $xy = z$, $z = 0$ and $(x - 1)^2 + (y - 1)^2 = 1$. **[Ans. π]**

8. Evaluate $\int_0^2 \int_1^z \int_0^{yz} xyz \, dx \, dy \, dz$. $\left[\textbf{Ans. } \frac{7}{2}\right]$

9. $\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \int_0^{xy} \cos \frac{z}{x} dz \, dy \, dx$. $\left[\textbf{Ans. } \left(\frac{\pi}{2} - 1\right)\right]$

10. Compute the volume of the solid bounded by $x^2 + y^2 = z$, $z = 2x$. **[Ans. 2π]**

11. Find the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = R^2$. $\left[\textbf{Ans. } \frac{\pi R^4}{2a}\right]$

12. Find the volume of the tetrahedron by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. $\left[\textbf{Ans. } \frac{abc}{6}\right]$

4.17 DRITCHLET'S* THEOREM

(*U.P.T.U., 2005*)

If V is a region bounded by $x \geq 0$, $y \geq 0$ and $x + y + z \leq 1$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx \, dy \, dz = \frac{\lceil(l) \lceil(m) \lceil(m)}{\lceil(l + m + n + 1)}$$

The given triple integral may be written as

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz$$

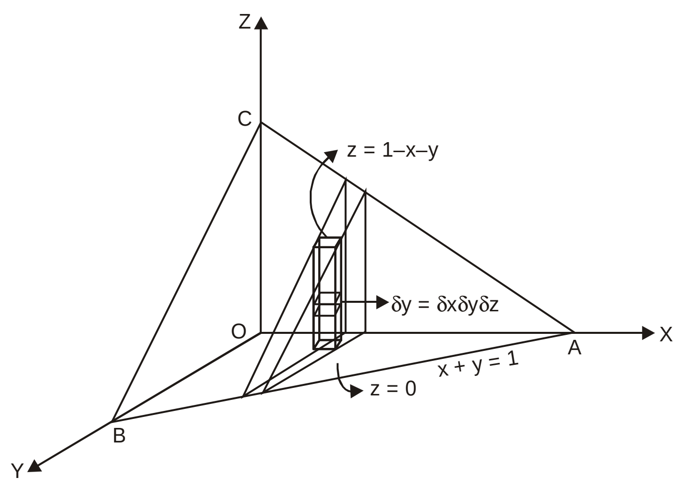


Fig. 4.52

* Peter Gustav Lyeune (1805–1859), German Mathematician.

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\frac{z^n}{n} \right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{n} \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} (1-x-y)^n dx\, dy$$

Put $y = (1 - x)\, t$, so that $dy = (1 - x)\, dt$

Then $$I = \frac{1}{n} \int_0^1 \int_0^1 x^{l-1} \{(1-x)\, t\}^m \cdot \{(1-x)-(1-x)\, t\}^n \, (1-x)\, dx\, dt$$

$$= \frac{1}{n} \int_0^1 x^{l-1} (1-x)^{(m+n-1)-1} dx \int_0^1 t^{m-1} (1-t)^{(n+1)-1}\, dt$$

$$= \frac{1}{n} \beta\,(l, m+n+1) \cdot \beta\,(m, n+1)$$

$$= \frac{1}{n} \frac{\Gamma(l) \cdot \Gamma(m+n+1)}{\Gamma(l+m+n+1)} \cdot \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{1}{n} \frac{\Gamma(l) \cdot \Gamma(m)\Gamma(n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{1}{n} \frac{\Gamma(l) \cdot \Gamma(m) \cdot n\Gamma(n)}{\Gamma(l+m+n+1)} \qquad \left[\because \Gamma(n+1) = n\Gamma(n)\right]$$

$$= \frac{\Gamma(l) \cdot \Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

Hence, $$\boxed{\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma(l) \cdot \Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}}$$

where V is the region given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x + y + z \leq 1$.

This integral is known as **Dritchlet's integral.** This is an important integral useful in evaluating multiple integrals.

Remark: It is to be noted that S is a region bounded by $x \geq 0$, $y \geq 0$ and $x + y \leq 1$ then

$$\iint_S x^{l-1}\, y^{m-1}\, dx\, dy = = \frac{\Gamma l\, \Gamma m}{\Gamma l+m+1}.$$

Note 1: The above integral can be generalised for more than three variables, *i.e.*,

$$\iint \ldots \int x_1^{p_1-1} \cdot x_2^{p_2-1} \cdot x_3^{p_3-1} \ldots x_n^{p_n-1}\, dx_1\, dx_2\, dx_3 \ldots dx_n = \frac{\Gamma(p_1)\,\Gamma(p_2)\,\Gamma(p_3) --- \Gamma(p_n)}{\Gamma(p_1+p_2+p_3+----+p_n+1)}$$

where the region of integration is given by $x_i = 1, 2, ..., n$; such that $x_1 + x_2 + ... + x_n \leq 1$.

Note 2: If the region of integration be $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x + y + z \leq h$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} h.$$

Example 1. Find the value of $\iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$ where x, y, z are always positive with $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$. (*U.P.T.U., 2005*)

Sol. Putting $\left(\frac{x}{a}\right)^p = u,\ \left(\frac{y}{b}\right)^q = v,\ \left(\frac{z}{c}\right)^r = w$, we have

$$x = au^{1/p},\ y = bv^{1/q},\ z = cw^{1/r},$$

$$\therefore \quad dx = \left(\frac{a}{p}\right) u^{\left(\frac{1}{p}-1\right)} du, dy = \left(\frac{b}{q}\right) v^{\left(\frac{1}{q}-1\right)} dv, dz = \left(\frac{c}{r}\right) w^{\left(\frac{1}{r}-1\right)} dw$$

$$\therefore \text{ The given integral} = \iiint \left(au^{\frac{1}{p}}\right)^{l-1} \left(bu^{\frac{1}{q}}\right)^{m-1} \left(cw^{\frac{1}{r}}\right)^{n-1} \left(\frac{a}{p}\right) u^{\frac{1}{p}-1} \cdot \frac{b}{q} v^{\frac{1}{q}-1} \cdot \frac{c}{r} w^{\frac{1}{r}-1} du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{\left(\frac{l}{p}-1\right)} v^{\frac{m}{q}-1} w^{\left(\frac{n}{r}-1\right)} du\, dv\, dw, \text{ where } u + v + w \le 1$$

$$= \frac{a^l b^m c^n}{pqr} \frac{\Gamma\left(\frac{l}{p}\right) \Gamma\left(\frac{m}{q}\right) \Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}.$$

Example 2. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol. The volume in the positive octant will be

$$V = \iiint dx\, dy\, dz$$

For points within positive octant, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$.

Put $\frac{x^2}{a^2} = u$ or $x = a\sqrt{u}$, $y = b\sqrt{v}$, $z = c\sqrt{w}$,

$$\therefore \quad dx = \frac{a}{2} u^{-\frac{1}{2}} du,\ dy = \frac{b}{2} v^{-\frac{1}{2}} dv,\ dz = \frac{c}{2} w^{-\frac{1}{2}} dw$$

$$V = \frac{abc}{8} \iiint u^{\left(\frac{1}{2}-1\right)} v^{\left(\frac{1}{2}-1\right)} w^{\left(\frac{1}{2}-1\right)} du\, dv\, dw, \text{ where } u + v + w \le 1$$

$$= \frac{abc}{8} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}$$

$$= \frac{abc}{8} \frac{\left(\sqrt{\pi}\right)^3}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} = \frac{\pi\, abc}{6}$$

$\therefore$ Total volume $= 8 \times \dfrac{\pi\, abc}{6} = \dfrac{4}{3}\pi\, abc.$

Example 3. The plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$ meets the axis in A, B and C. Apply Dritchlet's integral to find the volume of the tetrahedran $OABC$. Also find its mass if the density at any point is $kxyz$. (*U.P.T.U., 2004*)

Sol. Let $\dfrac{x}{a} = u$, $\dfrac{y}{b} = v$, $\dfrac{z}{c} = w$, then $u \geq 0$, $v \geq 0$, $w \geq 0$ and $u + v + w \leq 1$

Also, $dx = a\, du$, $dy = b\, dv$, $dz = c\, dw$.

$$\text{Volume } OABC = \iiint_D dx\, dy\, dz$$

$$= \iiint_D abc\, du\, dv\, dw, \text{ where } u + v + w < 1$$

$$= abc \iiint_D u^{1-1} v^{1-1} w^{1-1} du\, dr\, dw$$

$$= abc \frac{\left\lceil(1)\right.\left\lceil(1)\right.\left\lceil(1)\right.}{\left\lceil(1+1+1+1)\right.} = \frac{abc}{3!} = \frac{abc}{6}$$

$$\text{Mass} = \iiint_D kxyz\, dx\, dy\, dz = \iiint_{D'} k\,(au)(bv)(cw)\, abc\, du\, dv\, dw$$

$$= ka^2b^2c^2 \iiint_{D'} u^{2-1} v^{2-1} w^{2-1} du\, dv\, dw$$

$$= ka^2b^2c^2 \frac{\left\lceil 2\right.\left\lceil 2\right.\left\lceil 2\right.}{\left\lceil(2+2+2+1)\right.} = ka^2b^2c^2 \frac{1!1!1!}{6!} = \frac{ka^2b^2c^2}{720}.$$

Example 4. Find the mass of an octant of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$, the density at any point being $\rho = kxyz$. (*U.P.T.U., 2002*)

Sol. Putting $\dfrac{x^2}{a^2} = u$, $\dfrac{y^2}{b^2} = v$, $\dfrac{z^2}{c^2} = w$ and $u + v + w = 1$

So that $\dfrac{2xdx}{a^2} = du$, $\dfrac{2ydy}{b^2} = dv$, $\dfrac{2zdz}{c^2} = dw$, then

$$\text{mass} \iiint \rho dv = \iiint (kxyz) dx\, dy\, dz = k \iiint (xdx)(ydy)(zdz)\cdot$$

$$\text{Mass} = k \iiint \left(\frac{a^2du}{2}\right)\left(\frac{b^2dv}{2}\right)\left(\frac{c^2dw}{2}\right)$$

$$= \frac{ka^2b^2c^2}{8} \iiint du\, dv\, dw, \text{ where } u + v + w \leq 1$$

$$= \frac{ka^2b^2c^2}{8} \iiint u^{1-1} v^{1-1} w^{1-1} du\, dv\, dw = \frac{ka^2b^2c^2}{8} \frac{\left\lceil 1\right.\left\lceil 1\right.\left\lceil 1\right.}{\left\lceil 3+1\right.}$$

$$= \frac{ka^2b^2c^2}{8 \times 6} = \frac{ka^2b^2c^2}{48}.$$

Example 5. Find the volume of the solid surrounded by the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1. \qquad (U.P.T.U.,\ 2007)$$

Sol. Let $u=\left(\frac{x}{a}\right)^{\frac{2}{3}} \Rightarrow x=au^{\frac{3}{2}} \Rightarrow dx=\frac{3a}{2}u^{\frac{1}{2}}\,du$

$$v=\left(\frac{y}{b}\right)^{\frac{2}{3}} \Rightarrow y=bv^{\frac{3}{2}} \Rightarrow dy=\frac{3b}{2}v^{\frac{1}{2}}dv$$

$$w=\left(\frac{z}{c}\right)^{\frac{2}{3}} \Rightarrow z=cw^{\frac{3}{2}} \Rightarrow dz=\frac{3c}{2}w^{\frac{1}{2}}\,dw$$

$$\therefore \quad V=8\iiint dx\,dy\,dz=8\iiint\frac{27abc}{8}u^{\frac{1}{2}}\cdot v^{\frac{1}{2}}w^{\frac{1}{2}}du\,dv\,dw$$

$$=27abc\iiint u^{\frac{3}{2}-1},v^{\frac{3}{2}-1},w^{\frac{3}{2}-1}du\,dv\,dw$$

$$=27abc\cdot\frac{\Gamma\frac{3}{2}\Gamma\frac{3}{2}\cdot\Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+1\right)}$$

$$=27abc\cdot\frac{\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{11}{2}}$$

$$=27abc\cdot\frac{\pi^{\frac{3}{2}}}{8\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$=\frac{4\pi abc}{35}.$$

Example 6. Find the value of $\iiint\frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$ the region bounded by coordinate planes and the plane $x + y + z = 1$.

Sol. Here x varies from 0 to 1

y varies from 0 to $1 - x$

z varies from 0 to $1 - x - y$

Thus

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-x}\left[-\frac{1}{2(x+y+z+1)^2}\right]_0^{1-x-y} dx\, dy$$

$$= \int_0^1\int_0^{1-x} -\frac{1}{2}\left[\frac{1}{(x+y+1-x-y+1)^2}-\frac{1}{(x+y+1)^2}\right]dx\, dy$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-\frac{1}{(x+y+1)^2}\right]dx\, dy$$

$$= -\frac{1}{2}\int_0^1\left(\frac{y}{4}+\frac{1}{(x+y+1)}\right)_0^{1-x} dx = -\frac{1}{2}\int_0^1\left(\frac{1-x}{4}+\frac{1}{2}-\frac{1}{1+x}\right)dx$$

$$= -\frac{1}{2}\left[\frac{-(1-x)^2}{8}+\frac{x}{2}-\log(1+x)\right]_0^1 = \frac{1}{2}\left[\frac{1}{2}-\log 2+\frac{1}{8}\right]$$

$$= \frac{1}{2}\log 2-\frac{5}{16}.$$

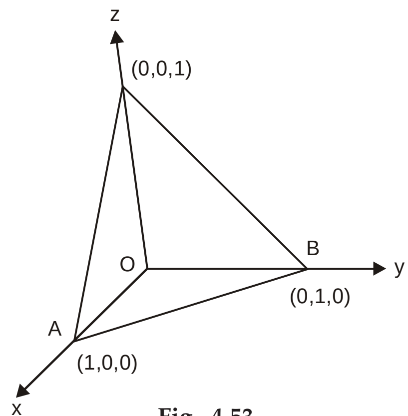


Fig. 4.53

Example 7. Find the area and the mass contained *m* the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^\alpha+\left(\frac{y}{b}\right)^\beta = 1$, where $\alpha > 0$, $\beta > 0$ given that density at any point $\rho(xy)$ is $k\sqrt{xy}$.

[*U.P.T.U., 2008*]

Sol. The area of the plane region is

$$A = \iint_S dx\, dy$$

Let $\left(\frac{x}{a}\right)^\alpha = u$ or $x = au^{1/\alpha}$ so $dx = \frac{a}{\alpha}u^{\frac{1}{\alpha}-1}\cdot du$

$\left(\frac{y}{b}\right)^\beta = v$ or $y = bv^{1/\beta}$ so $dy = \frac{b}{\beta}v^{\frac{1}{\beta}-1}\cdot dv$

$$\therefore \quad A = \iint_s \frac{ab}{\alpha\beta}u^{\frac{1}{\alpha}-1}.\, v^{\frac{1}{\beta}-1}du dv$$

$$A = \frac{ab}{\alpha\beta}\cdot\frac{\Gamma(1/\alpha)\,\Gamma(1/\beta)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}+1\right)} \quad \left| \text{As} \iint_s x^{l-1}y^{m-1}dx\, dy = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}\right.$$

Now the total mass M contained in the plane region A is :

$$M = \iint_S \rho(x,y)dx\, dy = \iint k\sqrt{xy}\, dx\, dy$$

$$= k\iint_{S'} \sqrt{a}\, u^{\frac{1}{2\alpha}} \cdot \sqrt{b}\, v^{\frac{1}{2\beta}} \cdot \frac{ab}{\alpha\beta} u^{\frac{1}{\alpha}-1} \cdot v^{\frac{1}{\beta}-1} \cdot du\, dv$$

$$= k\frac{(ab)^{3/2}}{\alpha\beta}\iint_{S'} u^{\frac{3}{2\alpha}-1} \cdot v^{\frac{3}{2\beta}-1} du\, dv$$

$$= k\frac{(ab)^{3/2}}{\alpha\beta} \cdot \frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\frac{3}{2\beta}\right)}{\Gamma\left(\frac{3}{2\alpha}+\frac{3}{2\beta}+1\right)}.$$

Example 8. Find the mass of the region in the xy-plane bounded by $x = 0$, $y = 0$, $x + y = 1$ with density $k\sqrt{xy}$.

Sol. Mass M contained in the plane region is

$$M = k\int_0^1\int_0^{1-x} \sqrt{xy}\, dx\, dy = k\int_0^1 x^{1/2}\left[\frac{y^{\frac{3}{2}}}{3/2}\right]_0^{1-x} dx = \frac{2k}{3}\int_0^1 x^{1/2}.\,(1-x)^{3/2} dx$$

$$= \frac{2k}{3}\int_0^1 x^{\frac{3}{2}-1}.\,(1-x)^{\frac{5}{2}-1} dx$$

$$= \frac{2k}{3}\beta\left(\frac{3}{2},\frac{5}{2}\right) = \frac{2k}{3}\cdot\frac{\Gamma(3/2)\,\Gamma(5/2)}{\Gamma(4)} = \frac{2k}{3}\cdot\frac{1/2.\sqrt{\pi}\cdot 3/2\cdot 1/2.\sqrt{\pi}}{3\cdot 2\cdot 1} = \frac{k\pi}{24}.$$

EXERCISE 4.7

1. Evaluate $\iiint dx\, dy\, dz$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$. $\left[\textbf{Ans. } \frac{\pi abc}{6}\right]$

2. Find the volume enclosed by the surface

$$\left(\frac{x}{a}\right)^{2n}+\left(\frac{y}{b}\right)^{2n}+\left(\frac{z}{c}\right)^{2n} = 1. \qquad \left[\textbf{Ans. } \frac{1}{6}a^2b^2c^2\right]$$

3. Find the volume of the solid bounded by coordinate planes and the surface

$$\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}} = 1. \qquad \left[\textbf{Ans. } \frac{abc}{90}\right]$$

4. Find the volume of the solid bounded by coordinate planes and the surface

$$\left(\frac{x}{a}\right)^{2n}+\left(\frac{y}{b}\right)^{2n}+\left(\frac{z}{c}\right)^{2n} = 1, \text{ } n \text{ being a positive integer.} \qquad \left[\textbf{Ans. } \frac{abc}{12n^2}\cdot\frac{\left(\Gamma\left(\frac{1}{2n}\right)\right)^3}{\Gamma\left(\frac{3}{2n}\right)}\right]$$

5. Evaluate $\iiint_V x^{\alpha-1} \cdot y^{\beta-1} z^{\Upsilon-1} dx\, dy\, dz$, where V is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the coordinate planes. [*U.P.T.U. (C.O.), 2003*]

$$\left[\textbf{Ans. } \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}{8\,\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\Upsilon}{2}\right)}\right]$$

6. Find the area enclosed by the curve $\left(\frac{x}{a}\right)^{2m} + \left(\frac{y}{b}\right)^{2n} = 1$ m, n being positive integers.

$$\left[\textbf{Ans. } \frac{ab}{4mn} \cdot \frac{\Gamma(1/2m)\,\Gamma(1/2m)}{\Gamma(1/2m + 1/2n + 1)}\right]$$

7. Find the area enclosed by the curve $\left(\frac{x}{a}\right)^{4} + \left(\frac{y}{b}\right)^{10} = 1\cdot$

$$\left[\textbf{Ans. } \frac{ab}{40} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{10}\right)}{\Gamma\left(\frac{27}{20}\right)}\right]$$

OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer of the choices given below :

1. The volume of the solid under the surface $az = x^2 + y^2$ and whose R is the circle $x^2 + y^2 = a^2$ is given as [*UPTU. 2008*]

(*i*) $\frac{\pi}{2a}$ (*ii*) $\frac{\pi a^3}{2}$

(*iii*) $\frac{4}{3}\pi a^3$ (*iv*) None of these

2. The value of triple integral $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dx\, dy\, dz$ is

(*i*) $\frac{4}{35}$ (*ii*) $-\frac{5}{38}$

(*iii*) $\frac{35}{4}$ (*iv*) $\frac{-17}{35}$

3. The volume bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 6$ and $z = 0$ is

(*i*) 32.5 (*ii*) 56

(*iii*) 33 (*iv*) 32

4. The area which is inside the cardioid $r = 2\,(1 + \cos\theta)$ and the outside the circle $r = 2$ is :

(*i*) $\pi + 8$ (*ii*) $\pi - 8$

(*iii*) $3\pi - 7$ (*iv*) None of these

B. Fill in the blanks:

1. The volume of the solid bounded by the surfaces; $z = 0$, $x^2 + y^2 = 1$, $x + y + z = 3$ is ....
2. In spherical coordinate system $dx\ dy\ dz$ = ...
3. In cylindrical coordinate system $dx\ dy\ dz$ = ...
4. The volume of region bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$ and $x = 1$ is ....
5. $\int_0^{\pi}\int_0^{x} \sin y\, dy\, dx$ = ...
6. $\int_0^{\infty} y^{n-1} . e^{-\lambda y}\, dy$ = ...
7. $\int_0^{\pi/2} \sin^m \theta . \cos^n \theta\, d\theta$ = ...
8. $\int_0^{\pi}\int_0^{a\theta} r^3 d\theta\, dr$ = ...

C. Indicate True and False for the following statements:

1. (*i*) Volume in cylindrical coordinates is $\iiint r dr d\theta\, dz$.

   (*ii*) Volume in spherical coordinates is $\iiint r^3 \sin\theta\, dr\, d\theta\, d\phi$.

   (*iii*) The total volume of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$ is multiplied by 4.

   (*iv*) In spherical coordinates $F(x, y, z) = F(r \sin\theta \cos\phi, r \sin\theta \sin\phi, z)$.

2. (*i*) Parallelopiped, ellipsoid and tetrahedron are regular three dimensional domain.

   (*ii*) In the area of the cardioid $r = a\,(1 + \cos\theta)$, r varies from 0 to $a(1 + \cos\theta)$ and θ varies from $-\pi$ to π.

   (*iii*) $\iint x^{l-1} . y^{m-1} dx\, dy = \dfrac{\Gamma l\, \Gamma m}{\Gamma (l+m+1)}$

   (*iv*) Triple integral is not generalization of double integral.

D. Match the following:

1. (*i*) $\beta\,(p, q)$	(*a*) $\Gamma\left(\frac{1}{2}\right)$
(*ii*) $\dfrac{\Gamma p\, \Gamma q}{\Gamma (p+q)}$	(*b*) $\int_0^{\infty} \dfrac{y^{p-1}}{(1+y)^{p+q}} \cdot dy$
(*iii*) $\sqrt{\pi}$	(*c*) $\beta\,(p, q)$
(*iv*) $\dfrac{\pi}{\sin p\pi}$	(*d*) $\Gamma p\, \Gamma (1-p)$

2. (*i*) $\iint_S F(x, y)\, dxdy$	(*a*) $\int_0^1 x^{m-1}(1-x)^{n-1} dx$
(*ii*) $\beta(m, n)$	(*b*) $n\Gamma n$
(*iii*) $\Gamma n+1$	(*c*) $n\pi/\sin n\pi$
(*iv*) $\Gamma 1+n\,\Gamma 1-n$	(*d*) $\iint_S F(u, v)\,\lvert J\rvert\, du\, dv$

ANSWERS TO OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer:

1. (*ii*) **2.** (*i*) **3.** (*iv*)

4. (*i*)

B. Fill in the blanks:

1. 3π **2.** $r^2 \sin\theta\, dr\, d\theta\, d\phi$] **3.** $r\, dr\, d\theta\, dz$

4. $\frac{4}{5}$ **5.** π **6.** $\frac{\Gamma n}{\lambda^n}$

7. $\frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$ **8.** $\frac{a^4\pi^5}{20}$

C. True and False:

1. (*i*) *T* (*ii*) *F* (*iii*) *T* (*iv*) *F*

2. (*i*) *T* (*ii*) *T* (*iii*) *T* (*iv*) *F*

D. Match the following:

1. (*i*) → (*b*), (*ii*) → (*c*), (*iii*) → (*a*), (*iv*) → (*d*)

2. (*i*) → (*d*), (*ii*) → (*a*), (*iii*) → (*b*), (*iv*) → (*c*).

❑❑❑

UNIT V

Vector Calculus

VECTOR DIFFERENTIAL CALCULUS

The vector differential calculus extends the basic concepts of (ordinary) differential calculus to vector functions, by introducing derivative of a vector function and the new concepts of gradient, divergence and curl.

5.1 VECTOR FUNCTION

If the vector $\vec{r}$ varies corresponding to the variation of a scalar variable t that is its length and direction be known and determine as soon as a value of t is given, then $\vec{r}$ is called a vector function of t and written as

$$\vec{r} = \vec{f}(t)$$

and read it as $\vec{r}$ equals a vector function of t.

Any vector $\vec{f}(t)$ can be expressed in the component form

$$\vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$

Where $f_1(t)$, $f_2(t)$, $f_3(t)$ are three scalar functions of t.

For example, $\quad \vec{r} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k}$

where $\quad f_1(t) = 5t^2,\ f_2(t) = t,\ f_3(t) = -t^3.$

5.2 VECTOR DIFFERENTIATION

Let $\vec{r} = \vec{f}(t)$ be a single valued continuous vector point function of a scalar variable t. Let O be the origin of vectors. Let $\overrightarrow{OP}$ represents the vector $\vec{r}$ corresponding to a certain value t to the scalar variable t. Then

$$\vec{r} = \vec{f}(t) \qquad ...(i)$$

Let $\overrightarrow{OQ}$ represents the vector $\vec{r} + \delta\vec{r}$ corresponding to the value $t + \delta t$ of the scalar variable t, where δt is infinitesimally small.

Then,

$$\vec{r} + \delta\vec{r} = \vec{f}(t + \delta t) \quad ...(i)$$

Subracting (*i*) from (*ii*)

$$\delta\vec{r} = \vec{f}(t + \delta t) - \vec{f}(t) \quad ...(iii)$$

Dividing both sides by δt, we get

$$\frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$$

Taking the limit of both side as $\delta t \to 0$.

We obtain
$$\lim_{t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$$

$$\boxed{\frac{d\vec{r}}{dt} = \lim_{\delta t\to 0}\frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t} = \frac{d}{dt}\vec{f}(t)}.$$

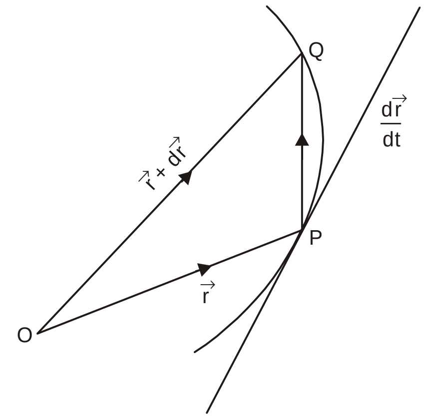


Fig. 5.1

5.3 SOME RESULTS ON DIFFERENTIATION

(*a*) $$\frac{d}{dt}(\vec{r}-\vec{s}) = \frac{d\vec{r}}{dt}-\frac{d\vec{s}}{dt}.$$

(*b*) $$\frac{d(s\vec{r})}{dt} = \frac{ds}{dt}\vec{r} + s\frac{d\vec{r}}{dt}$$

(*c*) $$\frac{d(\vec{a}\cdot\vec{b})}{dt} = \vec{a}\cdot\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}\cdot\vec{b}$$

(*d*) $$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b}+\vec{a}\times\frac{d\vec{b}}{dt}$$

(*e*) $$\frac{d}{dt}\{\vec{a}\times(\vec{b}\times\vec{c})\} = \frac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c})+\vec{a}\times\left(\frac{d\vec{b}}{dt}\times\vec{c}\right)+\vec{a}\times\left(\vec{b}\times\frac{d\vec{c}}{dt}\right).$$

Velocity $$\vec{v} = \dot{r} = \frac{d\vec{r}}{dt}.$$

Acceleration $$a = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}.$$

Example 1. A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t is the time. Find the component of its velocity and acceleration at $t = 1$ in the direction $i + j + 3k$.

Sol. Velocity
$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i}+y\hat{j}+z\hat{k})$$
$$= \frac{d}{dt}\left[(t^3+1)\hat{i}+t^2\hat{j}+(2t+5)\hat{k}\right]$$
$$= 3t^2\,\hat{i}+2t\,\hat{j}+2\hat{k}$$

$$= 3\hat{i} + 2\hat{j} + 2\hat{k}, \text{ at } t = 1.$$

Again unit vector in the direction of $\hat{i} + \hat{j} + 3\hat{k}$ is

$$= \frac{\hat{i} + \hat{j} + 3\hat{k}}{\left(\sqrt{1^2 + 1^2 + 3^2}\right)} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}$$

Therefore, the component of velocity at $t = 1$ in the direction of $\hat{i} + \hat{j} + 3\hat{k}$ is

$$\frac{(3\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (\hat{i} + \hat{j} + 3\hat{k})}{\sqrt{11}} = \frac{3 + 2 + 6}{\sqrt{11}} = \sqrt{11}$$

Acceleration $\quad \dfrac{d^2\vec{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right) = 6t\hat{i} + 2\hat{j} = 6\hat{i} + 2\hat{j}$, at $t = 1$

Therefore, the component of acceleration at $t = 1$ in the direction $\hat{i} + \hat{j} + 3\hat{k}$ is

$$\frac{(6\hat{i} + 2\hat{j}) \cdot (\hat{i} + \hat{j} + 3\hat{k})}{\sqrt{11}} = \frac{6 + 2}{\sqrt{11}} = \frac{8}{\sqrt{11}}.$$

Example 2. A particle moves along the curve $x = 4 \cos t$, $y = 4 \sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \pi/2$. Find also the magnitudes of the velocity and acceleration at any time t.

Sol. Let $\vec{r} = 4 \cos t\,\hat{i} + 4 \sin t\,\hat{j} + 6t\,\hat{k}$

at $\quad t = 0, \quad \vec{v} = 4\hat{j} + 6\hat{k}$

at $\quad t = \dfrac{\pi}{2}, \quad \vec{v} = -4\hat{i} + 6\hat{k}$

at $\quad t = 0, \quad |v| = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$

at $\quad t = \dfrac{\pi}{2}, \quad |v| = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}.$

Again, acceleration, $\quad \vec{a} = \dfrac{d^2\vec{r}}{dt^2} = -4 \cos t\,\hat{i} - 4 \sin t\,\hat{j}$

at $\quad t = 0, \quad \vec{a} = -4\hat{i}$

$\therefore$ at $\quad t = 0, \quad |a| = \sqrt{(-4)^2} = 4$

at $\quad t = \dfrac{\pi}{2}, \quad \vec{a} = -4\hat{j}$

at $\quad t = \dfrac{\pi}{2}, \quad |a| = \sqrt{(-4)^2} = 4.$

Example 3. If $\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}$, where a, b are constant vectors, then prove that

$$\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$$

Sol. $\quad \vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt} \quad$ ...(*i*)

$$\frac{d\vec{r}}{dt} = \vec{a}e^{nt} \cdot n + \vec{b}e^{-nt} \cdot (-n)$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}e^{nt}\cdot n^2 + \vec{b}e^{-nt}\cdot(-n)^2$$

$$\frac{d^2\vec{r}}{dt^2} = n^2\left[\vec{a}e^{nt} + \vec{b}e^{-nt}\right] = n^2 r \qquad \text{[From } (i)]$$

$$\Rightarrow \qquad \frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0. \qquad \textbf{Hence proved.}$$

Example 4. If $\vec{a}, \vec{b}, \vec{c}$ are constant vectors then show that $\vec{r} = \vec{a}t^2 + \vec{b}t + \vec{c}$ is the path of a point moving with constant acceleration.

Sol.

$$\vec{r} = \vec{a}t^2 + \vec{b}t + \vec{c}$$

$$\frac{d\vec{r}}{dt} = 2\vec{a}t + \vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = 2\vec{a} \text{ which is a constant vector.}$$

Hence, acceleration of the moving point is a constant. **Hence proved.**

EXERCISE 5.1

1. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = \sin 3t$.

Find the velocity and acceleration at $t = 0$.

[**Hint:** $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$]

$\left[\textbf{Ans.} \text{ Vel. } = \sqrt{10}, \text{ acc. } = 5\sqrt{13}\right]$

2. A particle moves along the curve

$$x = 3t^2,\ y = t^2 - 2t,\ z = t^2.$$

Find the velocity and acceleration at $t = 1$. $\left[\textbf{Ans. } \vec{v} = 2\sqrt{10},\ a = 2\sqrt{11}\right]$

3. Find the angle between the directions of the velocity and acceleration vectors at time t of a particle with position vector $\vec{r} = (t^2+1)\hat{i} - 2t\hat{j} + (t^2-1)\hat{k}$. $\left[\textbf{Ans.} \text{ arc cos } t\,\frac{\sqrt{2}}{\sqrt{2t^2+1}}\right]$

4. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

$\left[\textbf{Ans. } \frac{8\sqrt{14}}{7}; -\frac{\sqrt{14}}{7}\right]$

5. If $\vec{r}$ = (sec t) $\hat{i}$ + (tan t) $\hat{j}$ be the position vector of P. Find the velocity and acceleration of P at $t = \frac{\pi}{6}$. $\left[\textbf{Ans. } \frac{2}{3}\hat{i} + \frac{4}{3}\hat{j}, \frac{2}{3\sqrt{3}}(5\hat{i} + 4\hat{j})\right]$

5.4 SCALAR POINT FUNCTION

If for each point P of a region R, there corresponds a scalar denoted by $f(P)$, then f is called a "scalar point function" for the region R.

Example 1. The temperature $f(P)$ at any point P of a certain body occupying a certain region R is a scalar point function.

Example 2. The distance of any point $P(x, y, z)$ in space from a fixed point (x_0, y_0, z_0) is a scalar function.

$$f(P) = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\,.$$

Scalar field (*U.P.T.U., 2001*)

Scalar field is a region in space such that for every point P in this region, the scalar function f associates a scalar $f(P)$.

5.5 VECTOR POINT FUNCTION

If for each point P of a region R, there corresponds a vector $\vec{f}(P)$ then $\vec{f}$ is called "vector point function" for the region R.

Example. If the velocity of a particle at a point P, at any time t be $\vec{f}\,(P)$, then $\vec{f}$ is a vector point function for the region occupied by the particle at time t.

If the coordinates of P be (x, y, z) then

$$\vec{f}\,(P) = f_1\,(x, y, z)\; i + f_2\,(x, y, z)\; j + f_3\,(x, y, z)\; k.$$

Vector field (*U.P.T.U., 2001*)

Vector field is a region in space such that with every point P in the region, the vector function $\vec{f}$ associates a vector $\vec{f}(P)$.

Del operator: The linear vector differential (Hamiltorian) operator "del" defined and denoted as

$$\nabla = \hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

This operator also read nabla. It is not a vector but combines both differential and vectorial properties analogous to those of ordinary vectors.

5.6 GRADIENT OR SLOPE OF SCALAR POINT FUNCTION

If $f(x, y, z)$ be a scalar point function and continuously differentiable then the vector

$$\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}$$

is called the gradient of f and is written as grad f. (*U.P.T.U., 2006*)

Thus

$$\boxed{\text{grad } f = i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}=\vec{\nabla} f}$$

It should be noted that $\vec{\nabla} f$ is a vector whose three components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\cdot$ Thus, if f is a scalar point function, then $\vec{\nabla} f$ is a vector point function.

5.7 GEOMETRICAL MEANING OF GRADIENT, NORMAL

(U.P.T.U., 2001)

Consider any point P in a region throught which a scalar field $f(x, y, z) = c$ defined. Suppose that $\vec{\nabla} f \neq 0$ at P and that there is a f = const. surface S through P and a tangent plane T; for instance, if f is a temperature field, then S is an isothermal surface (level surface). If $\hat{n}$, at P, is choosen as any vector in the tangent plane T, then surely $\frac{df}{dS}$ must be zero.

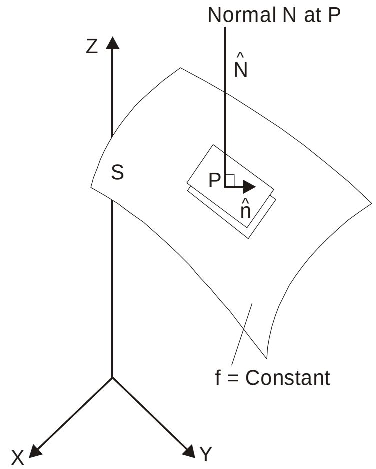


Fig. 5.2

Since $\frac{df}{dS} = \vec{\nabla} f \cdot \hat{n} = 0$

for every $\hat{n}$ at P in the tangent plane, and both $\vec{\nabla} f$ and $\hat{n}$ are non-zero, it follows that $\vec{\nabla} f$ is normal to the tangent plane T and hence to the surface S at P.

If letting $\hat{n}$ be in the tangent plane, we learn that $\vec{\nabla} f$ is normal to S, then to seek additional information about $\vec{\nabla} f$ it seems logical to let $\hat{n}$ be along the normal line at P,.

Then $\frac{df}{ds} = \frac{df}{dN}$, $\hat{N} = \hat{n}$ then

$$\frac{df}{dN} = \vec{\nabla} f \cdot \hat{N} = \left|\vec{\nabla} f\right| \cdot 1 \cos 0 = \left|\vec{\nabla} f\right| \cdot$$

So that the magnitude of $\vec{\nabla} f$ is the directional derivative of f along the normal line to S, in the direction of increasing f.

Hence, "The gradient $\left(\vec{\nabla} f\right)$ of scalar field $f(x, y, z)$ at P is vector normal to the surface f = const. and has a magnitude is equal to the directional derivative $\frac{df}{dN}$ in that direction.

5.8 DIRECTIONAL DERIVATIVE

Let $f = f(x, y, z)$ then the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the derivatives (rates of change) of f in the direction of the coordinate axes OX, OY, OZ respectively. This concept can be extended to define a derivative of f in a "given" direction $\overrightarrow{PQ}$.

Let P be a point in space and $\hat{b}$ be a unit vector from P in the given direction. Let s be the are length measured from P to another point Q along the ray C in the direction of $\hat{b}$. Now consider

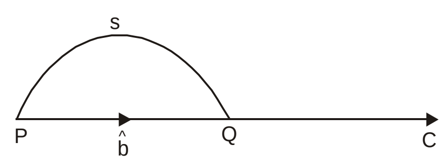


Fig. 5.3

$$f(s) = f(x, y, z) = f\{x(s), y(s), z(s)\}$$

Then $$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \quad ...(i)$$

Here $\frac{df}{ds}$ is called directional derivative of f at P in the direction $\hat{b}$ which gives the rate of change of f in the direction of b.

Since, $$\frac{dx}{ds}\hat{i}+\frac{dy}{ds}\hat{j}+\frac{dz}{ds}\hat{k} = \hat{b} = \text{unit vector} \qquad ...(ii)$$

Eqn. (i) can be rewritten as

$$\frac{df}{ds} = \left(\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{ds}\hat{i}+\frac{dy}{ds}\hat{j}+\frac{dz}{ds}\hat{k}\right)$$

$$\frac{df}{ds} = \left[\left(\hat{i}\frac{\partial}{\delta x}+\hat{y}\frac{\partial}{\delta y}+\hat{k}\frac{\partial}{\delta z}\right)f\right]\cdot\hat{b} = \nabla f\cdot\hat{b} \qquad ...(iii)$$

Thus the directional derivative of f at P is the component (dot product) of ∇f in the direction of (with) unit vector $\hat{b}$.

Hence the directional derivative in the direction of any unit vector $\vec{a}$ is

$$\boxed{\frac{df}{ds} = \nabla f\cdot\left(\frac{\vec{a}}{|\vec{a}|}\right)}$$

Normal derivative $\frac{df}{dn} = \nabla f\cdot\hat{n}$, where $\hat{n}$ is the unit normal to the surface f = constant.

5.9 PROPERTIES OF GRADIENT

Property I: $(\vec{a}\cdot\nabla)f = \vec{a}\cdot(\nabla f)$

Proof: L.H.S. $= (\vec{a}\cdot\nabla)f$

$$= \left\{\vec{a}\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\right\}f$$

$$= \left\{(\vec{a}\cdot\hat{i})\frac{\partial}{\partial x}+(\vec{a}\cdot\hat{j})\frac{\partial}{\partial y}+(\vec{a}\cdot\hat{k})\frac{\partial}{\partial z}\right\}f$$

$$= (\vec{a}\cdot\hat{i})\frac{\partial f}{\partial x}+(\vec{a}\cdot\hat{j})\frac{\partial f}{\partial y}+(\vec{a}\cdot\hat{k})\frac{\partial f}{\partial z} \qquad ...(i)$$

R.H.S. $= \vec{a}\cdot(\nabla f)$

$$= a\cdot\left(i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}\right)$$

$$= (a\cdot i)\frac{\partial f}{\partial x}+(a\cdot j)\frac{\partial f}{\partial y}+(a\cdot k)\frac{\partial f}{\partial z} \qquad ...(ii)$$

From (i) and (ii),

$$\boxed{(\vec{a}\cdot\nabla)f = \vec{a}\cdot(\nabla f).}$$

Property II: ***Gradient of a Constant***

The necessary and sufficient condition that scalar point function ϕ is a constant is that $\nabla\phi = 0$.

Proof: Let $\phi(x, y, z) = c$

Then, $$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0$$

$$\therefore \quad \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$
$$= i.0 + j.0 + k.0$$
$$= 0.$$

Hence, the condition is necessary.

Conversely: Let $\nabla\phi = 0$.

Then, $$i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = 0.i + 0.j + 0.k.$$

Equating the coefficients of i, j, k.

On both sides, we get $\frac{\partial\phi}{\partial x} = 0. \frac{\partial\phi}{\partial y} = 0, \frac{\partial\phi}{\partial z} = 0$

$\Rightarrow \phi$ is independent of x, y, z

$\Rightarrow \phi$ is constant.

Hence, the condition is sufficient.

Property III: ***Gradient of the Sum of Difference of Two Functions***

If f and g are any two scalar point functions, then

$$\nabla (f \pm g) = \nabla f \pm \nabla g$$

or $$\text{grad } (f \pm g) = \text{grad } f \pm \text{grad } g$$

Proof: $$\nabla (f \pm g) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(f \pm g)$$

$$= i\frac{\partial}{\partial x}(f \pm g) + j\frac{\partial}{\partial y}(f \pm g) + k\frac{\partial}{\partial z}(f \pm g)$$

$$= i\left(\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}\right) + j\left(\frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y}\right) + k\left(\frac{\partial f}{\partial z} \pm \frac{\partial g}{\partial z}\right)$$

$$= \left(i\frac{\partial f}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial f}{\partial z}\right) \pm \left(i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}\right)$$

$$\boxed{\nabla(f \pm g) = \text{grad } f \pm \text{grad } g.}$$

Property IV: ***Gradient of the Product of Two Functions***

If f and g are two scalar point functions, then

$$\nabla(fg) = f\nabla g + g\nabla f$$

or $$\text{grad } (fg) = f\,(\text{grad } g) + g\,(\text{grad } f).$$

Proof: $$\nabla(fg) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(fg)$$

$$= i \frac{\partial}{\partial x}(fg) + j \frac{\partial}{\partial y}(fg) + k \frac{\partial}{\partial z}(fg)$$

$$= i\left(f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x}\right) + j\left(f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y}\right) + k\left(f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z}\right)$$

$$= f\left(i \frac{\partial g}{\partial x} + j \frac{\partial g}{\partial y} + k \frac{\partial g}{\partial z}\right) + g\left(i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}\right)$$

$$\boxed{\nabla(fg) = f\nabla g + g\nabla f.}$$

Property V: ***Gradient of the Quotient of Two Functions***

If f and g are two scalar point functions, then

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}, \; g \neq 0$$

or

$$\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}, \neq 0.$$

Proof:

$$\nabla\left(\frac{f}{g}\right) = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right)\left(\frac{f}{g}\right)$$

$$= i \frac{\partial}{\partial x}\left(\frac{f}{g}\right) + j \frac{\partial}{\partial y}\left(\frac{f}{g}\right) + k \frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$

$$= i \frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2} + j \frac{g \frac{\partial f}{\partial y} - f \frac{\partial g}{\partial y}}{g^2} + k \frac{g \frac{\partial f}{\partial z} - f \frac{\partial g}{\partial z}}{g^2}$$

$$= \frac{g\left[i \frac{\partial f}{\partial z} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}\right] - f\left[i \frac{\partial g}{\partial x} + j \frac{\partial g}{\partial y} + k \frac{\partial g}{\partial z}\right]}{g^2}$$

$$\boxed{\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}.}$$

Example 1. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that (*U.P.T.U., 2007*)

(*i*) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector

(*ii*) $\text{grad } r = \frac{\vec{r}}{r}$

(*iii*) $\text{grad } \frac{1}{r} = -\frac{\vec{r}}{r^3}$

(*iv*) $\text{grad } r^n = nr^{n-2}\,\vec{r}$, where $r = |\vec{r}|$.

Sol. (*i*) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$

then $\vec{a} \cdot \vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = a_1x + a_2y + a_3z.$

$\therefore \qquad \nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= a_1 i + a_2 j + a_3 k = \vec{a}$. **Hence proved.**

(*ii*) $\qquad \text{grad } r = \Delta r = \Sigma\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}$

$= \Sigma\hat{i}\frac{x}{(x^2+y^2+z^2)^{1/2}} = \Sigma\hat{i}\frac{x}{r} = r$

Hence, $\qquad \text{grad } r = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{r} = \dfrac{\vec{r}}{r} = \hat{r}$.

(*iii*) $\qquad \text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\vec{r} = \frac{-1}{r^2}\frac{\vec{r}}{r}$

$= -\dfrac{\vec{r}}{r^3}$. **Proved.**

(*iv*) Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Now, $\qquad \text{grad } r^n = \nabla r^n = \Sigma\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}$

$= n\,(x^2 + y^2 + z^2)^{n/2-1}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$

$= n(x^2 + y^2 + z^2)^{(n-1)/2}\dfrac{\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{\left(x^2+y^2+z^2\right)^{1/2}}$

$= nr^{n-1}\dfrac{\vec{r}}{r}$

$= nr^{n-2}\vec{r}$.

Example 2. If $f = 3x^2y - y^3z^2$, find grad f at the point (1, –2, –1). (*U.P.T.U., 2006*)

Sol. $\qquad \text{grad } f = \vec{\nabla}f = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$

$= i\frac{\partial}{\partial x}\left(3x^2y-y^3z^2\right)+j\frac{\partial}{\partial y}(3x^2y-y^3z^2)+k\frac{\partial}{\partial z}(3x^2y-y^3z^2)$

$= i(6xy) + j(3x^2 - 3y^2z^2) + k(-2y^3z)$

grad ϕ at $(1, -2, -1) = i(6)(1)(-2) + j\,[(3)(1) - 3(4)(1)] + k(-2)(-8)(-1)$

$= -12i - 9j - 16k.$

Example 3. Find the directional derivative of $\dfrac{1}{r}$ in the direction $\vec{r}$ where $\vec{r} = xi + yj + zk$. (*U.P.T.U., 2002, 2005*)

Sol. Here $f(x, y, z) = \dfrac{1}{r} = \dfrac{1}{\sqrt{x^2+y^2+z^2}} = (x^2 + y^2 + z^2)^{-1/2}$

Now $\qquad \vec{\nabla}f = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left(x^2+y^2+z^2\right)^{-1/2}$

$$= \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-1/2} i + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\, j + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} k$$

$$= \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2x\right\} i + \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2y\right\}\; j + \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2z\right\} k$$

$$= \frac{-(xi+yj+zk)}{\left(x^2+y^2+z^2\right)^{3/2}}$$

and $\hat{a}$ = unit vector in the direction of $xi + yj + zk$

$$= \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}} \qquad \text{As } \hat{a} = \frac{\vec{r}}{|r|}$$

$\therefore$ Directional derivative $= \nabla f \cdot \hat{a}$ $= -\dfrac{xi+yj+zk}{(x^2+y^2+z^2)^{3/2}} \cdot \dfrac{xi+yj+zk}{\left(x^2+y^2+z^2\right)^{1/2}}$

$$= -\frac{(xi+yj+zk)^2}{(x^2+y^2+z^2)^2} = -\left(\frac{xi+yj+zk}{x^2+y^2+z^2}\right)^2.$$

Example 4. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2i - j - 2k$. In what direction the directional derivative will be maximum and what is its magnitude? Also find a unit normal to the surface $x^2yz + 4xz^2 = 6$ at the point $(1, -2, -1)$.

Sol. $\phi = x^2yz + 4xz^2$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2xyz + 4z^2$$

$$\frac{\partial \phi}{\partial y} = x^2 z,$$

$$\frac{\partial \phi}{\partial z} = x^2 y + 8xz$$

$$\text{grad } \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$= (2xyz + 4z^2)\, i + (x^2 z)\, j + (x^2 y + 8xz)k$$

$= 8i - j - 10k$ at the point $(1, -2, -1)$

Let $\hat{a}$ be the unit vector in the given direction.

Then $$\hat{a} = \frac{2i-j-2k}{\sqrt{4+1+4}} = \frac{2i-j-2k}{3}$$

$\therefore$ Directional derivative $$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{a}$$

$$= (8i - j - 10k) \cdot \left(\frac{2i-j-2k}{3}\right)$$

$$= \frac{16+1+20}{3} = \frac{37}{3}.$$

Again, we know that the directional derivative is maximum in the direction of normal which is the direction of grad ϕ. Hence, the directional derivative is maximum along grad $\phi = 8i - j - 10k$.

Further, maximum value of the directional derivative

$$= |\text{grad } \phi|$$

$$= |8i - j - 10k|$$

$$= \sqrt{64+1+100} = \sqrt{165}.$$

Again, a unit vector normal to the surface

$$= \frac{\text{grad } \phi}{|\text{grad } \phi|}$$

$$= \frac{8i - j - 10k}{\sqrt{165}}.$$

Example 5. What is the greatest rate of increase of $u = xyz^2$ at the point (1, 0, 3)?

Sol.

$$u = xyz^2$$

$$\therefore \quad \text{grad } u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$$

$$= yz^2\, i + xz^2\, \mathbf{j} + 2xyz\, k$$

$$= 9j \text{ at (1, 0, 3) point.}$$

Hence, the greatest rate of increase of u at (1, 0, 3)

$$= |\text{grad } u| \text{ at (1, 0, 3) point.}$$

$$= |9j| = 9.$$

Example 6. Find the directional derivative of

$$\phi = (x^2 + y^2 + z^2)^{-1/2}$$

at the points (3, 1, 2) in the direction of the vector $yz\, i + zx\, j + xy\, k$.

Sol.

$$\phi = (x^2 + y^2 + z^2)^{-1/2}$$

$$\therefore \quad \text{grad } \phi = i\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + j\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + k\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= i\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\right] + j\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)\right] + k\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)\right]$$

$$= -\frac{xi + yj + zk}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{3i + j + 2k}{(9+1+4)^{3/2}} \text{ at (3, 1, 2)}$$

$$= -\frac{3i + j + 2k}{14\sqrt{14}} \text{ at (3, 1, 2)}$$

Let $\hat{a}$ be the unit vector in the given direction, then

$$\hat{a} = \frac{yzi + zxj + xyk}{\sqrt{y^2z^2 + z^2x^2 + x^2y^2}}$$

$$= \frac{2i + 6j + 3k}{7} \text{ at } (3, 1, 2)$$

Now, $$\frac{d\phi}{ds} = \hat{a} \,.\, \text{grad } \phi$$

$$= \left(\frac{2i + 6j + 3k}{7}\right) \cdot \left(-\frac{3i + j + 2k}{14\sqrt{14}}\right)$$

$$= -\frac{(2)(3) + (6)(1) + (3)(2)}{7.14\sqrt{14}}$$

$$= -\frac{18}{7.14\sqrt{14}} = -\frac{9}{49\sqrt{14}}.$$

Example 7. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ, where Q is the point (5, 0, 4).

Sol. Here

Position vector of P = $i + 2j + 3k$

Position vector of Q = $5i + 0j + 4k$

$\therefore$ $$\overrightarrow{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$= (5i + 0j + 4k) - (i + 2j + 3k)$$

$$= 4i - 2j + k.$$

Let $\hat{a}$ be the unit vector along PQ, then

$$\hat{a} = \frac{4i - 2j + k}{\sqrt{16 + 4 + 1}} = \frac{4i - 2j + k}{\sqrt{21}}$$

Also, $$\text{grad } \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$= 2x\, i - 2y\, j + 4z\, k$$

$$= 2i - 4j + 12k \text{ at } (1, 2, 3)$$

Hence, $$\frac{d\phi}{ds} = \hat{a} \,.\, \text{grad } \phi$$

$$= \left(\frac{4i - 2j + k}{\sqrt{21}}\right) \cdot (2i - 4j + 12k)$$

$$= \frac{(4)(2) + (-2)(-4) + (1)(12)}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}}.$$

Example 8. For the function

$$\phi = \frac{y}{x^2 + y^2},$$

find the magnitude of the directional derivative making an angle 30° with the positive X-axis at the point (0, 1).

Sol. Here

$$\phi = \frac{y}{x^2+y^2}$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial \phi}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} \text{ and } \frac{\partial \phi}{\partial z} = 0$$

$$\therefore \quad \text{grad } \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$= i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} \qquad \left[\because \frac{\partial \phi}{\partial z} = 0\right]$$

$$= \frac{-2xy}{(x^2+y^2)^2} i + \frac{x^2-y^2}{(x^2+y^2)^2} j$$

$$= -j \text{ at } (0, 1).$$

Let $\hat{a}$ be the unit vector along the line making an angle 30° with the positive X-axis at the point (0, 1), then

$$\hat{a} = \cos 30° \; i + \sin 30° \; j.$$

Hence, the directional derivative is given by

$$\frac{d\phi}{ds} = \hat{a} \,.\, \text{grad } \phi$$

$$= (\cos 30° \; i + \sin 30° \; j) \cdot (-j)$$

$$= -\sin 30° = -\frac{1}{2}.$$

Example 9. Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, –1) has a maximum magnitude 64 in the direction parallel to Z-axis.

Sol.

$$\phi = axy^2 + byz + cz^2x^3$$

$$\therefore \quad \text{grad } \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$= (ay^2 + 3cz^2x^2)\, i + (2axy + bz)\, j + (by + 2czx^3)\, k$$

$$= (4a + 3c)\, i + (4a - b)\, j + (2b - 2c)\, k. \text{ at } (1, 2, -1).$$

Now, we know that the directional derivative is maximum along the normal to the surface, *i.e.*, along grad ϕ. But we are given that the directional derivative is maximum in the direction parallel to Z-axis, *i.e.*, parallel to the vector k.

Hence, the coefficients of i and j in grad ϕ should vanish and the coefficient of k should be positive. Thus

$$4a + 3c = 0 \qquad ...(i)$$

$$4a - b = 0 \qquad ...(ii)$$

and

$$2b - 2c > 0$$

$$b > c \qquad ...(iii)$$

Then

$$\text{grad } \phi = 2(b - c)\, k.$$

Also, maximum value of the directional derivative $= |\text{grad } \phi|$

$$\Rightarrow \quad 64 = |2(b-c)k| = 2(b-c) \qquad [\because b > c]$$

$$\Rightarrow \quad b - c = 32. \qquad \text{...}(iv)$$

Solving equations (*i*), (*ii*) and (*iv*), we obtain

$$a = 6,\ b = 24,\ c = -8.$$

Example 10. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$, at the point (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the values of a, b and c. (*U.P.T.U., 2001*)

Sol. Given $\qquad \phi = ax^2y + by^2z + cz^2x$

$$\therefore \quad \vec{\nabla}\phi = i\left(\frac{\partial\phi}{\partial x}\right) + j\left(\frac{\partial\phi}{\partial y}\right) + k\left(\frac{\partial\phi}{\partial z}\right)$$

$$= i(2axy + cz^2) + j(ax^2 + 2byz) + k(by^2 + 2czx)$$

$$\vec{\nabla}\phi \text{ at the point } (1, 1, 1) = i(2a + c) + j(a + 2b) + k(b + 2c).$$

We know that the maximum value of the directional derivative is in the direction of $\vec{\nabla} f$

i.e., $\quad |\nabla\phi| = 15 \Rightarrow (2a + c)^2 + (2b + a)^2 + (2c + b)^2 = (15)^2$

But, the directional derivative is given to be maximum parallel to the line.

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

$$\Rightarrow \quad \frac{2a+c}{2} = \frac{2b+a}{-2} = \frac{2c+b}{1}$$

$$\Rightarrow \quad 2a + c = -2b - a \Rightarrow 3a + 2b + c = 0 \qquad (i)$$

and $\quad 2b + a = -2(2c + b)$

$$\Rightarrow \quad 2b + a = -4c - 2b \Rightarrow a + 4b + 4c = 0 \qquad (ii)$$

Solving (*i*) and (*ii*), we get

$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k \text{ (say)}$$

$$\Rightarrow \quad a = 4k,\ b = -11k, \text{ and } c = 10k.$$

Now, $\quad (2a + c)^2 + (2b + a)^2 + (2c + b)^2 = (15)^2$

$$\Rightarrow \quad (8k + 10k)^2 + (-22k + 4k)^2 + (20k - 11k)^2 = (15)^2$$

$$\Rightarrow \quad k = \pm\frac{5}{9}$$

$$\Rightarrow \quad a = \pm\frac{20}{9},\ b = \pm\frac{55}{9} \text{ and } c = \pm\frac{50}{9}.$$

Example 11. Prove that $\nabla\int f(u)\,du = f(u)\,\nabla u$.

Sol. Let $\int f(u)\,du = F(u)$, a function of u so that $\frac{\partial F}{\partial u} = f(u)$

Then $\qquad \nabla\int f(u)\,du = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)F$

$$= i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$$

$$= i\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + j\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + k\frac{\partial F}{\partial u}\frac{\partial u}{\partial z}$$

$$= \frac{\partial F}{\partial u}\left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right)$$

$= f(u)\ \nabla u.$ **Hence proved.**

Example 12. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, – 1, 2) (*U.P.T.U., 2002*)

Sol. Let
$$\phi_1 = x^2 + y^2 + z^2 - 9$$
$$\phi_2 = x^2 + y^2 - z - 3$$

$$\therefore \quad \vec{\nabla}\phi_1 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2xi + 2yj + 2zk$$

$\nabla\phi_1$ at the point (2, – 1, 2) $= 4i - 2j + 4k$ (*i*)

$$\vec{\nabla}\phi_2 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 - z - 3) = 2xi + 2yj - k$$

$\nabla\phi_2$ at the point (2, – 1, 2) $= 4i - 2j - k$ (*ii*)

Let θ be the angle between normals (*i*) and (*ii*)

$$(4i - 2j + 4k) \cdot (4i - 2j - k) = \sqrt{16+4+16}\ \sqrt{16+4+1}\ \cos\theta$$

$$16 + 4 - 4 = 6\sqrt{21}\ \cos\theta \Rightarrow 16 = 6\sqrt{21}\ \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{8}{3\sqrt{21}} \Rightarrow \theta = \cos^{-1}\frac{8}{3\sqrt{21}}.$$

Example 13. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that grad u, grad v and grad w are coplanar vectors. (*U.P.T.U., 2002*)

Sol.
$$\text{grad } u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad } w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy)$$

$$= \hat{i}(z+y) + \hat{j}(z+x) + \hat{k}(y+x)$$

Now,

$$\text{grad } u\ (\text{grad } v \times \text{grad } w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+z+y & y+z+x & z+y+x \\ z+y & z+x & y+x \end{vmatrix} \quad \text{| Applying } R_2 \to R_2 + R_3$$

$$= 2\,(x + y + z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Hence, grad u, grad v and grad w are coplanar vectors.

Example 14. Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1, 1, 1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. (*U.P.T.U., 2003*)

Sol. $\vec{\nabla}\phi = \left[10xy + \frac{5}{2}z^2\right]i + (5x^2 - 10yz)\;j + (-5y^2 + 5zx)k$

$\nabla\,\phi$ at $P(1, 1, 1) = \frac{25}{2}i - 5j$

Direction Ratio of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ are 2, – 2, 1

Direction cosines of the line are $\frac{2}{\sqrt{(2)^2 + (-2)^2 + (1)^2}}, \frac{-2}{\sqrt{2^2 + (-2)^2 + 1}}, \frac{1}{\sqrt{2^2 + (-2)^2 + 1}}$

i.e., $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}$

Directional derivative in the direction of the line

$$= \left(\frac{25}{2}i - 5j\right)\cdot\left(\frac{2}{3}i - \frac{2}{3}j + \frac{1}{3}k\right)$$

$$= \frac{25}{3} + \frac{10}{3}$$

$$= \frac{35}{3}.$$

Example 15. Prove that $\nabla\cdot\left\{\frac{f(r)\vec{r}}{r}\right\} = \frac{1}{r^2}\frac{d}{dr}(r^2 f)$.

Sol. $$\nabla\cdot\left\{\frac{f(r)\vec{r}}{r}\right\} = \nabla\cdot\left\{f(r)\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{r}\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{f(r)x}{r}\right\} + \frac{\partial}{\partial y}\left\{\frac{f(r)y}{r}\right\} + \frac{\partial}{\partial z}\left\{\frac{f(r)z}{r}\right\}$$

Now, $$\frac{\partial}{\partial x}\left\{\frac{f(r)x}{r}\right\} = x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x} + \frac{f(r)}{r}$$

$$= x\left\{\frac{1}{r}\frac{df}{dr} - \frac{f(r)}{r^2}\right\}\frac{x}{r} + \frac{f(r)}{r} \quad \text{as} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \frac{x^2}{r^2}\frac{df(r)}{dr} - \frac{x^2}{r^3}f(r) + \frac{f(r)}{r}$$

Similarly, $$\frac{\partial}{\partial y}\left\{\frac{f(r)y}{r}\right\} = \frac{y^2}{r^2}\frac{df(r)}{dr} - \frac{y^2}{r^3}f(r) + \frac{f(r)}{r}$$

and $$\frac{\partial}{\partial z}\left\{\frac{f(r)z}{r}\right\} = \frac{z^2}{r^2}\frac{df(r)}{dr} - \frac{z^2}{r^3}f(r) + \frac{f(r)}{r}$$

Now using these results, we get

$$\nabla\cdot\left[\frac{f(r)\vec{r}}{r}\right] = \frac{df(r)}{dr} + \frac{2}{r}f(r)$$

$$= \frac{1}{r^2}\frac{d}{dr}(r^2 f)\cdot$$ **Hence proved.**

Example 16. Find $f(r)$ such that $\nabla f = \frac{\vec{r}}{r^5}$ and $f(1) = 0$.

Sol. It is given that

$$\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = \nabla f = \frac{\vec{r}}{r^5} = \frac{xi + yj + zk}{r^5}$$

So, $$\frac{\partial f}{\partial x} = \frac{x}{r^5},\ \frac{\partial f}{\partial y} = \frac{y}{r^5} \text{ and } \frac{\partial f}{\partial z} = \frac{z}{r^5}$$

We know that $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \frac{x}{r^5}dx + \frac{y}{r^5}dy + \frac{z}{r^5}dz$$

$$df = \frac{xdx + ydy + zdz}{r^5} = \frac{rdr}{r^5} = r^{-4}dr$$

Integrating $$f(r) = \frac{r^{-3}}{-3} + c$$

Since $$0 = f(1) = -\frac{1}{3} + c$$

So, $$c = \frac{1}{3}$$

Thus, $$f(r) = \frac{1}{3} - \frac{1}{3}\frac{1}{r^3}.$$

EXERCISE 5.2

1. Find grad f where $f = 2xz^4 - x^2y$ at $(2, -2, -1)$. **[Ans.** $10i - 4j - 16k$**]**

2. Find ∇f when $f = (x^2 + y^2 + z^2)\ e^{-\sqrt{x^2+y^2+z^2}}$. **[Ans.** $(2 - r)e^{-r}\ \vec{r}$**]**

3. Find the unit normal to the surface $x^2y + 2xz = 4$ at the point (2, –2, 3).

$$\left[\textbf{Ans. } \pm\frac{1}{3}(i - 2j - 2k)\right]$$

4. Find the directional derivative of $f = x^2yz + 4xz^2$ at (1, –2, –1) in the direction $2i - j - 2k$.

$$\left[\textbf{Ans. } \frac{37}{3}\right]$$

5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, –1, 2) .

$$\left[\textbf{Ans. } \theta = \cos^{-1}\left(\frac{8\sqrt{21}}{63}\right)\right]$$

6. Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $i + 2j + 2k$ at the point (1, 2, 0).

$$\left[\textbf{Ans. } \frac{10}{3}\right]$$

7. If $\nabla f = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$, find $f(x, y, z)$ if $f(1, -2, 2) = 4$. $\left[\textbf{Ans. } f = x^2yz^3 + 20\right]$

8. Find f given $\nabla f = 2xi + 4yj + 8zk$. $\left[\textbf{Ans. } f = x^2 + 2y^2 + 4z^2\right]$

9. Find the directional derivative of $\phi = (x^2 + y^2 + z^2)^{-1/2}$ at the point $P(3, 1, 2)$ in the direction of the vector, $yzi + xzj + xyk$.

$$\left[\textbf{Ans. } -\frac{9}{49\sqrt{14}}\right]$$

10. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)\cdot$

11. Show that $\nabla^2\left(\frac{x}{r^3}\right) = 0$, where r is the magnitude of position vector $\vec{r} = xi + yj + zk$.

[*U.P.T.U. (C.O.), 2002*]

12. Find the direction in which the directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at (1, 1) is zero.

$$\left[\textbf{Ans. } \frac{1+i}{\sqrt{2}}\right]$$

13. Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$, where $\vec{r} = xi + yj + zk$.

$$\left[\textbf{Ans. } -\frac{1}{r^2}\right]$$ (*U.P.T.U., 2003*)

14. Show that $\nabla r^{-3} = -3r^{-5}\vec{r}$.

15. If $\phi = \log|\vec{r}|$, show that $\nabla\phi = \frac{\vec{r}}{r^2}\cdot$ [*U.P.T.U., 2008*]

5.10 DIVERGENCE OF A VECTOR POINT FUNCTION

If $\vec{f}(x, y, z)$ is any given continuously differentiable vector point function then the divergence of $\vec{f}$ scalar function defined as (*U.P.T.U., 2006*)

$$\nabla \cdot \vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot \vec{f} = i\cdot\frac{\partial \vec{f}}{\partial x} + j\cdot\frac{\partial \vec{f}}{\partial y} + k\cdot\frac{\partial \vec{f}}{\partial z} = div\,\vec{f}.$$

5.11 PHYSICAL INTERPRETATION OF DIVERGENCE

Let $\vec{v} = v_x i + v_y j + v_z k$ be the velocity of the fluid at $P(x, y, z)$.

Here we consider the case of fluid flow along a rectangular parallelopiped of dimensions δx, δy, δz

Mass in $= v_x\, \delta y \delta z$ (along x-axis)

Mass out $= v_x(x + \delta x)\, \delta y \delta z$

$$= \left(v_x + \frac{\partial v_x}{\partial x}\delta x\right)\delta y \delta z$$

| By Taylor's theorem

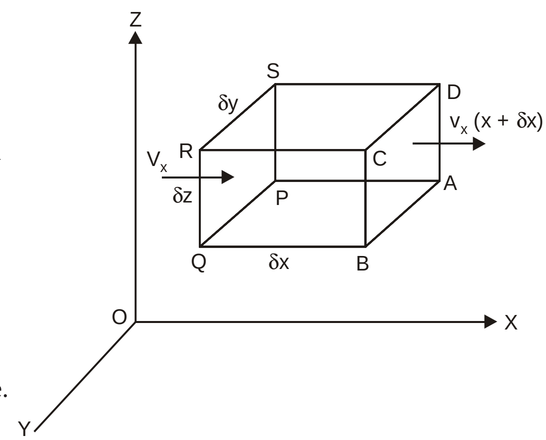


Fig. 5.4

Net amount of mass along x-axis

$$= v_x\, \delta y \delta z - \left(v_x + \frac{\partial v_x}{\partial x}\delta x\right)\delta y \delta z$$

$$= -\frac{\partial v_x}{\partial x}\delta x \delta y \delta z$$

| Minus sign shows decrease.

Similar net amount of mass along y-axis

$$= -\frac{\partial v_y}{\partial y}\delta x \delta y \delta z$$

and net amount of mass along z-axis $= -\frac{\partial v_z}{\partial z}\delta x \delta y \delta z$

$\therefore$ Total amount of fluid across parallelopiped per unit time $= -\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta x \delta y \delta z$

Negative sign shows decrease of amount

$\Rightarrow$ Decrease of amount of fluid per unit time $= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta x \delta y \delta z$

Hence the rate of loss of fluid per unit volume

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$= \nabla \cdot \vec{v} = \text{div } \vec{v}.$$

Therefore, div $\vec{v}$ represents the rate of loss of fluid per unit volume.

Solenoidal: For compressible fluid there is no gain no loss in the volume element

$\therefore$ $div\ \vec{v} = 0$

then $\vec{v}$ is called Solenoidal vector function.

5.12 CURL OF A VECTOR

If $\vec{f}$ is any given continuously differentiable vector point function then the curl of $\vec{f}$ (vector function) is defined as

$$\boxed{\text{Curl}\,\vec{f} = \nabla\times\vec{f} = i\times\frac{\partial\vec{f}}{\partial x}+j\times\frac{\partial\vec{f}}{\partial y}+k\times\frac{\partial\vec{f}}{\partial z}} \qquad (U.P.T.U., 2006)$$

Let $\vec{f} = f_x i + f_y j + f_z k$, then

$$\nabla\times\vec{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}.$$

5.13 PHYSICAL MEANING OF CURL

Here we consider the relation $\vec{v} = \vec{w}\times\vec{r}$, $\vec{w}$ is the angular velocity $\vec{r}$ is position vector of a point on the rotating body (*U.P.T.U., 2006*)

$$\text{curl}\ \vec{v} = \vec{\nabla}\times\vec{v}$$

$$= \vec{\nabla}\times(\vec{w}\times\vec{r})$$

$$= \bar{\nabla}\times[(w_1 i+w_2 j+w_3 k)\times(xi+yj+zk)] \qquad \begin{bmatrix}\vec{w}=w_1 i+w_2 j+w_3 k\\ \vec{r}=xi+yj+zk\end{bmatrix}$$

$$= \bar{\nabla}\times\begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \bar{\nabla}\times[(w_2 z-w_3 y)i-(w_1 z-w_3 x)j+(w_1 y-w_2 x)k]$$

$$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times[(w_2 z-w_3 y)i-(w_1 z-w_3 x)\ j+(w_1 y-w_2 x)k]$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_2 z-w_3 y & w_3 x-w_1 z & w_1 y-w_2 x \end{vmatrix}$$

$$= (w_1 + w_1)\ i - (-w_2 - w_2)\ j + (w_3 + w_3)\ k$$

$$= 2\ (w_1 i + w_2 j + w_3 k) = 2\vec{w}$$

Curl $\vec{v} = 2\vec{w}$ which shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

Irrotational vector: If curl $\vec{f} = 0$, then the vector $\vec{f}$ is said to be irrotational. Vice-versa, if $\vec{f}$ is irrotational then, curl $\vec{f} = 0$.

5.14 VECTOR IDENTITIES

Identity 1: grad $uv = u$ grad $v + v$ grad u

Proof:

$$\begin{aligned}
\text{grad } (uv) &= \nabla (uv) \\
&= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(uv) \\
&= \hat{i}\frac{\partial}{\partial x}(uv) + \hat{j}\frac{\partial}{\partial y}(uv) + \hat{k}\frac{\partial}{\partial z}(uv) \\
&= i\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) + j\left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) + k\left(u\frac{\partial v}{\partial z} + v\frac{\partial u}{\partial z}\right) \\
&= u\left(i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}\right) + v\left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right)
\end{aligned}$$

or $$\boxed{\text{grad } uv = u \text{ grad } v + v \text{ grad } u}.$$

Identity 2: $\boxed{\text{grad } (\vec{a}\cdot\vec{b}) = \vec{a} \times \text{curl } \vec{b} + \vec{b} \times \text{curl } \vec{a} + (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a}}$

Proof:

$$\begin{aligned}
\text{grad } (\vec{a}\cdot\vec{b}) &= \Sigma i\frac{\partial}{\partial x}(\vec{a}\cdot\vec{b}) = \Sigma i\left(\frac{\partial \vec{a}}{\partial x}\cdot\vec{b} + \vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right) \\
&= \Sigma i\left(\vec{b}\cdot\frac{\partial \vec{a}}{\partial x}\right) + \Sigma i\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right). \qquad ...(i)
\end{aligned}$$

Now, $$\vec{a} \times \left(i \times \frac{\partial \vec{b}}{\partial x}\right) = \left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i - (\vec{a}\cdot i)\frac{\partial \vec{b}}{dx}$$

$$\Rightarrow \quad \left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \vec{a} \times \left(i \times \frac{\partial \vec{b}}{\partial x}\right) + (\vec{a}\cdot i)\frac{\partial \vec{b}}{\partial x}$$

$$\Rightarrow \quad \Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \Sigma\vec{a} \times \left(i \times \frac{\partial \vec{b}}{\partial x}\right) + \Sigma(\vec{a}\cdot i)\frac{\partial \vec{b}}{\partial x}$$

$$\Rightarrow \quad \Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \vec{a} \times \Sigma\left(i \times \frac{\partial \vec{b}}{\partial x}\right) + \Sigma\left(\vec{a}\cdot i\frac{\partial}{\partial x}\right)\vec{b}$$

$$\Rightarrow \quad \Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)i = \vec{a} \times \text{curl}\,\vec{b} + (\vec{a}\cdot\nabla)\vec{b}\,\cdot \qquad ...(ii)$$

Interchanging $\vec{a}$ and $\vec{a}$, we get

$$\Sigma\left(\vec{b}\cdot\frac{\partial \vec{a}}{\partial x}\right)i = \vec{b} \times \text{curl}\,\vec{a} + (\vec{b}\cdot\nabla)\vec{a} \qquad ...(iii)$$

From equations (*i*), (*ii*) and (*iii*), we get

$$\boxed{\text{grad } (\vec{a}\cdot\vec{b}) = \vec{a} \times \text{curl } \vec{b} + \vec{b} \times \text{curl } \vec{a} + (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a}}.$$

Identity 3: $\text{div}\,(u\vec{a}) = u\,\text{div}\,\vec{a} + \vec{a}\cdot\text{grad}\,u$ (*U.P.T.U., 2004*)

Proof:

$$\begin{aligned}
\text{div}\,(u\vec{a}) &= \nabla\cdot(u\vec{a}) \\
&= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(u\vec{a}) \\
&= i\cdot\frac{\partial}{\partial x}(u\vec{a})+j\cdot\frac{\partial}{\partial y}(u\vec{a})+k\cdot\frac{\partial}{\partial z}(u\vec{a}) \\
&= i\cdot\left\{\frac{\partial u}{\partial x}\vec{a}+u\frac{\partial\vec{a}}{\partial x}\right\}+j\cdot\left\{\frac{\partial u}{\partial y}\vec{a}+u\frac{\partial\vec{a}}{\partial y}\right\}+k\cdot\left\{\frac{\partial u}{\partial z}\vec{a}+u\frac{\partial\vec{a}}{\partial z}\right\} \\
&= u\left\{i\cdot\frac{\partial\vec{a}}{\partial x}+j\cdot\frac{\partial\vec{a}}{\partial y}+k\cdot\frac{\partial\vec{a}}{\partial z}\right\}+\vec{a}\cdot\left\{i\frac{\partial u}{\partial x}+j\frac{\partial u}{\partial y}+k\frac{\partial u}{\partial z}\right\}
\end{aligned}$$

or

$$\boxed{\text{div}\,(u\vec{a}) = u\,\text{div}\,\vec{a} + \vec{a}\cdot\text{grad}\,u}.$$

Identity 4: $\text{div}\,(\vec{a}\times\vec{b}) = \vec{b}\cdot\text{curl}\,\vec{b} - \vec{a}\cdot\text{curl}\,\vec{b}$ [*U.P.T.U. (C.O.), 2003*]

Proof:

$$\begin{aligned}
\text{div}\,(\vec{a}\times\vec{b}) &= \nabla\cdot(\vec{a}\times\vec{b}) \\
&= \Sigma i\cdot\frac{\partial}{\partial x}(\vec{a}\times\vec{b}) \\
&= \Sigma i\cdot\left(\frac{\partial\vec{a}}{\partial x}\times\vec{b}+\vec{a}\times\frac{\partial\vec{b}}{\partial x}\right) \\
&= \Sigma i\cdot\left(\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right)+\Sigma i\cdot\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x}\right) \\
&= \Sigma\left(i\times\frac{\partial\vec{a}}{\partial x}\right)\cdot\vec{b}-\Sigma\left(i\times\frac{\partial\vec{b}}{\partial x}\right)\cdot\vec{a} \\
&= (\text{curl}\,\vec{a})\cdot\vec{b} - (\text{curl}\,b)\cdot\vec{a}
\end{aligned}$$

or

$$\boxed{\text{div}\,(\vec{a}\times\vec{b}) = \vec{b}\cdot\text{curl}\,\vec{a} - \vec{a}\cdot\text{curl}\,\vec{b}}.$$

Identity 5: $\text{curl}\,(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}$ [*U.P.T.U. (C.O.), 2003*]

Proof:

$$\begin{aligned}
\text{curl}\,(u\vec{a}) &= \nabla\times(u\vec{a}) \\
&= \Sigma i\times\frac{\partial}{\partial x}(u\vec{a}) \\
&= \Sigma i\times\left(\frac{\partial u}{\partial x}\vec{a}+u\frac{\partial\vec{a}}{\partial x}\right) \\
&= \Sigma\left(i\frac{\partial u}{\partial x}\right)\times\vec{a}+u\Sigma\left(i\times\frac{\partial\vec{a}}{\partial x}\right) \\
&= (\text{grad}\,u)\times\vec{a} + u\,\text{curl}\,\vec{a}
\end{aligned}$$

or

$$\boxed{\text{curl}\,(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}}.$$

Identity 6: $\text{curl}(\vec{a} \times \vec{b}) = \vec{a}\ \text{div}\ \vec{b} - \vec{b}\ \text{div}\ \vec{a} + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$

Proof:

$$\begin{aligned}
\text{curl}\,(\vec{a} \times \vec{b}) &= \nabla \times (\vec{a} \times \vec{b}) \\
&= \Sigma i \times \frac{\partial}{\partial x}(\vec{a} \times \vec{b}) \\
&= \Sigma i \times \left(\frac{\partial \vec{a}}{\partial x} \times \vec{b} + \vec{a} \times \frac{\partial \vec{b}}{\partial x}\right) \\
&= \Sigma i \times \left(\frac{\partial \vec{a}}{\partial x} \times \vec{b}\right) + \Sigma i \times \left(\vec{a} \times \frac{\partial \vec{b}}{\partial x}\right) \\
&= \Sigma\,(i.b)\frac{\partial \bar{a}}{\partial x} - \Sigma\left(i\cdot\frac{\partial \bar{a}}{\partial x}\right)\bar{b} + \Sigma\left(i\cdot\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - \Sigma\,(i.\bar{a})\frac{\partial \bar{b}}{\partial x} \\
&= \Sigma\,(i.\vec{b})\frac{\partial}{\partial x}\vec{a} - \Sigma\left(i\cdot\frac{\partial \vec{a}}{\partial x}\right)\vec{b} + \Sigma\left(i\cdot\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - \Sigma\left(\vec{a}\cdot\vec{i}\,\frac{\partial}{\partial x}\right)\vec{b} \\
&= (\vec{b}\cdot\nabla)\ \vec{a} - (\text{div}\ \vec{a})\ \vec{b} + (\text{div}\ \vec{b})\ \vec{a} - (\vec{a}.\nabla)\ \vec{b} \\
&= (\vec{b}\cdot\nabla)\ \vec{a} - (\vec{a}\cdot\nabla)\ \vec{b} + \vec{a}\ \text{div}\ \vec{b} - \vec{b}\ \text{div}\ \vec{a}.
\end{aligned}$$

or $\boxed{\text{curl}\,(\vec{a} \times \vec{b}) = \vec{a}\ \text{div}\ \vec{b} - \vec{b}\ \text{div}\ \vec{a} + (\vec{b}\cdot\nabla)\ \vec{a} - (\vec{a}\cdot\nabla)\vec{b}}$.

Identity 7: $\text{div grad}\ f = \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$

Proof:

$$\begin{aligned}
\text{div grad}\ f &= \nabla\cdot(\nabla f) \\
&= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

or $\boxed{\text{div grad}\ f = \nabla^2 f}$.

Identity 8: $\text{curl grad}\ f = 0$

Proof:

$$\begin{aligned}
\text{curl grad}\ f &= \nabla \times (\nabla f) \\
&= \Sigma i\frac{\partial}{\partial x} \times \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right) \\
&= \Sigma i \times \left(i\frac{\partial^2 f}{\partial x^2} + j\frac{\partial^2 f}{\partial x\partial y} + k\frac{\partial^2 f}{\partial x\partial z}\right) \\
&= \Sigma\left(k\frac{\partial^2 f}{\partial x\partial y} - j\frac{\partial^2 f}{\partial x\partial z}\right) \\
&= \left(k\frac{\partial^2 f}{\partial x\partial y} - j\frac{\partial^2 f}{\partial x\partial z}\right) + \left(i\frac{\partial^2 f}{\partial y\partial z} - k\frac{\partial^2 f}{\partial y\partial x}\right) + \left(j\frac{\partial^2 f}{\partial z\partial x} - i\frac{\partial^2 f}{\partial z\partial y}\right)
\end{aligned}$$

or $\boxed{\text{curl grad}\ f = 0}$.

Identity 9: div curl $\vec{f} = 0$

Proof: div curl $\vec{f} = \nabla \cdot (\nabla \times \vec{f})$

$$= \Sigma i \cdot \frac{\partial}{\partial x}\left\{ i \times \frac{\partial \vec{f}}{\partial x} + j \times \frac{\partial \vec{f}}{\partial y} + k \times \frac{\partial \vec{f}}{\partial z} \right\}$$

$$= \Sigma i \cdot \left\{ i \times \frac{\partial^2 \vec{f}}{\partial x^2} + j \times \frac{\partial^2 \vec{f}}{\partial x \partial y} + k \times \frac{\partial^2 \vec{f}}{\partial x \partial z} \right\}$$

$$= \Sigma \left\{ (i \times i) \cdot \frac{\partial^2 \vec{f}}{\partial x^2} + (i \times j) \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} + (i \times k) \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} \right\}$$

$$= \Sigma \left\{ k \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} - j \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} \right\}$$

$$= \left\{ k \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} - j \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} \right\} + \left\{ i \cdot \frac{\partial^2 \vec{f}}{\partial y \partial z} - k \cdot \frac{\partial^2 \vec{f}}{\partial y \partial x} \right\} + \left\{ j \cdot \frac{\partial^2 \vec{f}}{\partial z \partial x} - i \cdot \frac{\partial^2 \vec{f}}{\partial z \partial y} \right\}$$

$$\boxed{\text{div curl } \vec{f} = 0}.$$

Identity 10: grad div $\vec{f}$ = curl curl $\vec{f} + \frac{\partial^2 \vec{f}}{\partial x^2} + \frac{\partial^2 \vec{f}}{\partial y^2} + \frac{\partial^2 \vec{f}}{\partial z^2}$

Proof: Curl curl $\vec{f} = \nabla \times (\nabla \times \vec{f})$

$$= \Sigma i \frac{\partial}{\partial x} \times \left\{ i \times \frac{\partial \vec{f}}{\partial x} + j \times \frac{\partial \vec{f}}{\partial y} + k \times \frac{\partial \vec{f}}{\partial z} \right\}$$

$$= \Sigma i \times \left\{ i \times \frac{\partial^2 \vec{f}}{\partial x^2} + j \times \frac{\partial^2 \vec{f}}{\partial x \partial y} + k \times \frac{\partial^2 \vec{f}}{\partial x \partial z} \right\}$$

$$= \Sigma \left[\left\{ \left(i \cdot \frac{\partial^2 \vec{f}}{\partial x^2} \right) i - (i \cdot i) \cdot \frac{\partial^2 \vec{f}}{\partial x^2} \right\} + \left\{ \left(i \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} \right) j - (i \cdot j) \frac{\partial^2 \vec{f}}{\partial x \partial y} \right\} + \left\{ \left(i \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} \right) k - (i \cdot k) \frac{\partial^2 \vec{f}}{\partial x \partial z} \right\} \right]$$

$$= \Sigma \left[\left(i \cdot \frac{\partial^2 \vec{f}}{\partial x^2} \right) i + \left(i \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} \right) j + \left(i \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} \right) k \right] - \Sigma \frac{\partial^2 \vec{f}}{\partial x^2}$$

$$\Rightarrow \text{Curl curl } \vec{f} + \frac{\partial^2 \vec{f}}{\partial x^2} + \frac{\partial^2 \vec{f}}{\partial y^2} + \frac{\partial^2 \vec{f}}{\partial z^2} = \Sigma \left[\left(i \cdot \frac{\partial^2 \vec{f}}{\partial x^2} \right) i + \left(i \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} \right) j + \left(i \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} \right) k \right]. \quad ...(i)$$

Again, grad div $\vec{f} = \Sigma i \frac{\partial}{\partial x} \left\{ i \cdot \frac{\partial \vec{f}}{\partial x} + j \cdot \frac{\partial \vec{f}}{\partial y} + k \cdot \frac{\partial \vec{f}}{\partial z} \right\}$

$$= \Sigma i \left\{ i \cdot \frac{\partial^2 \vec{f}}{\partial x^2} + j \cdot \frac{\partial^2 \vec{f}}{\partial x \partial y} + k \cdot \frac{\partial^2 \vec{f}}{\partial x \partial z} \right\}$$

$$= \Sigma\left[\left(\hat{i}\cdot\frac{\partial^2\vec{f}}{\partial x^2}\right)\hat{i}+\left(\hat{j}\cdot\frac{\partial^2\vec{f}}{\partial x\partial y}\right)\hat{i}+\left(\hat{k}\cdot\frac{\partial^2\vec{f}}{\partial x\partial z}\right)\hat{i}\right]$$

$$= \Sigma\left[\left(\hat{i}\cdot\frac{\partial^2\vec{f}}{\partial x^2}\right)\hat{i}+\left(\hat{i}\cdot\frac{\partial^2\vec{f}}{\partial z\partial x}\right)\hat{j}+\left(\hat{i}\cdot\frac{\partial^2\vec{f}}{\partial y\partial z}\right)\hat{k}\right] \quad ...(ii)$$

From eqns. (*i*) and (*ii*), we prove

$$\text{grad div } \vec{f} = \text{curl curl } \vec{f} + \nabla^2 \vec{f}$$

Example 1. If $\vec{f} = xy^2\, i + 2x^2yz\, j - 3yz^2\, k$ then find div $\vec{f}$ and curl $\vec{f}$ at the point (1, – 1, 1).

Sol. We have $\vec{f} = xy^2\, i + 2x^2yz\, j - 3yz^2\, k$

$$\text{div } \vec{f} = \frac{\partial}{\partial x}(xy^2)+\frac{\partial}{\partial y}(2x^2yz)+\frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$$= (-1)^2 + 2(1)^2(1) - 6(-1)(1) \quad \text{at } (1, -1, 1)$$

$$= 1 + 2 + 6 = 9.$$

Again,

$$\text{curl } \vec{f} = \text{curl }[xy^2\, i + 2x^2yz\, j - 3yz^2\, k]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(-3yz^2)-\frac{\partial}{\partial z}(2x^2yz)\right\}+j\left\{\frac{\partial}{\partial z}(xy^2)-\frac{\partial}{\partial x}(-3yz^2)\right\} + k\left\{\frac{\partial}{\partial x}(2x^2yz)-\frac{\partial}{\partial y}(xy^2)\right\}$$

$$= i\,[-3z^2 - 2x^2y] + j\,[0 - 0] + k\,[4xyz - 2xy]$$

$$= (-3z^2 - 2x^2y)i + (4xyz - 2xy)\,k$$

$$= \{-3(1)^2 - 2(1)^2(-1)\}\; i + \{4(1)(-1)(1) - 2(1)(-1)\}\; k \text{ at } (1, -1, 1)$$

$$= -i - 2k.$$

Example 2. Prove that

(*i*) div $\vec{r}$ = 3. (*ii*) curl $\vec{r}$ = 0.

Sol. (*i*)

$$\text{div } \vec{r} = \nabla\cdot\vec{r}$$

$$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(xi+yj+zk)$$

$$= \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3.$$

(*ii*)

$$\text{curl } \vec{r} = \nabla\times\vec{r}$$

$$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times(xi+yj+zk)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right]+j\left[\frac{\partial}{\partial z}(x)-\frac{\partial}{\partial x}(z)\right]+k\left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]$$

$= i\ (0) + j\ (0) + k\ (0)$

$= 0 + 0 + 0 = 0.$

Example 3. Find the divergence and curl of the vector

$$(x^2 - y^2)\ i + 2xy\ j + (y^2 - xy)\ k.$$

Sol. Let $\vec{f} = (x^2 - y^2)\ i + 2xy\ j + (y^2 - xy)\ k.$

Then
$$\text{div } \vec{f} = \frac{\partial}{\partial x}(x^2-y^2)+\frac{\partial}{\partial y}(2xy)+\frac{\partial}{\partial z}(y^2-xy)$$

$= 2x + 2x + 0 = 4x$

and
$$\text{curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & y^2-xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(y^2-xy)-\frac{\partial}{\partial z}(2xy)\right]+j\left[\frac{\partial}{\partial z}(x^2-y^2)-\frac{\partial}{\partial x}(y^2-xy)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(2xy)-\frac{\partial}{\partial y}(x^2-y^2)\right]$$

$= i\ [2y - x) - 0] + j\ [0 - (-y)] + k\ [(2y) - (-\ 2y)]$

$= (2y - x)\ i + y\ j + 4y\ k.$ **Ans.**

Example 4. If $\vec{f}\ (x, y, z) = xz^3\ \hat{i}\ - 2x^2yz\ \hat{j}\ + 2yz^4\ \hat{k}$ find divergence and curl of $\vec{f}\ (x, y, z)$ (*U.P.T.U., 2006*)

Sol.
$$\text{div } \vec{f} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(xz^3 i - 2x^2yz\ j + 2yz^4\ k\right)$$

$$= \frac{\partial}{\partial x}(xz^3)-\frac{\partial}{\partial y}(2x^2yz)+\frac{\partial}{\partial z}(2yz^4)$$

$= z^3 - 2x^2z + 8yz^3$

$$\text{curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(2yz^4)+\frac{\partial}{\partial z}(2x^2yz)\right\}-j\left\{\frac{\partial}{\partial x}(2yz^4)-\frac{\partial}{\partial z}(xz^3)\right\}$$

$$+ k\left\{\frac{\partial}{\partial x}(-2x^2yz)-\frac{\partial}{\partial y}(xz^3)\right\}$$

$= i\ (2z^4 + 2x^2y) - j\ (0 - 3z^2x) + k\ (-\ 4xyz - 0)$

$= 2\ (x^2y + z^4)\ i + 3z^2xj - 4xyz\cdot k.$

Example 5. Find the directional derivative of $\nabla.\ \vec{u}$ at the point (4, 4, 2) in the direction of the corresponding outer normal of the sphere $x^2 + y^2 + z^2 = 36$ where $\vec{u} = x^4\, i + y^4\, j + z^4\, k$.

Sol. $\nabla.\ \vec{u} = \nabla.\ (x^4 i + y^4 j + z^4 k) = 4(x^3 + y^3 + z^3) = f$ (say)

$\therefore$ $(\nabla f)_{(4,\,4,\,2)} = 12\ (x^2 i + y^2 j + z^2 k)_{(4,\,4,\,2)} = 48(4i + 4j + k)$

Normal to the sphere $g \equiv x^2 + y^2 + z^2 = 36$ is

$$(\nabla g)_{(4,\,4,\,2)} = 2\ (xi + yj + zk)_{(4,\,4,\,2)} = 4(2i + 2j + k)$$

$$\hat{a} = \text{unit normal} = \frac{\nabla g}{|\nabla g|} = \frac{4(2i+2j+k)}{\sqrt{64+64+16}}$$

$$= \frac{2i+2j+k}{3}$$

The required directional derivative is

$$\nabla f.\,\hat{a} = 48\ (4i + 4j + k).\ \frac{2i+2j+k}{3}$$

$$= 16(8 + 8 + 1) = 272.$$

Example 6. A fluid motion is given by $\vec{v} = (y + z)i + (z + x)j + (x + y)k$ show that the motion is irrotational and hence find the scalar potential. (*U.P.T.U., 2003*)

Sol. Curl $\vec{v} = \Delta \times \vec{v}$

$$= \left(i\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left[(y+z)i + (z+x)j + (x+y)k\right]$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = i\,(1-1) - j\,(1-1) + k\,(1-1) = 0$$

Hence $\vec{v}$ is irrotational.

Now
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right) \cdot (idx + jdy + kdz)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi \cdot d\vec{r} = \nabla\phi \cdot d\vec{r} = \vec{v} \cdot d\vec{r} \qquad \left|\, \vec{v} = \nabla\phi \right.$$

$$= [(y + z)\ i + (z + x)\ j + (x + y)\ k].\ (idx + jdy + kdz)$$

$$= (y + z)\ dx + (z + x)\ dy + (x + y)\ dz$$

$$= ydx + zdx + zdy + xdy + xdz + ydz$$

On integrating
$$\phi = \int (ydx + xdy) + \int (zdy + ydz) + \int (zdx + xdz)$$

$$= \int d(xy) + \int d(yz) + \int d(zx)$$

$$\phi = xy + yz + zx + c$$

Thus, velocity potential $= xy + yz + zx + c.$

Example 7. Prove that $\vec{a}\times(\nabla\times\vec{r}) = \nabla(\vec{a}\cdot\vec{r})-(\vec{a}\cdot\nabla)\vec{r}$ where $\vec{a}$ is a constant vector and $\vec{r} = xi + yj + zk$. *(U.P.T.U., 2007)*

Sol. Let

$$\vec{a} = a_1 i + a_2 j + a_3 k$$
$$\vec{r} = r_1 i + r_2 j + r_3 k$$

$$\therefore \quad \nabla \times \vec{r} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r_1 & r_2 & r_3 \end{vmatrix} = i\left(\frac{\partial r_3}{\partial y}-\frac{\partial r_2}{\partial z}\right)-\left(\frac{\partial r_3}{\partial x}-\frac{\partial r_1}{\partial z}\right)j+\left(\frac{\partial r_2}{\partial x}-\frac{\partial r_1}{\partial y}\right)k$$

Now
$$\vec{a}\times(\nabla\times\vec{r}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \dfrac{\partial r_3}{\partial y}-\dfrac{\partial r_2}{\partial x} & \dfrac{\partial r_1}{\partial z}-\dfrac{\partial r_3}{\partial x} & \dfrac{\partial r_2}{\partial x}-\dfrac{\partial r_1}{\partial y} \end{vmatrix}$$

$$= \left\{\left(a_2\frac{\partial r_2}{\partial x}-a_2\frac{\partial r_1}{\partial y}\right)-\left(a_3\frac{\partial r_1}{\partial z}-a_3\frac{\partial r_3}{\partial x}\right)\right\}i$$
$$-\left\{\left(a_1\frac{\partial r_2}{\partial x}-a_1\frac{\partial r_1}{\partial y}\right)-\left(a_3\frac{\partial r_3}{\partial y}-a_3\frac{\partial r_2}{\partial z}\right)\right\}j$$
$$+\left\{\left(-a_1\frac{\partial r_3}{\partial x}+a_1\frac{\partial r_1}{\partial z}\right)-\left(a_2\frac{\partial r_3}{\partial y}-a_2\frac{\partial r_2}{\partial z}\right)\right\}k$$

$$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(a_1 r_1+a_2 r_2+a_3 r_3)-\left[a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right](r_1 i+r_2 j+r_3 k)$$

$$= \nabla\left\{(a_1 j+a_2 j+a_3 k)\cdot(r_1 j+r_2 j+r_3 k)\right\}$$
$$-\left\{(a_1 i+a_2 j+a_3 k)\cdot\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\right\}(r_1 i+r_2 j+r_3 k)$$

$$= \nabla(\vec{a}\cdot\vec{r})-(\vec{a}\cdot\nabla)\vec{r}\cdot$$ **Hence proved.**

Example 8. Find the directional derivative of $\nabla\cdot(\nabla f)$ at the point (1, –2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $f = 2x^3y^2z^4$. *(U.P.T.U., 2008)*

Sol.
$$\nabla f = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(2x^3y^2z^4) = (6x^2y^2z^4)i + (4x^3yz^4)j + 8x^3y^2z^3)k$$

$$\nabla.(\nabla f) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\{(6x^2\,y^2z^4)i + (4x^3\,yz^4)j + (8x^3\,y^2z^3)k\}$$

or
$$\nabla.(\nabla f) = 12x\,y^2z^4 + 4x^3\,z^4 + 24x^3\,y^2z^2 = F(x, y, z) \text{ (say)}$$

Now
$$\nabla F = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(12x\,y^2z^4 + 4x^3\,z^4 + 24x^3\,y^2z^2)$$
$$= (12y^2\,z^4 + 12x^2\,z^4 + 72x^2\,y^2z^2)i + (24xyz^4 + 48x^3\,yz^2)\,j$$
$$+ (48x\,y^2z^3 + 16x^3z^3 + 48x^3y^2z)k$$

$\therefore \qquad (\nabla F)_{(1,\,-2,\,1)} = 348i - 144j + 400k$

Let $\qquad g(x, y, z) = xy^2z - 3x - z^2 = 0$

$$\nabla g = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\, i + (2xyz)\, j + (xy^2 - 2z)\, k$$

$$(\nabla g)_{(1,\,-2,\,1)} = i - 4j + 2k$$

$$\hat{a} = \text{unit normal} = \frac{\nabla g}{|\nabla g|} = \frac{i - 4j + 2k}{\sqrt{1+16+4}} = \frac{i-4j+2k}{\sqrt{21}}$$

Hence, the required directional derivative is

$$\nabla F \,.\, \hat{a} = (348i - 144j + 400k)\,.\, \frac{(i - 4j + 2k)}{\sqrt{21}}$$

$$= \frac{348 + 576 + 800}{\sqrt{21}} = \frac{1724}{\sqrt{21}}.$$

Example 9. Determine the values of a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2).

Sol. Let $\qquad f \equiv ax^2 - byz - (a + 2)x = 0 \qquad \text{...}(i)$

$$g \equiv 4x^2y + z^3 - 4 = 0 \qquad \text{...}(ii)$$

$$\text{grad } f = \nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\{ax^2 - byz - (a + 2)x\}$$

$$= (2ax - a - 2)i + (-bz)j + (-by)k$$

$$(\nabla f)_{(1,\,-1,\,2)} = (a - 2)i - 2bj + bk \qquad \text{...}(iii)$$

$$\text{grad } g = \nabla g = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(4x^2y + z^3 - 4)$$

$$= (8xy)i + (4x^2)j + (3z^2)k$$

$$(\nabla g)_{(1,\,-1,\,2)} = -8i + 4j + 12k \qquad \text{...}(iv)$$

Since the surfaces are orthogonal so

$$\left(\vec{\nabla} f\right) . \left(\vec{\nabla} g\right) = 0$$

$\Rightarrow [(a - 2)i - 2bj + bk] \,.\, [-8i + 4j + 12k] = 0$

$$-8(a - 2) - 8b + 12b = 0 \;\Rightarrow\; -2a + b + 4 = 0 \qquad \text{...}(v)$$

But the point (1, –1, 2) lies on the surface (i), so

$$a + 2b - (a + 2) = 0 \Rightarrow 2b - 2 = 0 \Rightarrow b = 1$$

Putting the value of b in (v), we get

$$-2a + 1 + 4 = 0 \Rightarrow a = \frac{5}{2}$$

Hence, $a = \dfrac{5}{2}$, $\quad b = 1$.

Example 10. Prove that $\vec{A} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ is irrotational and find the scalar potential f such that $\vec{A} = \nabla f$.

Sol. $$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = (-x + x)i - (-y + y)j + (-z + z) = 0$$

Hence, $\vec{A}$ is irrotational.

Now $$\vec{A} = \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$$

Comparing on both sides, we get

$$\frac{\partial f}{\partial x} = (x^2 - yz),\ \frac{\partial f}{\partial y} = (y^2 - zx) \text{ and } \frac{\partial f}{\partial z} = (z^2 - xy)$$

$$\therefore \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz$$

$$= (x^2\, dx + y^2\, dy + z^2\, dz) - (yzdx + zxdy + xydz)$$

or $$df = \frac{1}{3}\, d(x^3 + y^3 + z^3) - d(xyz)$$

On integrating, we get

$$f = \frac{1}{3}\,(x^3 + y^3 + z^3) - xyz + c.$$

EXERCISE 5.3

1. Find div $\vec{A}$, when $\vec{A} = x^2zi - 2y^3z^2j + xy^2zk$. $\left[\textbf{Ans.}\ 2xz - 6y^2z^2 + xy^2\right]$

2. If $\vec{V} = \dfrac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$, find the value of div $\vec{V}$. (*U.P.T.U., 2000*)

$\left[\textbf{Ans.}\ 2 \text{ division } \sqrt{(x^2 + y^2 + z^2)}\right]$

3. Find the directional derivative of the divergence of $f(x, y, z) = xyi + xy^2j + z^2k$ at the point (2, 1, 2) in the direction of the outer normal to the sphere, $x^2 + y^2 + z^2 = 9$. $\left[\textbf{Ans.}\ \dfrac{13}{3}\right]$

4. Show that the vector field $\vec{f} = \vec{r}/|\vec{r}|^3$ is irrotational as well as solenoidal. Find the scalar potential. (*U.P.T.U., 2001, 2005*) $\left[\textbf{Ans.}\ -\dfrac{1}{\sqrt{x^2 + y^2 + z^2}}\right]$

5. If r is the distance of a point (x, y, z) from the origin, prove that curl $\left(k \times grad\frac{1}{r}\right)$ + grad $\left(k.grad\frac{1}{r}\right) = 0$, where k is the unit vector in the direction OZ. *(U.P.T.U., 2000)*

6. Prove that $\vec{A} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational. Find a scalar function $f(x, y, z)$ such that $\vec{A} = \nabla f$. $\left[\textbf{Ans. } f = 3x^2y + xz^3 - zy + c_2\right]$

7. Find the curl of $yzi + 3zxj + zk$ at $(2, 3, 4)$. $\left[\textbf{Ans. } -6i + 3j + 8k\right]$

8. If $f = x^2yz$, $g = xy - 3z^2$, calculate $\nabla.(\nabla f \times \nabla g)$. $\left[\textbf{Ans. } \text{zero}\right]$

9. Determine the constants a and b such that curl of $(2xy + 3yz)i + (x^2 + axz - 4z^2)j + (3xy + 2byz)k = 0$. $\left[\textbf{Ans. } a = 3, b = -4\right]$

10. Find the value of constant b such that
$\vec{A} = (bxy - z^3)i + (b - 2)x^2j + (1 - b)xz^2k$ has its curl identically equal to zero.
$\left[\textbf{Ans. } b = 4\right]$

11. Prove that $\vec{a} \,.\, \nabla\left(\frac{1}{r}\right) = \frac{\vec{a} \cdot \vec{r}}{r^3}$.

12. Prove that $\nabla\left(\vec{a} \cdot \vec{u}\right) = \left(\vec{a} \cdot \nabla\right)\vec{u} + \vec{a} \times \text{curl } \vec{u}\,\vec{a}$ is a constant vector.

13. Prove that $\nabla^2\left(\frac{x}{r^3}\right) = 0$.

14. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

15. If $u = x^2 + y^2 + z^2$ and $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $\left(u\vec{v}\right) = 5u$

16. Prove the curl $\left(\frac{\vec{a} \times \vec{r}}{r^3}\right) = -\frac{\vec{a}}{r^3} + \frac{3\vec{r}\left(\vec{a} \cdot \vec{r}\right)}{r^5}$.

17. Find the curl of $\vec{v} = e^{xyz}\left(\hat{i} + \hat{j} + \hat{k}\right)$ at the point $(1, 2, 3)$. $\left[\textbf{Ans. } e^6(i - 21j + 3k)\right]$

18. Prove that $\nabla \times \nabla f = 0$ for any $f(x, y, z)$.

19. Find curl of $\vec{A} = x^2yi - 2xzj + 2yzk$ at the point $(1, 0, 2)$. $\left[\textbf{Ans. } 4\hat{j}\right]$

20. Determine curl of $xyz^2i + yzx^2j + zxy^2\vec{k}$ at the point $(1, 2, 3)$.

$\left[\textbf{Ans. } xy(2z - x)\hat{i} + yz(2x - y)\hat{j} + zx(2y - z)\hat{k}; 10\hat{i} + 3\hat{k}\right]$

21. Find $f(r)$ such that $f(r)\,\vec{r}$ is solenoidal. $\left[\textbf{Ans. } \frac{c}{r^3}\right]$

22. Find a, b, c when $\vec{f} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational.
[**Ans.** $a = 4$, $b = 2$, $c = -1$]

23. Prove that $(y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)i + (3xy - 2xz + 2z)k$ are both solenoidal and irrotational. [*U.P.T.U., 2008*]

5.15 VECTOR INTEGRATION

Vector integral calculus extends the concepts of (ordinary) integral calculus to vector functions. It has applications in fluid flow design of under water transmission cables, heat flow in stars, study of satellites. Line integrals are useful in the calculation of work done by variable forces along paths in space and the rates at which fluids flow along curves (circulation) and across boundaries (flux).

5.16 LINE INTEGRAL

Let $\vec{F}(\vec{r})$ be a continuous vector point function. Then $\int_C \vec{F} \cdot d\vec{r}$, is known as the line integral of $\vec{F}(\vec{r})$ along the curve C.

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ where F_1, F_2, F_3 are the components of $\vec{F}$ along the coordinate axes and are the functions of x, y, z each.

Now,
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$\therefore$
$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$
$\therefore$
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$
$$= \int_C \left(F_1 dx + F_2 dy + F_3 dz\right).$$

Again, let the parametric equations of the curve C be
$$x = x\,(t)$$
$$y = y\,(t)$$
$$z = z\,(t)$$

then we can write
$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left[F_1(t)\frac{dx}{dt} + F_2(t)\frac{dy}{dt} + F_3(t)\frac{dz}{dt}\right] dt$$

were t_1 and t_2 are the suitable limits so as to cover the arc of the curve C.

Note: work done $= \int_C \vec{F} \cdot d\vec{r}$

Circulation: The line integral $\int_C \vec{F} \cdot d\vec{r}$ of a continuous vector point functional $\vec{F}$ along a closed curve C is called the circulation of $\vec{F}$ round the closed curve C.

This fact can also be represented by the symbol $\oint$.

Irrotational vector field: A single valued vector point Function $\vec{F}$ (Vector Field $\vec{F}$) is called irrotational in the region R, if its circulation round every closed curve C in that region is zero that is

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

or
$$\oint \vec{F} \cdot d\vec{r} = 0.$$

5.17 SURFACE INTEGRAL

Any integral which is to be evaluated over a surface is called a surface integral.

Let $\vec{F}(\vec{r})$ be a continuous vector point function. Let $\vec{r} = \vec{F}(u, v)$ be a smooth surface such that $\vec{F}(u, v)$ possesses continuous first order partial derivatives. Then the normal surface integral of $\vec{F}(\vec{r})$ over S is denoted by

$$\int_S \vec{F}(\vec{r}) \cdot d\vec{a} = \int_S \vec{F}(\vec{r}) \cdot \hat{n}\, dS$$

where $d\vec{a}$ is the vector area of an element dS and $\hat{n}$ is a unit vector normal to the surface dS.

Let F_1, F_2, F_3 which are the functions of x, y, z be the components of F along the coordinate axes, then

$$\begin{aligned}\text{Surface Integral} &= \int_S \vec{F} \cdot \hat{n}\, dS \\ &= \int_S \vec{F} \cdot d\vec{a} \\ &= \int\int_S \left(\vec{F_1}\hat{i} + \vec{F_2}\hat{j} + \vec{F_3}\hat{k}\right) \cdot \left(dydz\hat{i} + dzdx\hat{j} + dxdy\hat{k}\right) \\ &= \iint_S \left(F_1 dy\, dz + F_2 dz\, dx + F_3 dx\, dy\right).\end{aligned}$$

5.17.1 Important Form of Surface Integral

Let $dS = dS\left(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\,\hat{k}\right)$ ...(*i*)

where α, β and γ are direction angles of dS. It shows that $dS \cos \alpha$, $dS \cos \beta$, $dS \cos \gamma$ are orthogonal projections of the elementary area dS on yz. plane, zx-plane and xy-plane respectively. As the mode of sub-division of the surface S is arbitrary we have chosen a sub-division formed by planes parallel to coordinate planes that is yz-plane, zx-plans and xy plane.

Clearly, projection on the coordinate planes will be rectangles with sides dy and dz on yz plane, dz and dx on xz plane and dx and dy on xy plane.

Hence $\hat{i} \cdot dS = dS\left\{\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\,\hat{k}\hat{j}\right\} \cdot \hat{i}$

$\hat{i} \cdot \hat{n}\, dS = dS \cos \alpha = dy\, dz$

Hence $dS = \dfrac{dy\, dz}{\hat{i}.\hat{n}}.$

Similarly, multiplying both sides of (*i*) scalarly by $\hat{j}$ and $\hat{k}$ respectively, we have

$$dS = \frac{dz\, dx}{\hat{j} \cdot \hat{n}} \qquad \text{...}(ii)$$

and $$dS = \frac{dx\, dy}{\hat{k} \cdot \hat{n}}$$

Hence
$$\begin{aligned}\int \vec{F} \cdot \hat{n}\, dS &= \iint_{S_1} \vec{F} \cdot \hat{n} \frac{dy\, dz}{\hat{i} \cdot \hat{n}} && \text{...}(iii) \\ &= \iint_{S_2} \vec{F} \cdot \hat{n} \frac{dz\, dx}{\hat{j} \cdot \hat{n}} && \text{...}(iv) \\ &= \iint_{S_3} \vec{F} \cdot \hat{n} \frac{dx\, dy}{\hat{k} \cdot \hat{n}} && \text{...}(v)\end{aligned}$$

where S_1, S_2, S_3 are projections of S on yz, zx and xy plane respectively.

5.18 VOLUME INTEGRAL

Let $\vec{F}(\vec{r})$ is a continuous vector point function. Let volume V be enclosed by a surface S given by

$$\vec{r} = \vec{f}(u,v) \quad ...(i)$$

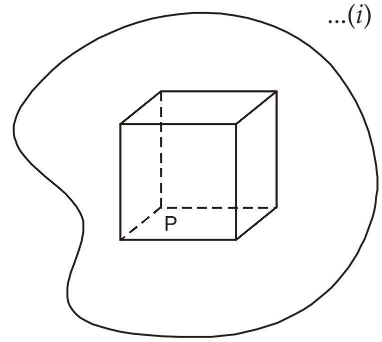


Fig. 5.5

sub-dividing the region V into n elements say of cubes having volumes

$$\Delta V_1, \Delta V_2, ... \Delta V_n$$

Hence $\Delta V_k = \Delta x_k \, \Delta y_k \, \Delta z_k$

$k = 1, 2, 3, ... \, n$

where (x_k, y_k, z_k) is a point say P on the cube. Considering the sum

$$\sum_{k=1}^{n} \vec{F}(x_k, y_k, z_k)\Delta V_k$$

taken over all possible cubes in the region. The limits of sum when $n \to \infty$ in such a manner that the dimensions ΔV_k tends to zero, if it exists is denoted by the symbol

$$\int_V \vec{F}(\vec{r}) dV \cdot or \int_V \vec{F} dV \;\; or \iiint_V \overline{F}\, dx\, dy\, dz$$

is called volume integral or space integral.

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

$$\int_V \vec{F}(\vec{r})\, dV = \hat{i}\iiint_V F_1\, dx\, dy\, dz + \hat{j}\iiint_V F_2 dx\, dy\, dz + \hat{k}\iiint_V F_3 dx\, dy\, dz$$

where F_1, F_2, F_3 which are function of x, y, z are the components of $\vec{F}$ along X, Y, Z axes respectively.

Independence of path

If in a conservative field $\vec{F}$

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

along any closed curve C.

Which is the condition of the independence of path.

Example 1. Evaluate $\int_C \vec{F} \cdot dr$ where $\vec{F} = x^2y^2\hat{i} + y\hat{j}$ and the curve C, is $y^2 = 4x$ in the xy-plane from (0, 0) to (4, 4).

Sol. We know that

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\therefore \quad d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\therefore \quad \vec{F} \cdot d\vec{r} = \left(x^2y^2\hat{i} + y\hat{j}\right) \cdot \left(dx\hat{i} + dy\hat{j}\right)$$

$$= x^2y^2dx + ydy$$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C \left(x^2y^2dx + ydy\right)$$

$$= \int_C x^2y^2 dx + \int_C y\, dy \cdot$$

But for the curve C, x and y both vary from 0 to 4.

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^4 x^2(4x)dx + \int_0^4 y\, dy \qquad [\because\ y^2 = 4x]$$

$$= 4\int_0^4 x^3 dx + \int_0^4 y\, dy$$

$$= 4\left(\frac{x^4}{4}\right)_0^4 + \left(\frac{y^2}{2}\right)_0^4$$

$$= 256 + 8 = 264.$$

Example 2. Evaluate $\int (x\,dy - y\,dx)$ around the circle $x^2 + y^2 = 1$.

Sol. Let C denote the circle $x^2 + y^2 = 1$, *i.e.*, $x = \cos t$, $y = \sin t$. In order to integrate around C, t varies from 0 to 2π.

$$\therefore \quad \int_C (x\,dy - y\,dy) = \int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \int_0^{2\pi}\left(\cos^2 t + \sin^2 t\right)dt$$

$$= \int_0^{2\pi} dt$$

$$= (t)_0^{2\pi}$$

$$= 2\pi.$$

Example 3. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (x^2 + y^2)\, i - 2xy\, j$, the curve C is the rectangle in the xy-plane bounded by $y = 0$, $x = a$, $y = b$, $x = 0$.

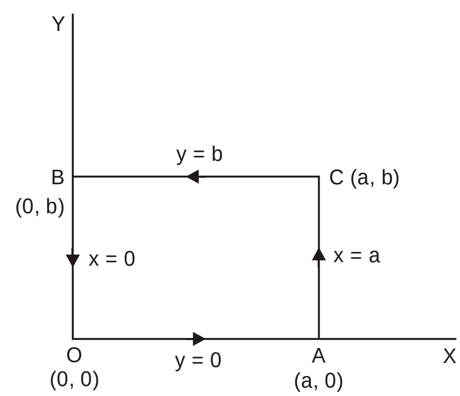


Fig. 5.6

Sol. $$\int_C \vec{F}\cdot d\vec{r} = \int_C \left\{\left(x^2 + y^2\right)i - 2xyj\right\}\cdot\left\{dxi + dyj\right\}$$

$$= \int_C \left\{\left(x^2 + y^2\right)dx - 2xy\,dy\right\} \qquad ...(i)$$

Now, C is the rectangle $OACB$.

On OA, $y = 0 \Rightarrow dy = 0$

On AC, $x = a \Rightarrow dx = 0$

On CB, $y = b \Rightarrow dy = 0$

On BO, $x = 0 \Rightarrow dx = 0$

$\therefore$ From (*i*),

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA}\left(x^2 + 0\right)dx + \int_{AC}(-2ay)dy + \int_{CB}\left(x^2 + b^2\right)dx + \int_{BO} 0dy$$

$$= \int_0^a x^2 dx - 2a\int_0^b y\,dy + \int_a^0 \left(x^2 + b^2\right)dx + \int_b^0 0\,dy$$

$$= \left(\frac{x^3}{3}\right)_0^a - 2a\left(\frac{y^2}{2}\right)_0^b + \left(\frac{x^3}{3} + b^2x\right)_a^0 + 0$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$
$$= -\,2ab^2.$$

Example 4. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = yz\, i + zx\, j + xy\, k$ and C is the portion of the curve $\vec{r} = (a \cos t)\, i + (b \sin t)\, j + (ct)\, k$ from $t = 0$ to $\frac{\pi}{2}$.

Sol. We have $\vec{r} = (a \cos t)\, i + (b \sin t)\, j + (ct)\, k.$

Hence, the parametric equations of the given curve are

$$x = a \cos t$$
$$y = b \sin t$$
$$z = ct$$

Also, $$\frac{d\vec{r}}{dt} = (-\,a \sin t)\, i + (b \cos t)\, j + ck$$

Now, $$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$
$$= \int_C (yzi + zxj + xyk) \cdot (-a \sin t i + b \cos t\, j + ck) dt$$
$$= \int_C (bct \sin t\, i + act \cos t\, j + ab \sin t \cos t\, k) \cdot (-a \sin t\, i + b \cos t\, j + ck) dt$$
$$= \int_C \left(-abc\, t \sin^2 t + abc\, t \cos^2 t + abc \sin t \cos t\right) dt$$
$$= abc \int_C \left[t\left(\cos^2 t - \sin^2 t\right) + \sin t \cos t\right] dt$$
$$= abc \int_C \left(t \cos 2t + \frac{\sin 2t}{2}\right) dt$$
$$= abc \int_0^{\frac{\pi}{2}} \left(t \cos 2\, t + \frac{\sin 2t}{2}\right) dt$$
$$= abc \left[t \frac{\sin 2t}{2} + \frac{\cos 2t}{4} - \frac{\cos 2t}{4}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{abc}{2} (t \sin 2t)_0^{\frac{\pi}{2}} = 0.$$

Example 5. Evaluate $\int_c \vec{F} . \vec{dr}$ where $\vec{F} = xyi + (x^2 + y^2)j$ and C is the x-axis from $x = 2$ to $x = 4$ and the line $x = 4$ from $y = 0$ to $y = 12$.

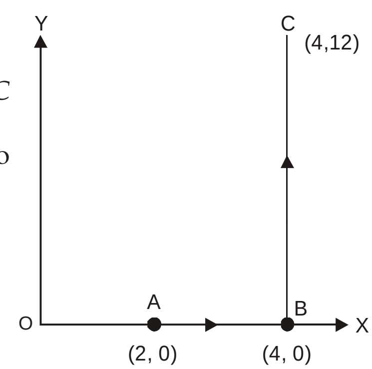


Fig. 5.7

Sol. Here the curve C consist the line AB and BC.

Since $\vec{r} = xi + yj$ (as $z = 0$)

$\vec{dr} = dxi + dyj$

so $$\int_C \vec{F} . \vec{dr} = \int_{ABC} \left\{xyi + \left(x^2 + y^2\right)j\right\} . \{dxi + dyj\}$$

$$= \int_{AB} \left\{xy\,dx + \left(x^2 + y^2\right)dy\right\} + \int_{BC} \left\{xy\,dx + \left(x^2 + y^2\right)dy\right\}$$

$$= \int_{x=2}^{4} 0\,.\,dx + \int_{y=0}^{12} 0 + \left(16 + y^2\right)dy \;=\; \int_0^{12}\left(16 + y^2\right)dy$$

$$= \left[16y + \frac{y^3}{3}\right]_0^{12} = [192 + 576] = 768.$$

Example 6. If $\vec{F} = (-2x + y)i + (3x + 2y)j$, compute the circulation of $\vec{F}$ about a circle C in the xy plane with centre at the origin and radius 1, if C is transversed in the positive direction.

Sol. Here the equation of circle is $x^2 + y^2 = 1$

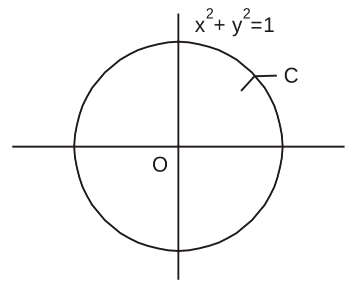


Fig. 5.8

Let $x = \cos\theta$, $y = \sin\theta$ | As $r = 1$

$$\vec{F} = (-2\cos\theta + \sin\theta)i + (3\cos\theta + 2\sin\theta)j$$

$$\vec{r} = xi + yj = (\cos\theta)i + (\sin\theta)i$$

So $d\vec{r} = \{(-\sin\theta)i + (\cos\theta)j\}d\theta$

Thus, the circulation along circle $C = \int_C \vec{F}\,.\,d\vec{r}$

$$= \int_{\theta=0}^{2\pi}\left\{(-2\cos\theta + \sin\theta)i + (3\cos\theta + 2\sin\theta)j\right\}.\left\{(-\sin\theta)i + (\cos\theta)j\right\}d\theta$$

$$= \int_0^{2\pi}\left(2\sin\theta\cos\theta - \sin^2\theta + 3\cos^2\theta + 6\sin\theta\cos\theta\right)d\theta$$

$$= \int_0^{2\pi}\left(8\sin\theta\cos\theta + 4\cos^2\theta - 1\right)d\theta = \int_0^{2\pi}(4\sin 2\theta + 2\cos 2\theta + 1)d\theta$$

$$= -2\left[\cos 2\theta\right]_0^{2\pi} + \left[\sin 2\theta\right]_0^{2\pi} + \left[\theta\right]_0^{2\pi} = -2\left[\cos 4\pi - \cos 0\right] + \left[\sin 4\pi - \sin 0\right] + 2\pi$$

$$= 2\pi.$$

Example 7. Compute the work done in moving a particle in the force field $\vec{F} = 3x^2\,i + (2xz - y)j + zk$ along.

(i) A straight line from $P(0, 0, 0)$ to $Q(2, 1, 3)$.

(ii) Curve C : defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol. (i) We know that the equation of straight line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\Rightarrow \quad \frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3}$$

or $$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}, \quad \text{so} \quad x = 2t, \quad y = t, \quad z = 3t$$

$$\therefore \qquad \vec{r} = xi + yj + zk = 2ti + tj + 3tk$$

$$\Rightarrow \qquad d\vec{r} = (2i + j + 3k)dt$$

and $$\vec{F} = (12t^2)i + (12t^2 - t)j + (3t)k$$

The work done $= \int_P^Q \vec{F} \,.\, d\vec{r} = \int_0^1 \left[(12t^2)i + (12t^2 - t)j + (3t)k\right] . \left[2i + j + 3k\right] dt$ | As t varies from 0 to 1

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)dt = \int_0^1 (36t^2 + 8t)dt$$

$$= \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1 = 12 + 4 = 16.$$

(*ii*) $$\vec{F} = 3x^2 i + \left(2x.\frac{3x^3}{8} - \frac{x^2}{4}\right)j + \frac{3x^3}{8}k = 3x^2 i + \left(\frac{3x^4}{4} - \frac{x^2}{4}\right)j + \frac{3x^3}{8}k$$

$$\vec{r} = xi + yj + zk = xi + \frac{x^2}{4}j + \frac{3x^3}{8}k$$

$$d\vec{r} = \left(i + \frac{x}{2}j + \frac{9x^2}{8}k\right)dx$$

Work done $= \int_{x=0}^{2} \vec{F} \,.\, dr = \int_0^2 \left[3x^2 i + \left(\frac{3x^4}{4} - \frac{x^2}{4}\right)j + \frac{3x^3}{8}k\right] . \left[i + \frac{x}{2}j + \frac{9x^2}{8}k\right]dx$

$$= \int_0^2 \left(3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27x^5}{64}\right)dx$$

$$= \left[x^3 + \frac{x^6}{16} - \frac{x^4}{32} + \frac{27x^6}{64 \times 6}\right]_0^2$$

$$= 8 + \frac{64}{16} - \frac{16}{32} + \frac{27 \times 64}{64 \times 6} = 8 + 4 - \frac{1}{2} + \frac{9}{2}$$

$$= 16.$$

Example 8. If V is the region in the first octant bounded by $y^2 + z^2 = 9$ and the plane $x = 2$ and $\vec{F} = 2x^2yi - y^2j + 4xz^2k$. Then evaluate $\iiint_V (\nabla \cdot \vec{F})\, dV$.

Sol. $\nabla \cdot \vec{F} = 4xy - 2y + 8xz$

The volume V of the solid region is covered by covering the plane region OAB while x varies from 0 to 2. Thus,

$$\iiint_V (\nabla \cdot \vec{F}) dV$$

$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz) dz\, dy\, dx$$

$$= \int_0^2 \int_0^3 \left[4xyz - 2yz + 4xz^2\right]_0^{\sqrt{9-y^2}} dy\, dx$$

$$= \int_0^2 \int_0^3 \left[(4xy - 2y)\sqrt{9-y^2} + 4x(9-y^2)\right] dy\, dx$$

$$= \int_0^2 \left[(4x-2)\left(-\frac{1}{3}(9-y^2)\right)^{\frac{3}{2}} + 4x\left(9y - \frac{y^3}{3}\right)\right]_0^3 dx$$

$$= \int_0^2 \left[9(4x-2) + 72x\right] dx$$

$$= \left[18x^2 - 18x + 36x^2\right]_0^2$$

$$= 180.$$

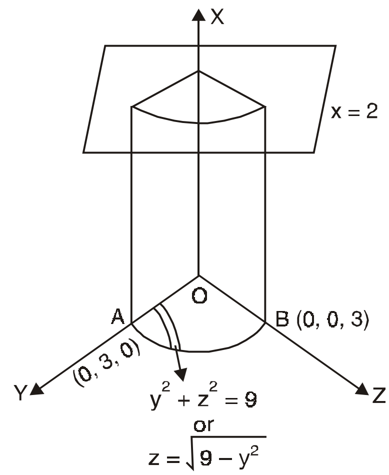


Fig. 5.9

Example 9. Evaluate $\iint_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)\cdot d\vec{S}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant. (*U.P.T.U., 2005*)

Sol. $\iint_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)\cdot d\vec{S} = \iint_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)\cdot\left(dy\,dz\,\hat{i} + \cdot\, dz\,dx\,\hat{j} + dx\,dy\,\hat{k}\right)$

$$= \iint_S \left(yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy\right)$$

$$= \int_0^a \int_0^{\sqrt{a^2-z^2}} yz\,dy\,dz + \int_0^a \int_0^{\sqrt{a^2-x^2}} zx\,dz\,dx + \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_0^a z\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-z^2}} dz + \int_0^a x\left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx + \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a z(a^2 - z^2)dz + \frac{1}{2}\int_0^a x(a^2 - x^2)dx + \frac{1}{2}\int_0^a y(a^2 - y^2)dy$$

$$= \frac{1}{2}\left(\frac{a^2z^2}{2} - \frac{z^4}{4}\right)_0^a + \frac{1}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a + \frac{1}{2}\left(\frac{a^2y^2}{2} - \frac{y^4}{4}\right)_0^a$$

$$= \frac{1}{2}\frac{a^4}{4} + \frac{1}{2}\frac{a^4}{4} + \frac{1}{2}\frac{a^4}{4} = \frac{3a^4}{8}.$$

Example 10. Evaluate $\iint_S \vec{F}\,.\,\hat{n}\,ds$, where $\vec{F} = zi + xj - 3y^2\,zk$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Sol. Since surface $S : x^2 + y^2 = 16$

Let $f \equiv x^2 + y^2 - 16$

$$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 - 16)$$

$$= 2xi + 2yi$$

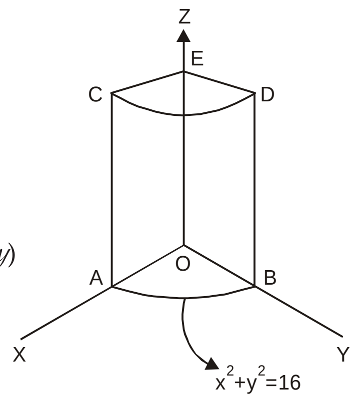


Fig. 5.10

unit normal $\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2xi + 2yj}{\sqrt{4x^2 + 4y^2}}$

$$\hat{n} = \frac{2(xi + yj)}{2\sqrt{x^2 + y^2}} = \frac{xi + yj}{\sqrt{16}} = \frac{xi + yi}{4} /$$

Now $\vec{F} \cdot \hat{n} = (zi + xj - 3y^2 zk). \left(\frac{xi + yj}{4}\right) = \frac{1}{4}(zx + xy)$

Here the surface S is perpendicular to xy-plane so we will take the projection of S on zx-plane. Let R be that projection.

$$\therefore \qquad ds = \frac{dx\,dz}{|\hat{n} \cdot j|} = \frac{dx\,dz}{\frac{y}{4}} = \frac{4dx\,dz}{y}$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_R \left(zi + xj - 3y^2 zk\right) \cdot \frac{(xi + yj)}{4} \cdot \frac{4}{y}\, dx\, dz$$

$$= \iint_R \left(\frac{zx + xy}{y}\right) dx\, dz$$

Since z varies from 0 to 5 and $y = \sqrt{16 - x^2}$ on S. x is also varies from 0 to 4.

$$\therefore \qquad \iint_R \left(\frac{zx + xy}{y}\right) dx\, dz = \int_{z=0}^{5} \int_{x=0}^{4} \left(\frac{xz}{\sqrt{16 - x^2}} + x\right) dx\, dz$$

$$= \int_0^5 \left[-z\sqrt{16 - x^2} + \frac{x^2}{2}\right]_0^4 dz = \int_0^5 (4z + 8)\, dz$$

$$= (2z^2 + 8z)_0^5 = 50 + 40 = 90.$$

Example 11. Evaluate $\iiint_V \phi\, dV$, where $\phi = 45x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$.

Sol. Putting $y = 0$, $z = 0$, we get $4x = 8$ or $x = 2$

Here x varies from 0 to 2

y varies from 0 to $4 - 2x$

and z varies from 0 to $8 - 4x - 2y$

Thus $$\iiint_V \phi\, dV = \iiint_V 45x^2 y\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2 y\, dx\, dy\, dz$$

$$= 45 \int_0^2 \int_0^{4-2x} x^2 y [z]_0^{8-4x-2y}\, dx\, dy$$

$$= 45 \int_0^2 \int_0^{4-2x} x^2 y (8 - 4x - 2y)\, dx\, dy$$

$$= 45\int_0^2 x^2\left[4y^2 - 2xy^2 - \frac{2}{3}y^3\right]_0^{4-2x} dx$$

$$= 45\int_0^2 x^2\left[4(4-2x)^2 - 2x(4-2x)^2 - \frac{2}{3}(4-2x)^3\right]dx$$

$$= \frac{45}{3}\int_0^2 x^2(4-2x)^3\,dx$$

$$= 15\int_0^2 x^2\left(64 - 8x^3 - 96x + 48x^2\right)dx$$

$$= 15\left[\frac{64x^3}{3} - \frac{8x^6}{6} - \frac{96x^4}{4} + \frac{48x^5}{5}\right]_0^2$$

$$= 15\left[\frac{512}{3} - \frac{256}{3} - 384 + \frac{1536}{5}\right]$$

$$= 128.$$

EXERCISE 5.4

1. Find the work done by a Force $\vec{F} = zi + xj + yk$ from $t = 0$ to 2π, where $\vec{r} = \cos t\, i + \sin t j + tk$.

$\left[\textbf{Hint: } \text{Work done } = \int_C \vec{F}\cdot d\vec{r}\right]$. **[Ans.** 3π**]**

2. Show that $\int_C \vec{F}\cdot d\vec{r} = -1$, where $\vec{F} = (\cos y)\, i - xj - (\sin y)\, k$ and C is the curve $y = \sqrt{1-x^2}$ in xy-plane from (1, 0) to (0, 1).

3. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\, i - (2xy + y)\, j$ moves a particle from origin to (1, 1) along a parabola $y^2 = x$. $\left[\textbf{Ans. } \frac{2}{3}\right]$

4. $\int_C xy^3 dS$ where C is the segment of the line $y = 2x$ in the xy plane from $A(-1, -2, 0)$ to $B(1, 2, 0)$. $\left[\textbf{Ans. } \frac{16}{\sqrt{5}}\right]$

5. $\vec{F} = 2xzi + \left(x^2 - y\right) j + \left(2z - x^2\right) k$ is conservative or not.

$\left[\textbf{Ans. } \nabla\times\overline{F} \neq 0, \text{so non-conservative}\right]$

6. If $\vec{F} = \left(2x^2 - 3z\right)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla\vec{F}dV$, where V is bounded by the plane $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. $\left[\textbf{Ans. } \frac{8}{3}\right]$

7. Show that $\iint_S \vec{F}\cdot\hat{n}dS = \frac{3}{2}$, where $\vec{F} = 4xzi - y^2j + yzk$ and S is the surface of the cube bounded by the planes;

$x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1.$ $\left[\textbf{Ans. } \frac{3}{2}\right]$

8. If $\vec{F} = 2y\hat{i} - 3\hat{j} + x^2\hat{k}$ and S is the surface of the parobolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, and $z = 6$ then evaluate $\iint_S F\cdot\hat{n}dS$. **[Ans.** 132**]**

9. If $\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$ evaluate $\oint \vec{A}\cdot d\vec{r}$ around the curve C consisting of $y = x^2$ and $y^2 = x$. $\left[\textbf{Ans. } \frac{2}{3}\right]$

10. Find the total work done in moving a particle in a force field $\vec{A} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + t,\ y = 2t^2,\ z = t^3$ from $t = 1$ to $t = 2$. **[Ans.** 303**]**

11. Find the surface integral over the parallelopiped $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 2,\ z = 3$ when $\vec{A} = 2xyi + yz^2j + xzk$. **[Ans.** 33**]**

12. Evaluate $\iiint_V \nabla\times\vec{A}dV$, where $\vec{A} = (x + 2y)\, i - 3zj + x\,\hat{k}$ and V is the closed region in the first octant bounded by the plane $2x + 2y + z = 4$. $\left[\textbf{Ans. } \frac{8}{3}(3i - j + 2k)\right]$

13. Evaluate $\iiint_V f dV$ where $f = 45x^2y$ and V denotes the closed region bounded by the planes $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0$. **[Ans.** 128**]**

14. If $\vec{A} = \left(2x^2 - 3z\right)\hat{i} - 2xyj - 4x\hat{k}$ and V is the closed region bounded by the planes $x = 0$, $y = 0,\ z = 0$ and $2x + 2y + z = 4$, evaluate $\iiint_V \left(\nabla\times\vec{A}\right)dV$. $\left[\textbf{Ans. } \frac{8}{3}(j - k)\right]$

15. If $\vec{A} = \left(x^3 - yz\right)\hat{i} - 2x^3yj + 2k$ evaluate $\iiint_V \left(\nabla\cdot\vec{A}\right)dV$ over the volume of a cube of side b. $\left[\textbf{Ans. } \frac{1}{3}b^3\right]$

16. Show that the integral

$\int_{(1,2)}^{(3,4)} \left(xy^2 + y^3\right)dx + \left(x^2y + 3xy^2\right)dy$ is independent of the path joining the points (1, 2) and (3, 4). Hence, evaluate the integral. **[Ans.** 254**]**

17. If $F = \nabla\phi$ show that the work done in moving a particle in the force field F from (x_1, y_1, z_1) to $B\ (x_2, y_2, z_2)$ is independent of the path joining the two points.

18. If $\vec{F} = (y - 2x)i + (3x + 2y)j$, find the circulation of $\vec{F}$ about a circle C in the xy-plane with centre at the origin and radius 2, if C is transversed in the positive direction. **[Ans.** 8π**]**

19. If $\vec{F}(2) = 2i - j + 2k,\ \vec{F}(3) = 4i - 2j + 3k$ then evaluate $\int_2^3 \vec{F}\cdot\frac{d\vec{F}}{dt}dt$. **[Ans.** 10**]**

20. Prove that $\vec{F} = (4xy - 3x^2z^2)i + 2x^2 j - 2x^3 zk$ is a conservative field.

21. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = xyi - x^2 j + (x + z)k$, S is the portion of plane $2x + 2y + z = 6$ included in the first octant. $\left[\textbf{Ans. } \frac{27}{4}\right]$

22. Find the volume enclosed between the two surfaces $S_1 : z = 8 - x^2 - y^2$ and $S_2 : z = x^2 + 3y^2$. $[\textbf{Ans. } 8\pi\sqrt{2}]$

5.19 GREEN'S* THEOREM

If C be a regular closed curve in the xy-plane bounding a region S and $P(x, y)$ and $Q(x, y)$ be continuously differentiable functions inside and on C then (*U.P.T.U., 2007*)

$$\iint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

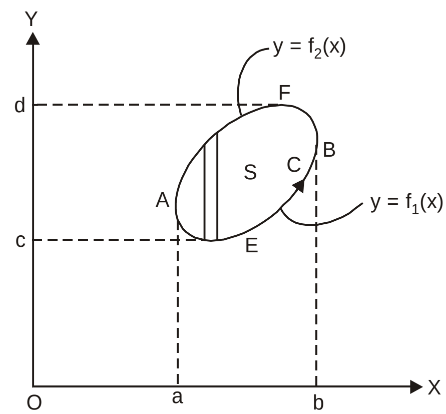


Fig. 5.11

Proof: Let the equation of the curves AEB and AFB are $y = f_1(x)$ and $y = f_2(x)$ respectively.

$$\text{Consider } \iint_S \frac{\partial P}{\partial y} dx\, dy = \int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy\, dx$$

$$= \int_a^b [P(x, y)]_{y=f_1(x)}^{y=f_2(x)} dx$$

$$= \int_a^b [P(x, f_2) - P(x, f_1)] dx$$

$$= -\int_b^a P(x, f_2) dx - \int_a^b P(x, f_1) dx$$

$$= -\int_{BFA} P(x, y)\, dx - \int_{AEB} P(x, y)\, dx$$

$$= -\int_{BFAEB} P(x, y)\, dx$$

$$\Rightarrow \quad \iint_S \frac{\partial P}{\partial y} dx\, dy = -\oint_C P(x, y)\, dx \quad \ldots(i)$$

Similarly, let the equations of the curve EAF and EBF be $x = f_1(y)$ and $x = f_2(y)$ respectively, then

$$\iint_S \frac{\partial Q}{\partial x} dx\, dy = \int_{y=c}^{d}\int_{x=f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x} dx\, dy = \int_c^d [Q(f_2, y) - Q(f_1, y)] dy$$

$$= \int_c^d Q(f_2, y) dy + \int_d^c Q(f_1, y) dy$$

$$\Rightarrow \quad \iint_S \frac{\partial Q}{\partial x} dx\, dy = \oint_C Q(x, y)\, dy \quad \ldots(ii)$$

* George Green (1793–1841), English Mathematician.

Adding eqns. (*i*) and (*ii*), we get

$$\boxed{\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy}$$

Vector form of Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \cdot k dS$$

Let $\vec{F} = Pi + Qj$, then we have

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

$\Rightarrow$ $$\left(\nabla \times \vec{F}\right) \cdot k = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Thus, $$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \cdot k\, dS}$$

Corollary: Area of the plane region S bounded by closed curve C.

Let $Q = x,\ P = -\,y$, then

$$\oint_C (xdy - ydx) = \iint_S (1+1) dxdy$$

$$= 2\int_S dx\, dy = 2A$$

Thus, $$\boxed{\text{area } A = \frac{1}{2}\oint_C (xdy - ydx)}$$

Example 1. Verify the Green's theorem by evaluating $\int_C \left[\left(x^3 - xy^3\right)dx + \left(y^3 - 2xy\, dy\right)\right]$ where C is the square having the vertices at the points (0, 0), (2, 0) (2, 2) and (0, 2). (*U.P.T.U., 2007*)

Sol. We have $$\int_C \left[\left(x^3 - xy^3\right)dx + \left(y^3 - 2xy\, dy\right)\right]$$

By Green's theorem, we have

$$\int_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy \quad ...(A)$$

Here $P = (x^3 - xy^3),\ Q = (y^3 - 2xy)$

$\therefore$ $$\frac{\partial P}{\partial y} = -\,3xy^2,\ \frac{\partial Q}{\partial x} = -\,2y$$

So $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{x=0}^{2}\int_{y=0}^{2}\left(-2y + 3xy^2\right) dx\, dy$

$= -2\int_{x=0}^{2} dx \int_{y=0}^{2} y dy + 3\int_{x=0}^{2} x dx \int_{y=0}^{2} y^2 dy$

$= -2\left[x\right]_0^2 \left[\frac{y^2}{2}\right]_0^2 + 3\left[\frac{x^2}{2}\right]_0^2 \left[\frac{y^3}{3}\right]_0^2 = -8 + 16$

$\Rightarrow \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = 8$ ...(*i*)

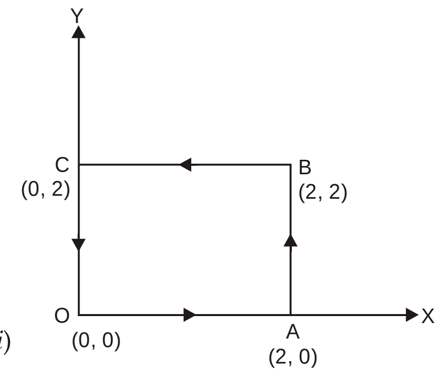


Fig. 5.12

Now, the line integral

$$\int_C (Pdx + Qdy) = \int_C \left[\left(x^3 - xy^3\right)dx + \left(y^3 - 2xy\right)dy\right]$$

$$= \int_{OA} \left[\left(x^3 - xy^3\right)dx + \left(y^3 - 2xy\right)dy\right] + \int_{AB} \left[\left(x^3 - xy^3\right)dx + \left(y^3 - 2xy\, dy\right)\right]$$

$$+ \int_{BC} \left[\left(x^3 - xy^3\right)dx + \left(y^3 - 2xy\right)dy\right] + \int_{CO} \left[\left(x^3 - xy^3\right)dx + \left(y^3 - 2xy\, dy\right)\right]$$

But along OA, $y = 0 \Rightarrow dy = 0$ and $x = 0$ to 2
along AB, $x = 2 \Rightarrow dx = 0$ and $y = 0$ to 2
along BC, $y = 2 \Rightarrow dy = 0$ and $x = 2$ to 0
along CO, $x = 0 \Rightarrow dx = 0$ and $y = 2$ to 0

$$\therefore \int_C (Pdx + Qdy) = \int_0^2 x^3 dx + \int_0^2 \left(y^3 - 4y\right)dy + \int_2^0 \left(x^3 - 8x\right)dx + \int_2^0 y^3 dy$$

$$= \left[\frac{x^4}{4}\right]_0^2 + \left[\frac{y^4}{4} - 2y^2\right]_0^2 + \left[\frac{x^4}{4} - 4x^2\right]_2^0 + \left[\frac{y^4}{4}\right]_2^0$$

$$= 4 - 4 + 12 - 4$$

$\Rightarrow \int_C (Pdx + Qdy) = 8$ ...(*ii*)

Thus from eqns. (*i*) and (*ii*) relation (*A*) satisfies. Hence, the Green's theorem is verified.

Hence proved.

Example 2. Verify Green's theorem in plane for $\oint_C \left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy$, where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$.

Sol. By Green's theorem

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

i.e., Line Integral (*LI*) = Double Integral (*DI*)

Here, $P = x^2 - 2xy,\ Q = x^2y + 3$

$$\frac{\partial P}{\partial y} = -2x, \frac{\partial Q}{\partial x} = 2xy$$

So the R.H.S. of the Green's theorem is the double integral given by

$$DI = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_S \left[(2xy - (-2x))\right] dx\, dy$$

The region S is covered with y varying from $-2\sqrt{2}\sqrt{x}$ of the lower branch of the parabola to its upper branch $2\sqrt{2}\sqrt{x}$ while x varies from 0 to 2. Thus

$$DI = \int_{x=0}^{2}\int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\, dy\, dx$$

$$= \int_0^2 xy^2 + 2xy \Big|_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= 8\sqrt{2}\int_0^2 x^{\frac{3}{2}} dx = \frac{128}{5}$$

The L.H.S. of the Green's theorem result is the line integral

$$LI = \oint_C \left(x^2 - 2xy\right)dx + \left(x^2y + 3\right)dy \cdot$$

Here C consists of the curves *OA, ADB. BO.* so

$$LI = \oint_C = \int_{OA+ADB+BO}$$

$$= \int_{OA} + \int_{ADB} + \int_{BO} = LI_1 + LI_2 + LI_3$$

Along *OA*: $y = -2\sqrt{2}\sqrt{x}$, so $dy = -\sqrt{\frac{2}{x}}dx$

$$LI_1 = \int_{OA} \left(x^2 - 2xy\right)dx + \left(x^2y + 3\right)dy$$

$$= \int_0^2 \left[x^2 - 2x\left(-2\sqrt{2}\sqrt{x}\right)\right]dx$$

$$+\left[x^2\left(-2\sqrt{2}\sqrt{x}\right) + 3\right]\left(-\sqrt{\frac{2}{x}}\right)dx$$

$$= \int_0^2 \left(5x^2 + 4\sqrt{2}\cdot x^{3/2} - 3\sqrt{2}x^{-\frac{1}{2}}\right)dx$$

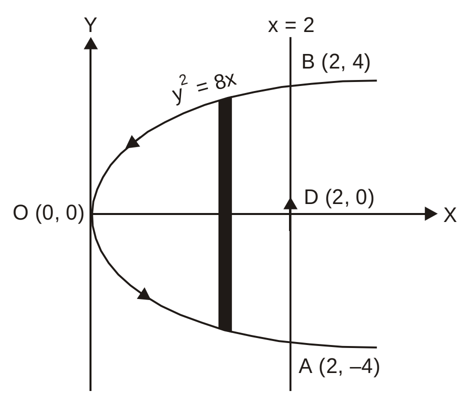


Fig. 5.13

$$= \left[\frac{5x^3}{3} + 4\sqrt{2}\,\frac{2}{5}x^{\frac{5}{2}} - 3\sqrt{2}\cdot 2\sqrt{x}\right]_0^2$$

$$= \frac{40}{3} + \frac{64}{5} - 12$$

Along *ADB* : $x = 2,\ dx = 0$

$$LI_2 = \int_{ADB}\left(x^2 - 2xy\right)dx + \left(x^2y + 3\right)dy$$

$$= \int_{-4}^{4}(4y + 3)\,dy = 24$$

Along *BO* : $y = 2\sqrt{2}\sqrt{x}$, with x : 2 to 0.

$$dy = \sqrt{\frac{2}{x}}dx$$

$$LI_3 = \int_{BO}\left(x^2 - 2xy\right)dx + \left(x^2y + 3\right)dy$$

$$= \int_2^0 \left(5x^2 - 4\sqrt{2}x^{\frac{3}{2}} + 3\sqrt{2}x^{-\frac{1}{2}}\right)dx$$

$$= -\frac{40}{3} + \frac{64}{5} - 12$$

$$LI = LI_1 + LI_2 + LI_3 = \left(\frac{40}{3} + \frac{64}{5} - 12\right) + (24) + \left(-\frac{40}{3} + \frac{64}{5} - 12\right) = \frac{128}{5}$$

⇒ Hence the Green's theorem is verified.

Example 3. Apply Green's theorem to evaluate $\int_C\left[\left(2x^2 - y^2\right)dx + \left(x^2 + y^2\right)dy\right]$ where C is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$.

(*U.P.T.U., 2005*)

Sol. By Green's theorem

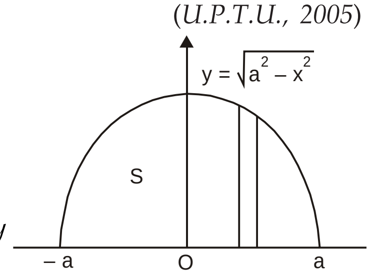


Fig. 5.14

$$\int_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$= \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\left[\frac{\partial}{\partial x}\left(x^2 + y^2\right) - \frac{\partial}{\partial y}\left(2x^2 - y^2\right)\right]dx\,dy$$

$$= \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}}(2x + 2y)\,dx\,dy = \int_{-a}^{a}\left[2xy + \frac{2y^2}{2}\right]_0^{\sqrt{a^2-x^2}}dx.$$

$$= \int_{-a}^{a}\left[2x\sqrt{a^2 - x^2} + \left(a^2 - x^2\right)\right]dx = 0 + 2\int_0^a\left(a^2 - x^2\right)dx$$

[First integral vanishes as function is odd]

$$= 2\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\left[a^3 - \frac{a^3}{3}\right] = \frac{4}{3}a^3.$$

Example 4. Verify Green's theorem in the plane for

$$\oint_C \left(2x - y^3\right)dx - xy\, dy$$

where C is the boundary of the annulus (doubly connected) region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

Sol. Here $P = 2x - y^3$, $Q = xy$ so that

$$\frac{\partial P}{\partial y} = -3y^2, \quad \frac{\partial Q}{\partial x} = y$$

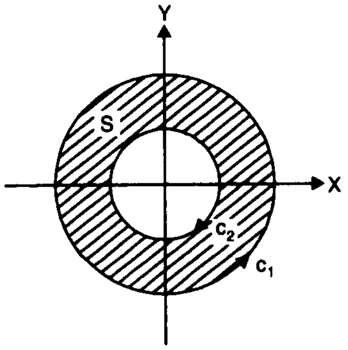


Fig. 5.15

Thus R.H.S. of Green's theorem is

$$= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\, dy$$

$$= \iint_S \left(y + 3y^2\right)dx\, dy$$

where S is the annulus region.

Put $x = r\cos\theta$, $y = r\sin\theta$, so that θ varies from 0 to 2π and r from 1 to 3.

$$\text{R.H.S.} = \int_0^{2\pi}\int_1^3 \left(r\sin\theta + 3r^2\sin^2\theta\right) r\, dr\, d\theta$$

$$= \frac{26}{3}\int_0^{2\pi} \sin\theta\, d\theta + 60\int_0^{2\pi} \frac{1 - \cos 2\theta}{2}\, d\theta = 60\pi$$

$$\text{L.H.S.} = \int_C P dx + Q dy = \int_{C_1 + C_2} \left(2x - y^3\right)dx - xy\, dy$$

$$\Rightarrow \quad \text{L.H.S.} = \int_{C_1}\left[\left(2x - y^3\right)dx - xy\, dy\right] - \int_{C_2}\left[\left(2x - y^3\right)dx - xy\, dy\right]$$

$$= 2\int_{C_1}\left[\left(2\sqrt{9 - y^2} - y^3\right)\left(-\frac{y\, dy}{\sqrt{9 - y^2}}\right) - y\sqrt{9 - y^2}\, dy\right]$$

$$-2\int_{C_2}\left[\left(2\sqrt{1 - y^2} - y^3\right)\left(-\frac{y\, dy}{\sqrt{1 - y^2}}\right) - y\sqrt{1 - y^2}\, dy\right]$$

$\left(\text{As circle is 2 semi circle and } dx = \frac{-y\, dy}{x} \text{ from circles}\right)$

$$= 2\int_{-3}^{3}\left[-2y + \frac{y^4}{\sqrt{9 - y^2}} - y\sqrt{9 - y^2}\right]dy \;\; -2\int_{-1}^{1}\left[-2y + \frac{y^4}{\sqrt{1 - y^2}} - y\sqrt{1 - y^2}\right]dy$$

$$= 4\int_0^3 \frac{y^4 dy}{\sqrt{9 - y^2}} - 4\int_0^1 \frac{y^4}{\sqrt{1 - y^2}}\, dy$$

(Other integral is zero being odd function of y.)

Let $y = 3\sin\theta$ in first and $y = \sin\phi$ in second integral.

$$\text{L.H.S.} = 4\times 81\int_0^{\frac{\pi}{2}} \frac{\sin^4\theta\cos\theta}{\cos\theta}d\theta - 4\int_0^{\frac{\pi}{2}}\frac{\sin^4\phi\cos\phi}{\cos\phi}d\phi$$

$$= 4\times 81\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} - 4\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} = 60\pi$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence Green's theorem is verified.

Example 5. Find the area of the loop of the folium of Descartes

$$x^3 + y^3 = 3axy,\ a > 0.$$

Sol. Let $y = tx.$ ...(*i*)

$\therefore$ $x^3 + t^3x^3 = 3ax.tx$

giving $$x = \frac{3at}{1+t^3} \quad ...(ii)$$

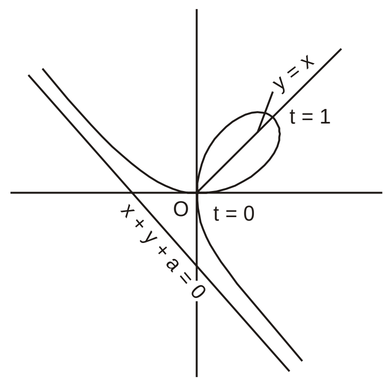


Fig. 5.16

Hence, required area $= \frac{1}{2}\int_C (x\,dy - y\,dx)$

$$= \frac{1}{2}\int_C x^2\cdot\frac{x\,dy - y\,dx}{x^2}$$

$$= \frac{1}{2}\int_C x^2 d\left(\frac{y}{x}\right)$$

$$= \frac{1}{2}\int_C x^2 dt, \text{ as } y = tx$$

$$= \frac{1}{2}\int_C \frac{9a^2t^2}{(1+t^3)^2}dt \text{ using } (ii)$$

$$= 3a^2\int_0^1 \frac{3t^2}{(1+t^3)^2}dt, \text{ by summetry}$$

$$= 3a^2\left[-\frac{1}{1+t^3}\right]_0^1 = \frac{3}{2}a^2.$$

Example 6. Using Green's theorem, find the area of the region in the first quadrant bounded by the curves $y = x$, $y = \frac{1}{x}$, $y = \frac{x}{4}$. (*U.P.T.U., 2008*)

Sol. By Green's theorem the area of the region is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

$$= \frac{1}{2}\left[\int_{C_1}(xdy - ydx) + \int_{C_2}(xdy - ydx) + \int_{C_3}(xdy - ydx)\right] \quad ...(i)$$

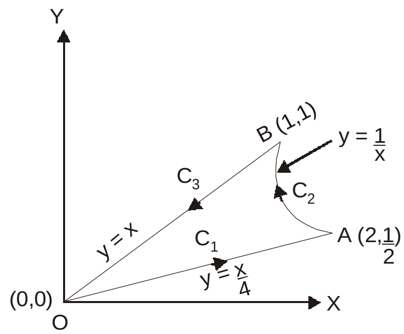


Fig. 5.17

Now along the curve $C_1 : y = \frac{x}{4}$ or $dy = \frac{dx}{4}$ and x varies from 0 to 2.

$$\int_{C_1} (xdy - ydx) = \int_0^2 \left(\frac{x}{4} dx - \frac{x}{4} dx \right) = 0 \quad ...(ii)$$

along, the curve $C_2 : y = \frac{1}{x}$, $dy = -\frac{1}{x^2}\, dx$ and x varies from 2 to 1.

$$\int_{C_2} (xdy - ydx) = \int_2^1 \left\{ x.\left(\frac{1}{-x^2} \right) dx - \frac{1}{x} dx \right\} = \int_2^1 - \frac{2}{x} dx$$

or

$$\int_{C_2} (xdy - ydx) = -2[\log x]_2^1 = -2[\log 1 - \log 2] = 2 \log 3 \quad ...(iii)$$

along, the curve $C_3 : y = x$, $dy = dx$ and x varies from 1 to 0.

$$\int_{C_3} (xdy - ydx) = \int_1^0 (xdx - xdx) = 0 \quad ...(iv)$$

Using (*ii*), (*iii*) and (*iv*) in (*i*), we get the required area

$$A = \frac{1}{2} [0 + 2 \log 2 + 0] = \log 2.$$

Example 7. Verify Green's theorem in the *xy*-plane for $\int_C \left[\left(2xy - x^2\right) dx + \left(x^2 + y^2\right) dy \right]$, where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$.

Sol. Here $P(x, y) = 2xy - x^2$

$$Q(x, y) = x^2 + y^2$$

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 2x$$

By Green's theorem, we have

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy \quad ...(i)$$

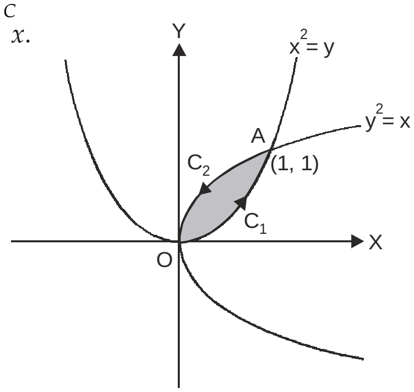


Fig. 5.18

$\therefore$ R.H.S. $= \iint_S (2x - 2x)\, dx\, dy = 0$

and L.H.S. $= \int_C \left[\left(2xy - x^2\right) dx + \left(x^2 + y^2\right) dy\right]$

$$= \int_{C_1} \left[\left(2xy - x^2\right) dx + \left(x^2 + y^2\right) dy\right] + \int_{C_2} \left[\left(2xy - x^2\right) dx + \left(x^2 + y^2\right) dy\right] \quad \ldots(ii)$$

Along C_1 : $y = x^2$ *i.e.*, $dy = 2xdx$ and x varies from 0 to 1

$$\int_{C_1} \left[\left(2xy - x^2\right)dx + \left(x^2 + y^2\right)dy\right] = \int_0^1 \left(2x^3 - x^2 + 2x^3 + 2x^5\right)dx$$

$$= \int_0^1 \left(4x^3 - x^2 + 2x^5\right)dx = \left[x^4 - \frac{x^3}{3} + \frac{x^6}{3}\right]_0^1 = 1$$

Along C_2 : $y^2 = x$, $2y\, dy = dx$ or $dy = \dfrac{dx}{2x^{1/2}}$ and x varies from 1 to 0.

$$\int_{C_2} \left[\left(2xy - x^2\right)dx + \left(x^2 + y^2\right)dy\right] = \int_1^0 \left[\left(2x \cdot x^{1/2} - x^2\right)dx + \left(x^2 + x\right). \frac{dx}{2x^{1/2}}\right]$$

$$= \int_1^0 \left(2x^{3/2} - x^2 + \frac{1}{2}x^{3/2} + \frac{1}{2}x^{1/2}\right)dx$$

$$= \int_1^0 \left(\frac{5}{2}x^{3/2} - x^2 + \frac{1}{2}x^{1/2}\right)dx$$

$$= \frac{5}{2}\left[\frac{x^{5/2}}{5/2}\right]_1^0 - \left[\frac{x^3}{3}\right]_1^0 + \frac{1}{2}\left[\frac{x^{3/2}}{3/2}\right]_1^0$$

$$= -1 + \frac{1}{3} - \frac{1}{3} = -1$$

Using the above values in (*ii*), we get

L.H.S. = 1 – 1 = 0

Thus L.H.S. = R.H.S.; Hence, the Green's theorem is verified.

EXERCISE 5.5

1. Using Green's theorem evaluate $\int_C \left(x^2 y\, dx + x^2 dy\right)$, where C is the boundary described counter clockwise of the triangle with vertices (0, 0), (1, 0), (1, 1). (*U.P.T.U., 2003*)

$\left[\textbf{Ans. } \dfrac{5}{12}\right]$

2. Verify Green's theorem in plane for $\int_C \left\{\left(xy + y^2\right)dx + x^2 dy\right\}$, where C is the closed curve of the region bounded by $y = x^2$ and $y = x$.

$\left[\textbf{Ans. } -\dfrac{1}{20}\right]$

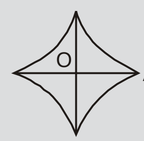
3. Evaluate $\int_C [(\cos x \sin y - xy)dx + \sin x \cdot \cos y\, dy]$ by Green's theorem where C is the circle $x^2 + y^2 = 1$. **[Ans.** 0]

4. Evaluate by Green's theorem $\int_C \{e^{-x} \sin y\, dx + e^{-x} \cos y\, dy\}$, where C is the rectangle with vertices (0, 0) (π, 0), $\left(\pi, \frac{1}{2}\pi\right)$ $\left(0, \frac{1}{2}\pi\right)$. **[Ans.** $2(e^{-\pi}-1)$]

5. Find the area of the ellipse by applying the Green's theorem that for a closed curve C in the xy-plane.
[**Hint:** Parametric eqn. of ellipse $x = a \cos \phi$, $y = a \sin \phi$ and ϕ vary from $\phi_1 = 0$ to $\phi_2 = 2\pi$]
[Ans. $\pi\, ab$]

6. Verify the Green's theorem to evaluate the line integral $\int_C (2y^2 dx + 3x\, dy)$, where C is the boundary of the closed region bounded by $y = x$ and $y = x^2$. $\left[\textbf{Ans. } \frac{27}{4}\right]$

7. Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ with $a > 0$.

[**Hint:** $x = a \cos^3 \phi$, $y = a \sin^3 \phi$, ϕ varies from $\phi_1 = 0$ to $\phi_1 = \pi/2$ 

]

$\left[\textbf{Ans. } \frac{3\pi a^2}{8}\right]$

8. Verify Green's theorem $\int_C (3x + 4y)\, dx + (2x - 3y)\, dy$ with C; $x^2 + y^2 = 4$.

[Ans. Common value: – 8π]

9. Find the area of the loop of the folium of descartes $x^3 + y^3 = 3axy$, $a > 0$. $\left[\textbf{Ans. } \frac{3a^2}{2}\right]$

[**Hint:** Put $y = tx$, $t : 0$ to ∞]

10. Evaluate the integral $\oint_C (x^2 - \cos hy)\, dx + (y + \sin x)\, dy$, where C: $0 \le x \le \pi$, $0 \le y \le 1$.

[Ans. π (cos *h* 1 – 1)]

11. Verify Green's theorem in the xy-plane for $\int_C \left[\left(3x^2 - 8y^2\right)dx + (4y - 6xy)dy\right]$, where C is the region bounded by parabolas $y = \sqrt{x}$ and $y = x^2$. $\left[\textbf{Ans. } \frac{3}{2}\right]$

12. Find the area of a loop of the four - leafed rose $r = 3 \sin 2\theta$. $\left[\textbf{Hint: } A = \frac{1}{2}\int_0^{\pi/2} r^2 d\theta = \frac{9\pi}{8}\right]$

13. Verify the Green's theorem for $\int_C \left(y^2 dx + x^2 dy\right)$, where C is the boundary of the square $-1 \le x \le 1$ and $-1 \le y \le 1$.

14. Using Green's theorem in the plane evaluate $\int_C \left[2\tan^{-1}(y/x)dx + \log\left(x^2+y^2\right)dy\right]$, where C is the boundary of the circle $(x-1)^2 + (y+1)^2 = 4$.

5.20 STOKE'S THEOREM

If $\vec{F}$ is any continuously differentiable vector function and S is a surface enclosed by a curve C, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS.$$

where $\hat{n}$ is the unit normal vector at any point of S. (*U.P.T.U., 2006*)

Proof: Let S is surface such that its projection on the xy, yz and xz planes are regions bounded by simple closed curves. Let equation of surface $f(x, y, z) = 0$, can be written as

$$z = f_1(x, y)$$
$$y = f_2(x, z)$$
$$x = f_3(y, z)$$

Let $\vec{F} = F_1 i + F_2 j + F_3 k$

Then we have to show that

$$\iint_S \nabla\times\{F_1 i + F_2 j + F_3 k\}\cdot\hat{n}\,dS = \int_C \vec{F}\cdot d\vec{r}$$

Considering integral $\iint_S \nabla\times(F_1 i)\cdot\hat{n}\,dS$, we have

$$[\nabla\times(F_1\, i)]\cdot\hat{n}\,dS = \left[\left\{i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right\}\times F_1 i\right]\cdot\hat{n}dS \quad \ldots(i)$$

$$= \left[\frac{\partial F_1}{\partial z}j - \frac{\partial F_1}{\partial y}k\right]\cdot\hat{n}dS$$

$$= \left[\frac{\partial F_1}{\partial z}\hat{n}\cdot j - \frac{\partial F_1}{\partial y}\hat{n}\cdot k\right]dS \quad \ldots(ii)$$

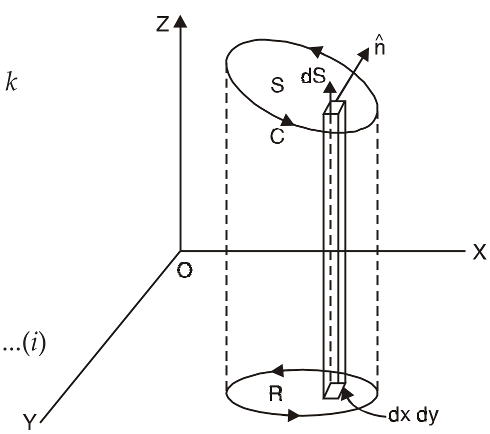


Fig. 5.19

$$\vec{r} = xi + yj + zk$$

Also, $$= xi + yj + f_1(x, y)k$$

So, $$\frac{\partial\vec{r}}{\partial y} = j + \frac{\partial f_1}{\partial y}k \qquad [\text{As } z = f(x, y)]$$

But $\frac{\partial\vec{r}}{\partial y}$ is tangent to the surface S. Hence, it is perpendicular to $\hat{n}$.

So, $$\hat{n}\cdot\frac{\partial r}{\partial y} = \hat{n}\cdot j + \frac{\partial f_1}{\partial y}\hat{n}\cdot k = 0$$

Hence, $$\hat{n}\cdot j = -\frac{\partial f_1}{\partial y}\hat{n}\cdot k = -\frac{\partial z}{\partial y}\hat{n}\cdot k$$

Hence, (*ii*) becomes

$$[\nabla \times (F_1\, i)]\cdot\hat{n}\ dS = -\left[\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y}\right]\hat{n}\cdot k\, dS \qquad ...(iii)$$

But on surface S

$$F_1(x, y, z) = F_1[x, y, f_1(x, y)]$$

$$= F(x, y) \qquad ...(iv)$$

$$\therefore \quad \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y} \qquad ...(v)$$

Hence, relation (*iii*) with the help of relation (*v*) gives

$$[\nabla \times (F_1\, i)]\cdot\hat{n}\ dS = -\frac{\partial F}{\partial y}(\hat{n}\cdot k)\, dS = -\frac{\partial F}{\partial y}\, dx\, dy$$

$$\iint_S (\nabla \times F_1 i)\cdot\hat{n}\, dS = \iint_R -\frac{\partial \vec{F}}{\partial y}\, dx\, dy \qquad ...(vi)$$

where R is projection of S on xy-plane.

Now, by Green's theorem in plane, we have

$$\int_{C_1}\vec{F}dx = -\iint_R \frac{\partial \vec{F}}{\partial y}\, dx\, dy,$$

where C_1 is the boundary of R.

As at each point (x, y) of the curve C_1 the value of F is same as the value of F_1 at each point (x, y, z) on C and dx is same for both curves. Hence, we have

$$\int_{C_1}\vec{F}\, dx = \int_C \vec{F}_1\, dx\,.$$

Hence, $$\int_C F_1\, dx = -\iint_R \frac{\partial \vec{F}}{\partial y}\, dx\, dy \qquad ...(vii)$$

From eqns. (*vi*) and (*vii*), we have

$$\iint_S \{\nabla \times F_1 i\}\cdot\hat{n}\, dS = \int_C \vec{F}_1\, dx \qquad ...(viii)$$

Similarly, taking projection on other planes, we have

$$\iint_S \{\nabla \times F_2 j\}\cdot\hat{n}\, dS = \int_C \vec{F}_2\, dy\,. \qquad ...(ix)$$

$$\iint_S \{\nabla \times F_3 k\}\cdot\hat{n}\, dS = \int_C \vec{F}_3\, dz \qquad ...(x)$$

Adding eqns. (*viii*), (*ix*), (*x*), we get

$$\iint_S \nabla \times \{F_1 i + F_2 j + F_3 k\}\cdot\hat{n}\, dS = \int_C \{F_1 dx + F_2 dy + F_3 dz\}$$

$$\Rightarrow \quad \boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot\hat{n}dS}\,.$$

5.21 CARTESIAN REPRESENTATION OF STOKE'S THEOREM

Let $$\vec{F} = F_1 i + F_2 j + F_3 k$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left\{\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right\} i + \left\{\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right\} j + \left\{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right\} k$$

So the relation

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \hat{n} \, dS,$$

is transformed into the form

$$\int_C \{F_1 dx + F_2 dy + F_3 dz\} = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\, dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz\, dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy\right].$$

Example 1. Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\, i - 2xy\, j$ taken round the rectangle bounded by $x = \pm a$, $y = 0$, $y = b$. (*U.P.T.U., 2002*)

Sol. We have $\vec{F} \cdot dr = \{(x^2 + y^2)\, i - 2xy\, j\} \cdot \{dx\, i + dy\, j\}$
$= (x^2 + y^2)\, dx - 2xy\, dy$

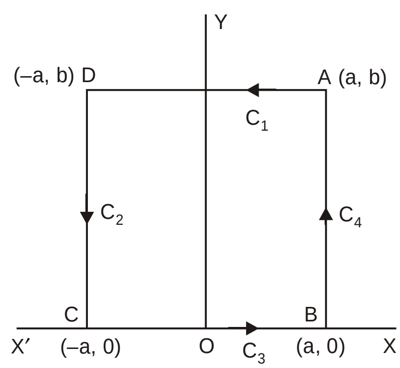


Fig. 5.20

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}$$

$$= I_1 + I_2 + I_3 + I_4$$

$$\therefore \quad I_1 = \int_{C_1} \{(x^2 + y^2) dx - 2xy\, dy\}$$

$$= \int_a^{-a} \{(x^2 + b^2)\, dx - 0\} \qquad \begin{bmatrix} \because y = b \\ \therefore dy = 0 \end{bmatrix}$$

$$= \left(\frac{x^3}{3} + b^2 x\right)_a^{-a}$$

$$= -\left(\frac{2}{3} a^3 + 2b^2 a\right)$$

$$I_2 = \int_{C_2} \{(x^2 + y^2) dx - 2xy\, dy\}$$

$$= \int_b^0 \{(-a)^2 + y^2\} 0 - 2(-a)\, y\, dy \qquad \begin{bmatrix} \because x = -a \\ \therefore dx = 0 \end{bmatrix}$$

$$= 2a \int_b^0 y\, dy$$

$$= 2a\left(\frac{y^2}{2}\right)_b^0 = -\,ab^2$$

$$I_3 = \int_{C_3}(x^2+y^2)dx - 2xy\,dy$$

$$= \int_{C_3} x^2 dx \qquad \left[\begin{array}{l}\because y=0\\ \therefore dy=0\end{array}\right]$$

$$= \int_{-a}^{+a} x^2 dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{2a^3}{3}$$

$$I_4 = \int_{C_4} -2ay\,dy \qquad \left[\begin{array}{l}\because x=0\\ \therefore dx=0\end{array}\right]$$

$$= -2a\int_0^b y\,dy = -2a\left(\frac{y^2}{2}\right)_0^b$$

$$= -\,ab^2$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = I_1+I_2+I_3+I_4$$

$$= -\left(\frac{2a^3}{3}+2b^2a\right) - ab^2 + \frac{2}{3}a^3 - ab^2$$

$$= -\,4ab^2 \qquad \ldots(i)$$

Again,

$$\text{curl }\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= -\,4yk$$

$$\hat{n} = k$$

$$\therefore \quad \hat{n}\cdot\text{curl }\vec{F} = k\cdot(-\,4yk) = -\,4y$$

$$\therefore \quad \iint_S \hat{n}\cdot\text{curl }\vec{F}\,dS = \int_{-a}^{a}\int_0^b -\,4y\,dx\,dy$$

$$= \int_{-a}^{+a} -\,4\left(\frac{y^2}{2}\right)_0^b dx$$

$$= -\,2b^2(x)_{-a}^{a}$$

$$= -\,4ab^2. \qquad \ldots(ii)$$

From eqns. (*i*) and (*ii*), we verify Stoke's theorem.

Example 2. Verify Stoke's theorem when $\vec{F} = yi + zj + xk$ and surface S is the part of the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane.

Sol. Stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl }\vec{F})\cdot\hat{n}dS$$

Here, C is unit circle $x^2 + y^2 = 1,\ z = 0$

Also, $\vec{F} \cdot d\vec{r} = (yi + zj + xk) \cdot (dxi + dyj + dzk)$

$= ydx + zdy + xdz$

$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C ydx + \int_C zdy + \int_C xdz$

Again, on the unit circle C, $z = 0$

$dz = 0$

Let $x = \cos\phi, \quad \therefore\ dx = -\sin\phi . d\phi$

and $y = \sin\phi, \quad \therefore\ dy = \cos\phi . d\phi$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C y\, dx$$

$$= \int_0^{2\pi} \sin\phi\, (-\sin\phi)\, d\phi$$

$$= -\int_0^{2\pi} \sin^2\phi\, d\phi$$

$$= -\pi \qquad ...(i)$$

$$\text{Again,} \quad \text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -i - j - k$$

Using spherical polar coordinates

$$\hat{n} = \sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k$$

$$\therefore \quad \text{curl } \vec{F} \cdot \hat{n} = -(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta).$$

$$\text{Hence,} \quad \iint_S (\text{curl } \vec{F}) \cdot \hat{n}\, dS = -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta\, d\theta\, d\phi$$

$$= -\int_{\theta=0}^{\pi/2} [\sin\theta\sin\phi - \sin\theta\cos\phi + \phi\cos\theta]_0^{2\pi} \sin\theta\, d\theta$$

$$= -2\pi \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$= -\pi \int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= \frac{\pi}{2} (\cos 2\theta)_0^{\pi/2}$$

$$= -\pi \qquad ...(ii)$$

From eqns. (*i*) and (*ii*), we verify Stoke's theorem.

Example 3. Verify Stoke's theorem for $\vec{F} = xzi - yj + x^2yk$, where S is the surface of the region bounded by $x = 0,\ y = 0,\ z = 0,\ 2x + y + 2z = 8$ which is not included in the xz-plane. (*U.P.T.U., 2006*)

Sol. Stoke's theorem states that

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n} dS$$

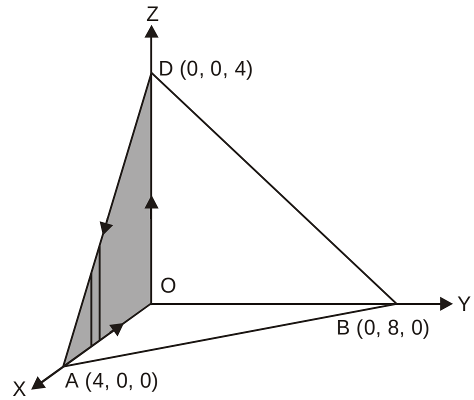


Fig. 5.21

Here C is curve consisting of the straight lines AO, OD and DA.

$$\text{L.H.S.} = \oint_C \vec{F}\cdot d\vec{r} = \int_{AO+OD+DA}$$

$$= \int_{AO} + \int_{OD} + \int_{DA} = LI_1 + LI_2 + LI_3$$

On the straight line AO: $y = 0$, $z = 0$, $\vec{F} = 0$, so

$$LI_1 = \int_{AO} \vec{F}\cdot d\vec{r} = 0$$

On the straight line OD : $x = 0$, $y = 0$. $\vec{F} = 0$, so

$$LI_2 = \int_{OD} \vec{F}\cdot d\vec{r} = 0$$

On the straight line DA: $x + z = 4$ and $y = 0$, so

$$\vec{F} = xzi = x\,(4 - x)\,i$$

$$LI_3 = \int_{DA} \vec{F}\cdot d\vec{r} = \int_0^4 x(4-x)\,i\cdot dxi = \int_0^4 x(4-x)\,dx = \frac{32}{3}$$

$$LI = 0 + 0 + \frac{32}{3} = \frac{32}{3}$$

Here the surface S consists of three surfaces (planes) $S_1 : OAB$, $S_2 : OBD$, $S_3 : ABD$, so that

$$\text{R.H.S.} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS = \iint_{S_1+S_2+S_3}$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = SI_1 + SI_2 + SI_3$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & -y & x^2y \end{vmatrix} = x^2i + x(1 - 2y)\,j$$

On the surface S_1: Plane OAB : $z = 0$, $\hat{n} = -\hat{k}$, so

$$(\nabla\times\vec{F})\cdot\hat{n} = \left[x^2i + x\,(1-2y)\,j\right]\cdot(-k) = 0$$

$$\text{SI}_1 = \iint_{S_1} (\nabla\times\vec{F})\cdot\hat{n}dS = 0$$

On surface S_2: Plane OBD : Plane $x = 0$, $\hat{n} = -\,i$, so

$$\nabla\times\vec{F} = 0$$

$$\text{SI}_2 = \iint_{S_2} (\nabla\times\vec{F})\cdot\hat{n}dS = 0$$

On surface S_3: Plane ABD: $2x + y + 2z = 8$.

Unit normal $\hat{n}$ to the surface $S_3 = \dfrac{\nabla(2x+y+2z)}{|\nabla(2x+y+2z)|}$

$$\hat{n} = \frac{2i + j + 2k}{\sqrt{4+1+4}} = \frac{2i + j + 2k}{3}$$

$$(\nabla \times \vec{F}) \cdot \hat{n} = \frac{2}{3}x^2 + \frac{1}{3}x(1-2y)$$

To evaluate the surface integral on the surface S_3, project S_3 on to say xz-plane *i.e.*, projection of ABD on xz-plane is AOD

$$dS = \frac{dx\,dz}{n \cdot j} = \frac{dx\,dz}{1/3} = 3dx\,dz$$

Thus

$$SI_3 = \iint_{S_3} (\nabla \times \vec{F}) \cdot \hat{n}dS$$

$$= \iint_{AOD} \left[\frac{2}{3}x^2 + \frac{x}{3}(1-2y)\right] 3\,dx\,dz$$

$$= \int_{x=0}^{4} \int_{z=0}^{4-x} \left[2x^2 + x(1-2y)\right] dz\,dx$$

since the region AOD is covered by varying z from 0 to $4 - x$, while x varies from 0 to 4. Using the equation of the surface S_3, $2x + y + 2z = 8$, eliminate y, then

$$SI_3 = \int_0^4 \int_0^{4-x} \left\{2x^2 + x[1 - 2(8 - 2x - 2z)]\right\} dz\,dx$$

$$= \int_0^4 \int_0^{4-x} (6x^2 - 15x + 4xz)\,dz\,dx$$

$$= \int_0^4 \left[6x^2 z - 15xz + \frac{4xz^2}{2}\right]_0^{4-x} dx$$

$$= \int_0^4 (23x^2 - 4x^3 - 28x)\,dx = \frac{32}{3}$$

Thus L.H.S. = *L.I.* = R.H.S. = S.I.

Hence Stoke's theorem is verified.

Example 4. Evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}dS$ over the surface of intersection of the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ which is included in the first octant, given that $\vec{F} = 2yzi - (x + 3y - 2)\,j + (x^2 + z)k$.

Sol. By Stoke's theorem the given surface integral can be converted to a line integral *i.e.*,

$$SI = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}dS = \oint_C \vec{F} \cdot d\vec{r} = LI$$

Here C is the curve consisting of the four curves C_1: $x^2 + z^2 = a^2$, $y = 0$; C_2: $x^2 + y^2 = a^2$, $z = 0$, C_3: $x = 0$, $y = a$, $0 \le z \le a$: C_4: $x = 0$, $z = a$, $0 \le y \le a$.

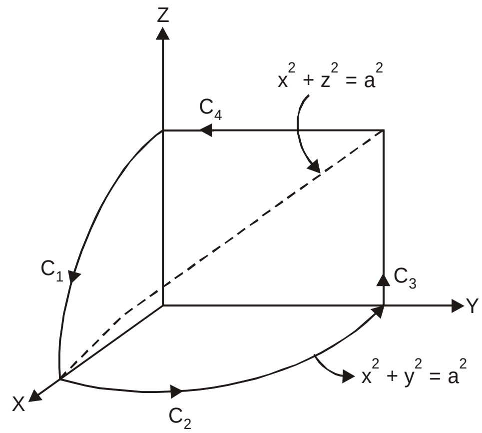


Fig. 5.22

$$LI = \oint_C \vec{F} \cdot d\vec{r} = \int_{C_1+C_2+C_3+C_4} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= LI_1 + LI_2 + LI_3 + LI_4$$

On the curve C_1: $y = 0;\ x^2 + z^2 = a^2$

$$LI_1 = \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} (x^2 + z)\, dz$$

$$= \int_a^0 \left[(a^2 - z^2) + z\right] dz = -\frac{2}{3}a^3 - \frac{a^2}{2}$$

On the curve C_2: $z = 0,\ x^2 + y^2 = a^2$

$$LI_2 = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} -(x + 3y - 2)\, dy$$

$$= -\int_0^a \left(\sqrt{a^2 - y^2} + 3y - 2\right) dy$$

$$= -\frac{\pi a^2}{4} - \frac{3}{2}a^2 + 2a$$

On the curve C_3: $x = 0,\ y = a,\ 0 \le z \le a$

$$LI_3 = \int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^a z dz = \frac{a^2}{2}$$

On C_4; $x = 0,\ z = a,\ 0 \le y \le a$

$$LI_4 = \int \vec{F} \cdot d\vec{r} = \int_a^0 (2 - 3y)\, dy = -2a + \frac{3a^2}{2}$$

$$SI = \iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n} dS = LI = \left(\frac{-2a^3}{3} - \frac{a^2}{2}\right)$$

$$+ \left(-\frac{\pi a^2}{4} - \frac{3a^2}{2} + 2a\right) + \frac{a^2}{2} + \left(-2a + \frac{3a^2}{2}\right)$$

$$SI = \frac{-a^2}{12}(3\pi + 8a).$$

Example 5. Evaluate $\oint_S \vec{F}\cdot d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x+z)\,\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0) (1, 0, 0) and (1, 1, 0) (*U.P.T.U., 2001*)

Sol. Since z-coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy-plane and $\hat{n} = \hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j} + 2\,(x-y)\,\hat{k}$$

$$\therefore \quad \text{curl}\ \vec{F}\cdot\hat{n} = \left[\hat{j} + 2\,(x-y)\,\hat{k}\right]\cdot\hat{k} = 2\,(x-y)$$

The equation of line OB is $y = x$.

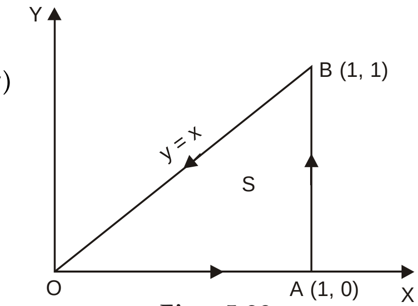


Fig. 5.23

By Stoke's theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS$

$$= \int_0^1\int_0^x 2\,(x-y)\,dy\,dx$$

$$= \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1\left(x^2 - \frac{x^2}{2}\right)dx = \int_0^1 x^2 dx = \frac{1}{3}.$$

Example 6. Apply Stoke's theorem to prove that

$\int_C (y\,dx + z\,dy + x\,dz) = -2\sqrt{2}\pi a^2$, where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0$, $x + y = 2a$ and begins of the point $(2a, 0, 0)$.

Sol. The given curve C is

$$x^2 + y^2 + z^2 - 2ax - 2ay = 0$$
$$x + y = 2a$$

$$\Rightarrow \quad (x-a)^2 + (y-a)^2 + z^2 = \left(a\sqrt{2}\right)^2$$
$$x + y = 2a$$

which is the curve of intersection of the sphere

$$(x-a)^2 + (y-a)^2 + z^2 = \left(a\sqrt{2}\right)^2$$

and the plane $\quad x + y = 2a.$

Clearly, the centre of the sphere is $(a, a, 0)$ and radius is $a\sqrt{2}$.

Also, the plane passes through $(a, a, 0)$.

Hence, the circle C is a great circle.

$\therefore$ Radius of circle C = Radius of sphere = $\sqrt{2a}$

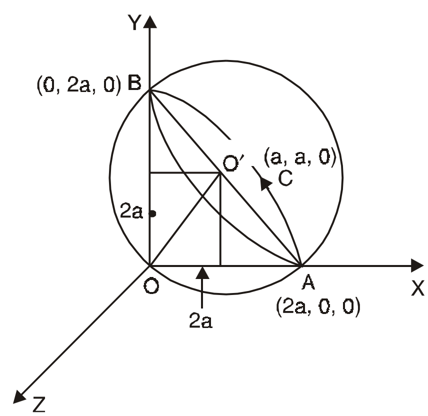


Fig. 5.24

Now, $\int_C (y\,dx + z\,dy + x\,dz) = \int_C (yi + zj + xk)\cdot(dxi + dyj + dzk)$

$$= \int_C (yi + zj + xk) \cdot d\vec{r}$$

$$= \int_C curl\ (yi + zj + xk) \cdot \hat{n} dS . \qquad \text{[Using Stoke's theorem]}$$

But $\quad$ curl $(yi + zj + xk) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\ i - j - k.$

Since S is the surface of the plane $x + y = 2a$ bounded by the circle C. Then

$$\hat{n} = \frac{\nabla(x + y - 2a)}{|\nabla(x + y - 2a)|}$$

$$= \frac{i + j}{\sqrt{2}}$$

$\therefore \quad$ curl $(yi + zj + xk) \cdot \hat{n} = (-\ i - j - k) \cdot \left(\dfrac{i + j}{\sqrt{2}}\right)$

$$= -\frac{1+1}{\sqrt{2}} = -\sqrt{2} .$$

Hence, the given line integral $= \int_S -\sqrt{2}\ dS$

$$= -\ \sqrt{2}\ \text{(Area of the circle } C).$$

$$= -\ \sqrt{2}\ \pi\ (\sqrt{2}a)^2 = -2\sqrt{2}\pi a^2.$$

Example 7. Evaluate $\int (xy\, dx + xy^2\, dy)$ taken round the positively oriented square with vertices (1, 0), (0, 1), (– 1, 0) and (0, – 1) by using Stoke's theorem and verify the theorem.

Sol. We have

$$\int_C (xy\ dx + xy^2 dy)$$

$$= \int_C (xy\ i + xy^2 j) \cdot (dx\ i + dy\ j)$$

$$= \int_C (xy\ i + xy^2 j) \cdot d\vec{r}$$

$$= \iint_S \text{curl}\ (xy\ i + xy^2 j) \cdot \hat{n} dS$$

by Stoke's theorem, where S is the area of the square $ABCD$.

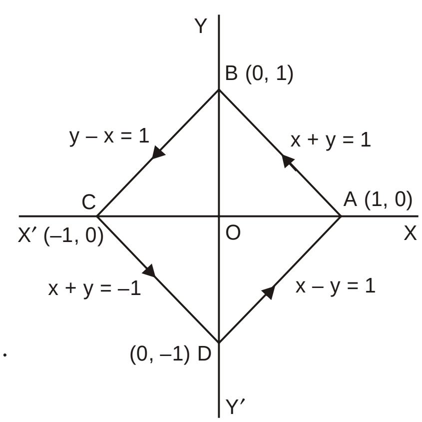


Fig. 5.25

Now, $\quad$ curl $(xy\ i + xy^2\ j) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$

$$= (y^2 - x)\ k$$

$$\therefore \quad \text{curl}\,(xy\,i + xy^2 j)\cdot \hat{n} = (y^2 - x)\,k\cdot k = (y^2 - x)$$

$$\therefore \iint_S \text{curl}\,(xy\,i + xy^2 j)\cdot \hat{n}\,dS = \iint_S (y^2 - x)\,dS$$

$$= \iint_S (y^2 - x)\,dx\,dy$$

$$= \iint_S y^2\,dx\,dy - \iint_S x\,dx\,dy$$

$$= 4\int_0^1\int_0^{1-x} y^2 dx\,dy - S\bar{x} \qquad \text{[By symmetry]}$$

$$= 4\int_0^1\int_0^{1-x} y^2 dx\,dy - S\cdot 0 \qquad [\because\ \bar{x} = x\text{-coordinate of the } C.G. \text{ of } ABCD = 0]$$

$$= 4\int_0^1\int_0^{1-x} y^2 dx\,dy$$

$$= 4\int_0^1 \left(\frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \frac{4}{3}\int_0^1 (1-x)^3 dx = \frac{1}{3}. \qquad ...(i)$$

Verification of Stoke's theorem: The given line integral

$$= \int_C (xy\,dx + xy^2 dy),$$

where C is the boundary of the square $ABCD$. Now C can be broken up into four parts namely:

(*i*) the line AB whose equation is $x + y = 1$,

(*ii*) the line BC whose equation is $y - x = 1$,

(*iii*) the line CD whose equation is $x + y = -1$, and

(*iv*) the line DA whose equation is $x - y = 1$.

Hence, the given line integral

$$= \int_{AB}(xy\,dx + xy^2 dy) + \int_{BC}(xy\,dx + xy^2 dy) + \int_{CD}(xy\,dx + xy^2 dy) + \int_{DA}(xy\,dx + xy^2 dy)$$

$$= \left\{\int_0^1 x\,(1-x)\,dx + \int_0^1 (1-y)\,y^2 dy\right\} + \left\{\int_0^{-1} x\,(1+x)\,dx + \int_1^0 (y-1)\,y^2 dy\right\}$$

$$+ \left\{\int_{-1}^0 x\,(-x-1)\,dx + \int_0^1 -y^2(1+y)\,dy\right\} + \left\{\int_0^1 x\,(x-1)\,dx + \int_{-1}^0 y^2(1+y)\,dy\right\}$$

$$= 2\int_0^1 x\,(x-1)\,dx + 2\int_0^{-1} x\,(1+x)\,dx + 2\int_0^1 (1-y)\,y^2\,dy + 2\int_{-1}^0 y^2\,(1+y)\,dy$$

$$= 2\left(\frac{x^2}{3}-\frac{x^2}{2}\right)_0^1 + 2\left(\frac{x^3}{3}+\frac{x^2}{2}\right)_0^{-1} + 2\left(\frac{y^3}{3}-\frac{y^4}{4}\right)_0^1 + 2\left(\frac{y^3}{3}+\frac{y^4}{4}\right)_{-1}^0$$

$$= 2\left(\frac{1}{3}-\frac{1}{2}\right)+2\left(-\frac{1}{3}+\frac{1}{2}\right)+2\left(\frac{1}{3}-\frac{1}{4}\right)+2\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= 4\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= \frac{1}{3}. \qquad ...(ii)$$

From eqns. (*i*) and (*ii*), it is evident that

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \text{curl } \vec{F}\cdot \hat{n}\, dS$$

Hence, Stoke's theorem is verified.

Example 8. Verify Stoke's theorem for the function

$$\vec{F} = (x + 2y)\, dx + (y + 3x)\, dy$$

where C is the unit circle in the xy-plane.

Sol. Let $\vec{F} = F_1\, i + F_2\, j + F_3\, k$

$$\vec{F}\cdot dr = (F_1\, i + F_2\, j + F_3\, k)\cdot(dx\, i + dy\, j + dz\, k)$$
$$= F_1 dx + F_2 dy + F_3 dz$$

Here, $F_1 = x + 2y$, $F_2 = y + 3x$, $F_3 = 0$

Unit circle in xy-plane is $x^2 + y^2 = 1$

or $x = \cos\phi,\ dx = -\sin\phi\, d\phi$

$y = \sin\phi,\ dy = \cos\phi\, d\phi.$

Hence, $$\int_C \vec{F}\cdot d\vec{r} = \int (x+2y)\, dx + (y+3x)\, dy$$

$$= \int_0^{2\pi} [-(\cos\phi + 2\sin\phi)\sin\phi\, d\phi + (\sin\phi + 3\cos\phi)\cos\phi\, d\phi]$$

$$= \int_0^{2\pi} \{-\sin\phi\cos\phi - 2\sin^2\phi + \sin\phi\cos\phi + 3\cos^2\phi\}\, d\phi$$

$$= \int_0^{2\pi} (3\cos^2\phi - 2\sin^2\phi)\, d\phi$$

$$= \int_0^{2\pi} \{(3\cos^2\phi - 2(1-\cos^2\phi))\}\, d\phi$$

$$= \int_0^{2\pi} \{5\cos^2\phi - 2\}\, d\phi$$

$$= \int_0^{2\pi} \left[\frac{5(1+\cos 2\phi)}{2} - 2\right] d\phi$$

$$= \int_0^{2\pi} \left\{\frac{1}{2} + \frac{5}{2}\cos 2\phi\right\} d\phi$$

$$= \frac{1}{2}\left[\phi + \frac{5}{2}\sin 2\phi\right]_0^{2\pi}$$

$$= \frac{1}{2}[2\pi + 0] = \pi.$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y & y+3x & 0 \end{vmatrix}$$

$$= i\,\{0\} - j\,\{0\} + k\,\left\{\frac{\partial}{\partial x}(y+3x) - \frac{\partial}{\partial y}(x+2y)\right\}$$

$$= k\,(3-2) = k.$$

Hence, $$\iint_S \text{curl } \vec{F}\cdot\hat{n}\,dS = \int (k\cdot k)\,dS$$

$$= \int dS$$

$$= \iint dx\,dy$$

$$= \int x dy$$

$$= \int_0^{2\pi} \cos^2\phi\,d\phi$$

$$= \frac{1}{2}\int_0^{2\pi} (1+\cos 2\phi)\,d\phi$$

$$= \frac{1}{2}\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{2\pi}$$

$$= \pi.$$

So Stoke's theorem is verified.

EXERCISE 5.6

1. Evaluate $\iint_S (\nabla \times \vec{A}) \cdot \vec{n} dS$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane and $\vec{A} = (x^2 + y - 4)\, i + 3xyj + (2xz + z^2)\, k$. **[Ans.** -16π**]**

2. If $\vec{F} = (y^2 + z^2 + x^2)\, i + (z^2 + x^2 - y^2)\, j + (x^2 + y^2 - z^2)\, k$ evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$ taken over the surface $S = x^2 + y^2 - 2ax + az = 0,\ z \geq 0$. **[Ans.** $2\pi a^3$**]**

3. Evaluate $\iint_S \nabla \times (yi + zj + xk) \cdot \hat{n}\, dS$ over the surface of the paraboloid $z = 1 - x^2 - y^2, z \geq 0$. **[Ans.** π**]**

4. $\vec{F} = (2x - y)\, i - yz^2 j - y^2 zk$, where S upper half surface of the sphere $x^2 + y^2 + z^2 = 1$. **[Hint:** Here C, $x^2 + y^2 = 1$, $z = 0$**]** **[Ans.** π**]**

5. Using Stoke's theorem or otherwise, evaluate

$$\int_C \left[(2x - y)dx - yz^2 dy - y^2 z\, dz \right].$$

where C is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius. **[Ans.** π**]**

6. Use the Stoke's theorem to evaluate

$$\int_C \left[(x + 2y)dx + (x - z)\, dy + (y - z)dz \right].$$

where C is the boundary of the triangle with vertices (2, 0, 0) (0, 3, 0) and (0, 0, 6) oriented in the anti-clockwise direction. **[Ans.** 15**]**

7. Verify Stoke's theorem for the Function $\vec{F} = x^2 i - xy\, j$ integrated round the square in the plane $z = 0$ and bounded by the lines $x = 0,\ y = 0,\ x = a,\ y = a$.

[Ans. Common value $\frac{-a^3}{2}$**]**

8. Verify Stoke's theorem for $\vec{F} = (x^2 + y - 4)i + 3xy\, j + (2xz + z^2)k$ over the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above the xy plane. **[Ans.** Common value -16π**]**

9. Verify Stoke's theorem for the function $\vec{F} = zi + xj + yk$, where C is the unit circle in xy plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$. *(U.P.T.U., 2002)*

[Ans. Common value π**]**

10. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by Stoke's theorem for $\vec{F} = yzi + zxj + xyk$ and C is the curve of intersection of $x^2 + y^2 = 1$ and $y = z^2$. **[Ans.** 0**]**

5.22 GAUSS'S DIVERGENCE THEOREM

If $\vec{F}$ is a continuously differentiable vector point function in a region V and S is the closed surface enclosing the region V, then

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V div\,\vec{F}\,dV \qquad ...(i)$$

where $\hat{n}$ is the unit outward drawn normal vector to the surface S. (*U.PT.U., 2006*)

Proof: Let i, j, k are unit vectors along X, Y, Z axes respectively. Then $\vec{F} = F_1 i + F_2 j + F_3 k$, where F_1, F_2, F_3, and their derivative in any direction are assumed to be uniform, finite and continuous. Let S is a closed surface which is such that any line parallel to the coordinate axes cuts S at the most on two points. Let z coordinates of these points be $z = F_1(x, y)$ and $z = F_2(x, y)$, we have assumed that the equations of lower and upper portions S_2 and S_1 of S are $z = F_2(x, y)$ and $z = F_1(x, y)$ respectively.

The result of Gauss divergence theorem (*i*) incomponent form is

$$\iiint_S (F_1 i + F_2 j + F_3 k)\cdot\hat{n}ds = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dV \qquad ...(ii)$$

Now, consider the integral

$$I_1 = \iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$$

$$= \iint_R \left[\int_{F_2}^{F_1} \frac{\partial F_3}{\partial z}dz\right]dx\,dy$$

where R is projection of S on xy-plane.

$$I_1 = \iint_R \left[F_3(x,y,z)\right]_{F_2(x,y)}^{F_1(x,y)} dx\,dy$$

$$= \iint_R \left[F_3(x,y,F_1) - F_3(x,y,F_2)\right]dx\,dy$$

$$\iint_R F_3(x,y,F_1)\,dx\,dy - \iint_R F_3(x,y,F_2)\,dx\,dy$$

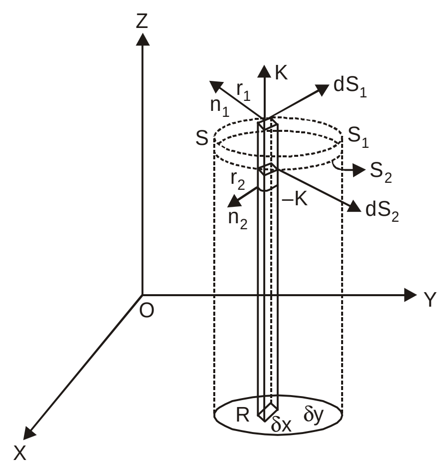


Fig. 5.26

For the upper portion S_1 of S,

$$dx\,dy = k\cdot\hat{n}_1\cdot dS_1$$

where $\hat{n}_1$ is unit normal vector to surface dS_1 in outward direction.

For the lower portion S_2 of S.

$$dx\,dy = -k\cdot\hat{n}_1\cdot dS_2$$

where $\hat{n}_2$ is unit normal vector to surface dS_2 in outward direction.

Thus, we have

$$\iint_R F_3(x,y,F_1)\,dx\,dy = \iint_{S_1} F_3\,k\cdot\hat{n}_1\,dS_1$$

and

$$\iint_R F_3(x,y,F_2)\,dx\,dy = -\iint_{S_2} F_3\,k\cdot\hat{n}_2\,dS_2$$

So $\iint_R F_3(x, y, F_1)\, dx\, dy - \iint_R F_3(x, y, F_2)\, dx\, dy$

$$= \iint_{S_1} F_3\,(k \cdot \hat{n}_1)\, dS_1 + \iint_{S_2} F_3\,(k \cdot \hat{n}_2)\, dS_2$$

$$= \iint F_3\; k\,.\,(\hat{n}_1\; dS_1 + \hat{n}_2\; dS_2)$$

$$= \iint_S F_3\,(k \cdot \hat{n})\, dS \qquad [\because \hat{n}S = \hat{n}_1 S_1 + \hat{n}_2 S_2]$$

Hence,

$$I_1 = \iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz$$

$$= \iint_{S_1} F_3\,(k \cdot \hat{n})\, dS \qquad \text{...}(iii)$$

Similarly, projecting S on other coordinate planes, we have

$$\iiint_V \frac{\partial F_3}{\partial y}\, dx\, dy\, dz = \iint_S F_2\,(j \cdot \hat{n})\, dS \qquad \text{...}(iv)$$

$$\iiint_V \frac{\partial F_1}{\partial x}\, dx\, dy\, dz = \iint_S F_1\,(i \cdot \hat{n})\, dS \qquad \text{...}(v)$$

Adding eqns. (*iii*), (*iv*), (*v*)

$$\iiint_V \left\{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right\} dx\, dy\, dz = \iint_S \{F_1(i\,.\,\hat{n}) + F_2(j\,.\,\hat{n}) + F_3(k\,.\,\hat{n})\}\, dS$$

$$\Rightarrow \iiint_V \left\{i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right\} \cdot \{F_1 i + F_2 j + F_3 k\}\; dx\; dy\; dz$$

$$= \iint_S \{F_1\; i + F_2\; j + F_3\; k\} \cdot \hat{n}\; dS$$

$$\Rightarrow \qquad \iiint_V \text{div}\, \vec{F}\; dV = \iint_S \vec{F} \cdot \hat{n}\; dS$$

$$= \iint_S \vec{F} \cdot \hat{n}\; dS$$

or

$$\boxed{\iint_S \vec{F} \cdot \hat{n}\; ds = \iiint_V \text{div}\, \vec{F}\; dV}\,.$$

5.23 CARTESIAN REPRESENTATION OF GAUSS'S THEOREM

Let $\vec{F} = F_1 i + F_2 j + F_3 k$

where F_1, F_2, F_3 are functions of x, y, z.

and $dS = dS\,(\cos\,\alpha\; i + \cos\,\beta\; j + \cos\,\gamma\; k)$

where α, β, γ are direction angles of dS. Hence, $dS \cos\,\alpha$, $dS \cos\,\beta$, $dS \cos\,\gamma$ are the orthogonal projections of the elementary area dS on yz-plane, zx-plane and xy-plane respectively. As the mode of sub-division of surface is arbitrary, we choose a sub-division formed by planes parallel to yz-plane, zx-plane and xy-plane. Clearly, its projection on coordinate planes will be rectangle with sides dy and dz on yz-plane, dz and dx on zx-plane, dx and dy on xy-plane.

Hence, projected surface elements are $dy\,dz$ on yz-plane, $dz\,dx$ on zx-plane and $dx\,dy$ on xy-plane.

$$\therefore \quad \int_S \vec{F}\cdot\hat{n}\,dS = \iint_S \left[F_1 dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy\right] \quad \text{...(}i\text{)}$$

By Gauss divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \text{div}\,\vec{F}\,dV. \quad \text{...(}ii\text{)}$$

In cartesian coordinates,

$$dV = dx\,dy\,dz.$$

Also,

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

Hence,

$$\int_V \text{div}\,\vec{F}\,dV = \iiint_V \left\{\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right\} dx\,dy\,dz \quad \text{...(}iii\text{)}$$

Hence cartesian form of Gauss theorem is,

$$\iint_S \left\{F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy\right\} = \iiint_V \left\{\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right\} dx\,dy\,dz.$$

Example 1. Find $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = (2x+3z)\,\hat{i} - (xz+y)\,\hat{j} + \left(y^2+2z\right)\hat{k}$ and S is the surface of the sphere having centre at (3, – 1, 2) and radius 3. (*U.P.T.U., 2000, 2005*)

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \text{div}\,\vec{F}\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x+3z)+\frac{\partial}{\partial y}(-xz-y)+\frac{\partial}{\partial z}\left(y^2+2z\right)\right] dV$$

$$= \iiint_V (2-1+2)\,dV = 3\iiint_V dV = 3V$$

But V is the volume of a sphere of radius 3.

$$\therefore \quad V = \frac{4}{3}\pi(3)^3 = 36\pi.$$

Hence

$$\iint_S \vec{F}\cdot\hat{n}\,dS = 3\times 36\pi = 108\pi.$$

Example 2. Evaluate $\iint_S \left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}\,dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane.

Sol. Let V be the volume enclosed by the surface S. Then by divergence theorem, we have

$$\iint_S \left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}\,dS = \iiint_V \text{div}\left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right) dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}\left(y^2z^2\right)+\frac{\partial}{\partial y}\left(z^2x^2\right)+\frac{\partial}{\partial z}\left(z^2y^2\right)\right] dV$$

$$= \iiint_V 2zy^2\, dV = 2\iiint_V zy^2\, dV$$

Changing to spherical polar coordinates by putting

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$
$$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$$

To cover V, the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

$$\therefore \quad 2\iiint_V zy^2\, dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)\left(r^2\sin^2\theta\sin^2\phi\right)r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\sin^3\theta\cos\theta\sin^2\phi\left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

$$= \frac{1}{12}\int_0^{2\pi}\sin^2\phi\cdot d\phi = \frac{1}{12}\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{\pi}{12}.$$

Example 3. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, dS$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.

Sol. If V is the volume enclosed by S, then V is bounded by the surfaces $z = 0$, $z = 4$, $z^2 = x^2 + y^2$.

By divergence theorem, we have $\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \text{div}\,\vec{F}\, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}\left(xyz^2\right) + \frac{\partial}{\partial z}(3z)\right]dV = \iiint_V \left(4z + xz^2 + 3\right)dV$$

$$= \int_0^4\int_{-z}^{z}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}\left(4z + xz^2 + 3\right)dx\, dy\, dz$$

$$= 2\int_0^4\int_{-z}^{z}\int_0^{\sqrt{z^2-y^2}}(4z+3)\, dx\, dy\, dz,\ \text{ since } \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} x\, dx = 0$$

$$= 2\int_0^4\int_{-z}^{z}(4z+3)\sqrt{z^2-y^2}\, dy\, dz = 4\int_0^4\int_0^z (4z+3)\sqrt{z^2-y^2}\, dy\, dz$$

$$= 4\int_0^4 (4z+3)\left[\frac{y\sqrt{z^2-y^2}}{2} + \frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$= 4\int_0^4 (4z+3)\left[\frac{z^2}{2}\sin^{-1}1\right]dz = 4\times\frac{\pi}{4}\int_0^4\left(4z^3 + 3z^2\right)dz$$

$$= \pi\left[z^4 + z^3\right]_0^4 = \pi(256+64) = 320\pi.$$

Example 4. By transforming to a triple integral evaluate

$$I = \iint_S \left(x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy\right) \qquad (U.P.T.U.,\ 2006)$$

where S is the closed surface bounded by the planes $z = 0$, $z = b$ and the cylinder $x^2 + y^2 = a^2$.

Sol. By divergence theorem, the required surface integral I is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(x^2 y\right) + \frac{\partial}{\partial z}\left(x^2 z\right)\right] dV$$

$$= \int_{z=0}^{b} \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} \left(3x^2 + x^2 + x^2\right) dx\, dy\, dz$$

$$= 4 \times 5 \int_{z=0}^{b} \int_{y=0}^{a} \int_{x=0}^{\sqrt{(a^2-y^2)}} x^2\, dx\, dy\, dz = 20 \int_{z=0}^{b} \int_{y=0}^{a} \left[\frac{x^3}{3}\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy\, dz$$

$$= \frac{20}{3} \int_{z=0}^{b} \int_{y=0}^{a} \left(a^2 - y^2\right)^{\frac{3}{2}} dy\, dz = \frac{20}{3} \int_{y=0}^{a} \left[\left(a^2 - y^2\right)^{\frac{3}{2}} z\right]_{z=0}^{b} dy = \frac{20}{3} \int_{y=0}^{a} b\left(a^2 - y^2\right)^{\frac{3}{2}} dy\,.$$

Put $y = a \sin t$ so that $dy = a \cos t\, dt$.

$$\therefore \quad I = \frac{20}{3} b \int_0^{\frac{\pi}{2}} a^3 \cos^3 t\, (a \cos t) dt = \frac{20}{3} a^4 b \int_0^{\frac{\pi}{2}} \cos^4 t\, dt = \frac{20}{3} a^4 b\, \frac{3}{4.2}\, \frac{\pi}{2} = \frac{5}{4} \pi a^4 b\,.$$

Example 5. Verify divergence theorem for $\vec{F} = (x^2 - yz)\, i + (y^2 - zx)\, j + (z^2 - xy)\, k$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. [*U.P.T.U. (C.O.), 2006*]

Sol. We have div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right) = 2x + 2y + 2z.$

$\therefore$ Volume integral $= \iiint_V \nabla \cdot \vec{F}\, dV = \iiint_V 2(x + y + z)\, dV$

$$= 2\int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x + y + z)\, dx\, dy\, dz = 2\int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^2}{2} + yx + zx\right]_{x=0}^{a} dy\, dz$$

$$= 2\int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{a^2}{2} + ay + az\right] dy\, dz = 2\int_{z=0}^{c} \left[\frac{a^2}{2} y + a\frac{y^2}{2} + azy\right]_{y=0}^{b} dz$$

$$= 2\int_{z=0}^{c} \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right] dz$$

$$= 2\left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab\frac{z^2}{2}\right]_0^c = [a^2bc + ab^2c + abc^2] = abc\,(a + b + c).$$

Surface integral: Now we shall calculate

$$\iint_S \vec{F} \cdot \hat{n}\, dS$$

Over the six faces of the rectangular parallelopiped.

Over the face *DEFG*,

$$\hat{n} = i,\ x = a.$$

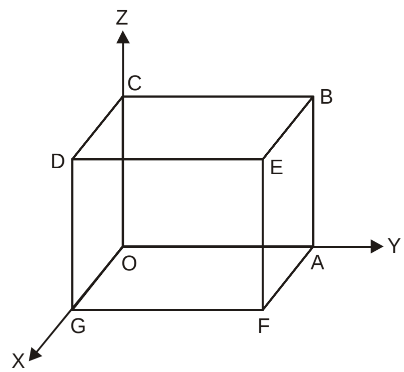


Fig. 5.27

Therefore, $\iint_{DEFG} \hat{F} \cdot \hat{n}\, dS$

$$= \int_{z=0}^{c} \int_{y=0}^{b} \left[\left(a^2 - yz\right) i + \left(y^2 - 2a\right) j + \left(z^2 - ay\right) k\right] \cdot i\, dy\, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} \left(a^2 - yz\right) dy\, dz = \int_{z=0}^{c} \left[a^2 y - z\frac{y^2}{2}\right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c} \left[a^2 b - \frac{zb^2}{2}\right] dz = \left[a^2 bz - \frac{z^2}{4} b^2\right]_0^c = a^2 bc - \frac{c^2 b^2}{4}.$$

Over the face $ABCO$, $\hat{n} = -i$, $x = 0$. Therefore

$$= \iint_{ABCO} \hat{F} \cdot \hat{n}\, dS = \iint \left[(0 - yz) i + y^2 j + z^2 k\right] \cdot (-i)\, dy\, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} yz\, dy\, dz = \int_{z=0}^{c} \left[\frac{y^2 z}{2}\right]_{y=0}^{b} dz = \int_{z=0}^{c} \frac{b^2}{2} z dz = \frac{b^2 c^2}{4}$$

Over the face $ABEF$, $\hat{n} = j$, $y = b$. Therefore

$$\iint_{ABEF} \vec{F} \cdot \hat{n}\, dS = \int_{z=0}^{c} \int_{x=0}^{a} \left[\left(x^2 - bz\right) i + \left(b^2 - zx\right) j + \left(z^2 - bx\right) k\right] \cdot j\, dx\, dz$$

$$= \int_{z=0}^{c} \int_{x=0}^{a} \left(b^2 - zx\right) dx\, dz = b^2 ca - \frac{a^2 c^2}{4}.$$

Over the face $OGDC$, $\hat{n} = -j$, $y = 0$. Therefore

$$\iint_{OGDC} \vec{F} \cdot \hat{n}\, dS = \int_{z=0}^{c} \int_{x=0}^{a} zx\, dx\, dz = \frac{c^2 a^2}{4}.$$

Over the face $BCDE$, $\hat{n} = k$, $z = c$. Therefore

$$\iint_{BCDE} \vec{F} \cdot \hat{n}\, dS = \int_{y=0}^{b} \int_{x=0}^{a} \left(c^2 - xy\right) dx\, dy = c^2 ab - \frac{a^2 b^2}{4}.$$

Over the face $AFGO$, $\hat{n} = -k$, $z = 0$. Therefore

$$\iint_{AFGO} \vec{F} \cdot \hat{n}\, dS = \int_{y=0}^{b} \int_{x=0}^{a} xy\, dx\, dy = \frac{a^2 b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \left(a^2 bc - \frac{c^2 b^2}{4} + \frac{c^2 b^2}{4}\right) + \left(b^2 ca - \frac{a^2 c^2}{4} + \frac{a^2 c^2}{4}\right) + \left(c^2 ab - \frac{a^2 b^2}{4} + \frac{a^2 b^2}{4}\right)$$

$$= abc\,(a + b + c).$$

Hence, the theorem is verified.

Example 6. If $\vec{F} = 4xzi - y^2 j + yzk$ and S is the surface bounded by $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$, evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$.

Sol. By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV, \text{ where } V \text{ is the volume enclosed by the surface } S$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}\left(-y^2\right)+\frac{\partial}{\partial z}(yz)\right] dV = \iiint_V (4z-2y+y)\,dV$$

$$= \iiint_V (4z-y)\,dx\,dy\,dz = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left[2z^2-yz\right]_{z=0}^{1} dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1}(2-y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[2y-\frac{y^2}{2}\right]_{y=0}^{1} dx = \int_0^1\left[2-\frac{1}{2}\right]dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}.$$

Example 7. Evaluate $\int_S \left(y^2z^2i+z^2x^2j+z^2y^2k\right)\cdot\hat{n}\,dS,$ where S is the part of the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane and bounded by this plane.

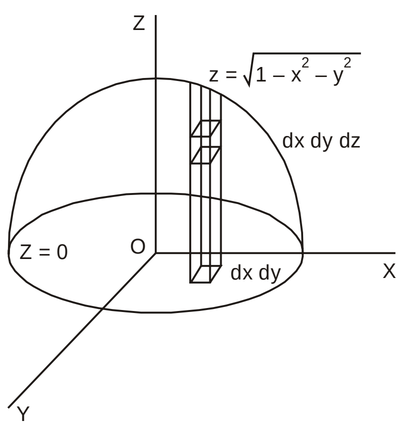
Sol. We have $\vec{F} = y^2z^2\,i + z^2x^2\,j + z^2y^2\,k$

$$\text{div }\vec{F} = \frac{\partial}{\partial x}\left(y^2z^2\right)+\frac{\partial}{\partial y}\left(z^2x^2\right)+\frac{\partial}{\partial z}\left(z^2y^2\right)$$

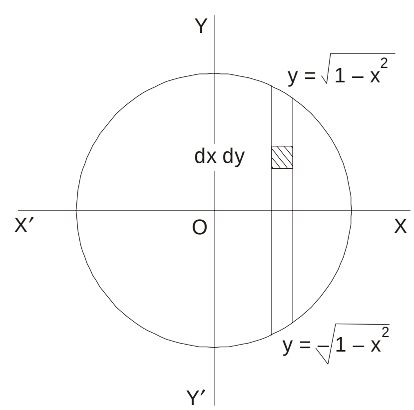
$$= 2zy^2$$

$\therefore$ Given integral $= \iiint_V 2zy^2\,dV$ [By Gauss's divergence theorem]

where V is the volume enclosed by the surface S, *i.e.*, it is the hemisphere $x^2 + y^2 + z^2 = 1$, above the xy-plane.



(*i*)



(*ii*)

Fig. 5.28

From the above Figure 5.28 (*i*) and (*ii*), it is evident that

limits of z are from 0 to $\sqrt{1-x^2-y^2}$

limits of y are from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

and limits of x are from –1 to + 1.

$\therefore$ Given integral

$$= 2\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} zy^2 dx\, dy\, dz$$

$$= 2\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} y^2 dx\, dy$$

$$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(1-x^2-y^2\right) y^2 dx\, dy$$

$$= \int_{x=-1}^{1}\left[\left(1-x^2\right)\frac{y^3}{3}-\frac{y^5}{5}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 2\int_{x=-1}^{1}\left\{\frac{1}{3}\left(1-x^2\right)^{\frac{5}{2}}-\frac{1}{5}\left(1-x^2\right)^{\frac{5}{2}}\right\} dx$$

$$= 4\int_0^1\left\{\frac{\left(1-x^2\right)^{5/2}}{3}-\frac{\left(1-x^2\right)^{5/2}}{5}\right\} dx$$

$$= \frac{8}{15}\int_0^1\left(1-x^2\right)^{\frac{5}{2}} dx$$

$$= \frac{8}{15}\int_0^{\frac{\pi}{2}}\left(1-\sin^2\theta\right)^{\frac{5}{2}}\cos\theta\, d\theta \qquad \text{[Putting } x = \sin\theta]$$

$$= \frac{8}{15}\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$

$$= \frac{8}{15}\cdot\frac{5\pi}{32}=\frac{\pi}{12}\cdot$$

Example 8. Evaluate $\int_S \vec{F}\cdot\hat{n}\, dS$ where $\vec{F} = (x + y^2)\, i - 2x\, j + 2yz\, k$ where S is surface bounded by coordinate planes and plane $2x + y + 2z = 6$.

Sol. We know from Gauss divergence theorem,

$$\int\int_S \vec{F}\cdot\hat{n}\, dS = \int\int\int_V \text{div}\, \vec{F}\, dV$$

$$\vec{F} = (x + y^2)\, i - 2x\, j + 2yz\, k$$

$$\text{div}\ \vec{F} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left\{\left(x+y^2\right)i-2x\, j+2yz\, k\right\}$$

$$= \frac{\partial}{\partial x}\left(x+y^2\right)+\frac{\partial}{\partial y}(-2x)+\frac{\partial}{\partial z}(2yz)$$

$$= 1 + 2y$$

Let

$$I = \int\int\int_S \vec{F}\cdot\hat{n}\, dS = \int\int\int_V \text{div}\, \vec{F}\, dV$$

$$= \int\int\int_V (1 + 2y)\, dV$$

$$= \iiint (1 + 2y)\, dx\, dy\, dz$$

Limit of z is 0 to $\frac{6-2x-y}{2}$

Limit of y is 0 to $6 - 2x$

Limit of x is 0 to 3

Hence,
$$I = \iiint (1+2y)\, dx\, dy\, dz$$
$$= \iint (1+2y)\, \{z\}_0^{(6-2x-y)/2}\, dx\, dy$$
$$= \frac{1}{2}\iint (1+2y)(6-2x-y)\, dx\, dy$$
$$= \frac{1}{2}\iint \left\{6-2x+11y-4xy-2y^2\right\} dx\, dy$$
$$= \frac{1}{2}\int \left\{6y-2xy+\frac{11}{2}y^2-2xy^2-\frac{2}{3}y^3\right\}_0^{6-2x} dx$$
$$= \frac{1}{2}\int \left[6(6-2x)-2x(6-2x)+\frac{11}{2}(6-2x)^2-2x(6-2x)^2-\frac{2}{3}(6-2x)^3\right] dx$$
$$= \frac{1}{2}\int \left\{-\frac{8}{3}x^3+26x^2-84x+90\right\} dx$$
$$= \frac{1}{2}\left[-\frac{2}{3}x^4+\frac{26}{3}x^3-42x^2+90x\right]_0^3$$
$$= \frac{1}{2}\left[-54+234-378+270\right]$$
$$= \frac{1}{2}[72] = 36\,.$$

Example 9. Verify Gauss divergence theorem for
$$\iint_S \left\{\left(x^3-yz\right) dy\, dz - 2x^2y\, dz\, dx + z\, dx\, dy\right\}$$
over the surface of cube bounded by coordinate planes and the planes $x = y = z = a$

Sol. Let $\vec{F} = F_1 i + F_2 j + F_3 k.$

From Gauss divergence theorem, we know
$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \left[F_1 dy\, dz + F_2 dz\, dx + F_3 dx\, dy\right] = \iiint_V \text{div}\, \vec{F}\, dV \quad ...(i)$$

Here, $F_1 = x^3 - yz,\ F_2 = -2x^2y,\ F_3 = z$

So, $\vec{F} = (x^3 - yz)\, i - 2x^2y\, j + z\, k$
$$\text{div}\ \vec{F} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left\{\left(x^3-yz\right)i-2x^2y\, j+z\, k\right\}$$
$$= \frac{\partial}{\partial x}\left(x^3-yz\right)+\frac{\partial}{\partial y}\left(-2x^2y\right)+\frac{\partial}{\partial z}(z)$$
$$= 3x^2 - 2x^2 + 1 = x^2 + 1$$

Hence,
$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \left(x^2+1\right) dV$$

$$= \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a (x^2+1)\,\{z\}_0^a\,dx\,dy$$

$$= a\int_0^a\int_0^a (x^2+1)\,dx\,dy$$

$$= a\int_0^a (x^2+1)\,\{y\}_0^a\,dx$$

$$= a^2\int_0^a (x^2+1)\,dx$$

$$= a^2\left\{\frac{a^3}{3}+x\right\}_0^a$$

$$= a^2\left\{\frac{a^3}{3}+a\right\} = \frac{a^5}{3}+a^3 \qquad ...(ii)$$

Verification by direct integral: Outward drawn unit vector normal to face *OEFG* is $-i$ and dS is $dy\ dz$.

If I_1 is integral along this face,

$$I_1 = \int_S \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot(-i)\,dy\,dz$$

$$= \iint_S (x^3 - yz)\,dy\,dz \qquad \text{[As } x = 0 \text{ for this face]}$$

$$= \int_0^a\int_0^a yz\,dy\,dz$$

$$= \int_0^a y\left\{\frac{z^2}{2}\right\}_0^a dy$$

$$= \frac{a^2}{2}\int_0^a y\,dy = \frac{a^2}{2}\left[\frac{y^2}{2}\right]_0^a = \frac{a^4}{4}$$

For face *ABCD*, its equation is $x = a$ and $\hat{n}\,dS = \hat{i}\,dy\,dz$,

If I_2 is integral along this face

$$I_2 = \iint_S \vec{F}\cdot i\,dy\,dz$$

$$= \iint_S (x^3 - yz)\,dy\,dz$$

$$= \int_0^a\int_0^a (a^3 - yz)\,dy\,dz$$

$$= \int_0^a \left\{a^3 z - y\frac{z^2}{z}\right\}_0^a dy$$

$$= \int_0^a \left\{a^3 a - y\frac{a^2}{2}\right\} dy$$

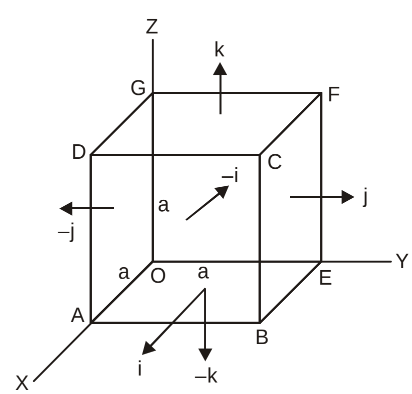


Fig. 5.29

$$= \left[a^4 y - \frac{a^2}{2}\frac{y^2}{2} \right]_0^a$$

$$= a^5 - \frac{a^4}{4}$$

If I_3 is integral along face *OGDA* whose equation is

$$y = 0$$

$$\hat{n}\, dS = -\, j\, dxdz$$

Hence,
$$I_3 = \iint_S \vec{F} \cdot (-j)\, dx\, dz$$

$$= -\iint_S -2x^2 y\, dx\, dz$$

$$= 0, \text{ as } y = 0.$$

If I_4 is integral along face *BEFC* whose equation is

$$y = a$$

$$\hat{n}\, dS = j\, dx\, dz$$

Then
$$I_4 = -\iint_S 2x^2 y\, dx\, dz$$

$$= -2a \int_0^a \int_0^a x^2\, dx\, dz$$

$$= -2a \int_0^a x^2 \, \{z\}_0^a\, dx$$

$$= -2a^2 \int_0^a x^2\, dx$$

$$= -2a^2 \left[\frac{x^3}{3} \right]_0^a = -\frac{2}{3} a^5 .$$

If I_5 is integral along face *OABE* whose equation is

$$z = 0$$

$$\hat{n}\, dS = -\, k\, dx\, dy$$

$$I_5 = \iint_S \vec{F} \cdot (-\, k\, dx\, dy)$$

$$= -\iint_S z\, dx\, dy = 0 \; as\; z = 0.$$

If I_6 is integral along face *CFGD* whose equation is

$$z = a$$

$$\hat{n}\, dS = k\, dx\, dy$$

$$I_6 = \iint_S z\, dx\, dy = \int_0^a \int_0^a a\, dx\, dy$$

$$= a \int_0^a [y]_0^a\, dx = a^2 \int_0^a dx = a^3$$

Total surface
$$I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

$$= \frac{a^4}{4} + a^5 - \frac{a^4}{4} + 0 - \frac{2}{3}a^5 + 0 + a^3$$

$$= \frac{a^5}{3} + a^3 \qquad ...(iii)$$

which is equal to volume integral. Hence Gauss theorem is verified.

Example 10. Evaluate by Gauss divergence theorem

$$\iint_S \left\{xz^2\, dy\, dz + \left(x^2 y - z^3\right) dz\, dx + \left(2xy + y^2 z\right) dx\, dy\right\}$$

where S is surface bounded by $z = 0$ and $z = \sqrt{a^2 - x^2 - y^2}$.

Sol. Let $\vec{F} = F_1 i + F_2 j + F_3 k$.

Cartesian form of Gauss divergence theorem is

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \left[F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy\right] = \iiint_V \text{div}\, \vec{F}\, dV$$

Here, $F_1 = xz^2;\ F_2 = x^2y - z^3,\ F_3 = 2xy + y^2z.$

Hence, $\vec{F} = xz^2\, i + (x^2y - z^3)\, j + (2xy - y^2z)\, k$

$$\text{div}\ \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \{xz^2\, i + (x^2y - z^3)\, j + (2xy + y^2z)\, k\}$$

$$= \frac{\partial}{\partial x}\left(xz^2\right) + \frac{\partial}{\partial y}\left(x^2 y - z^3\right) + \frac{\partial}{\partial z}\left(2xy + y^2 z\right)$$

$$= z^2 + x^2 + y^2.$$

Let

$$I = \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \text{div}\, \vec{F}\, dV$$

$$= \iiint \left(x^2 + y^2 + z^2\right) dx\, dy\, dz$$

Limit of z is 0 to $\sqrt{a^2 - x^2 - y^2}$

Limit of y is $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$

Limit of x is $-a$ to a

$$I = \iiint \left(x^2 + y^2 + z^2\right) dx\, dy\, dz$$

$$= \iint \left[\left(x^2 + y^2\right) z + \frac{z^3}{3}\right]_0^{\sqrt{a^2 - x^2 - y^2}} dx\, dy$$

$$= \iint \left[\left(x^2 + y^2\right)\sqrt{a^2 - x^2 - y^2} + \frac{\left(a^2 - x^2 - y^2\right)^{\frac{3}{2}}}{3}\right] dx\, dy$$

$$= \iint \sqrt{a^2 - x^2 - y^2} \left\{x^2 + y^2 + \frac{a^2 - x^2 - y^2}{3}\right\} dx\, dy$$

$$= \frac{1}{3}\iint \sqrt{a^2-x^2-y^2}\,\left\{3x^2+3y^2+a^2-x^2-y^2\right\}dx\,dy$$

$$= \frac{1}{3}\iint \sqrt{a^2-x^2-y^2}\,\left\{2x^2+2y^2+a^2\right\}dx\,dy$$

$$= \frac{1}{3}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left\{\left(2x^2+a^2\right)\sqrt{a^2-x^2-y^2}+2y^2\sqrt{a^2-x^2-y^2}\right\}dx\,dy$$

$$= \frac{2}{3}\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left\{\left(2x^2+a^2\right)\sqrt{a^2-x^2-y^2}+2y^2\sqrt{a^2-x^2-y^2}\right\}dx\,dy$$

Let $y = \sqrt{a^2-x^2}\,\sin\theta$

$dy = \sqrt{a^2-x^2}\,\cos\theta\,d\theta$

$$I = \frac{2}{3}\int_{-a}^{a}\int_{0}^{\frac{\pi}{2}}\left[\left(2x^2+a^2\right)\left(a^2-x^2\right)\cos^2\theta+2\left(a^2-x^2\right)^2\sin^2\theta\cos^2\theta\right]dx\,d\theta$$

$$= \frac{2}{3}\int_{-a}^{a}\left[\left(2x^2+a^2\right)\left(a^2-x^2\right)\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}+2\left(a^2-x^2\right)^2\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(3)}\right]dx$$

$$= \frac{2}{3}\int_{-a}^{a}\left[\left(2x^2+a^2\right)\left(a^2-x^2\right)\frac{\pi}{4}+2\left(a^2-x^2\right)^2\frac{\pi}{16}\right]dx$$

$$= \frac{4}{3}\times\frac{\pi}{8}\int_{0}^{a}\left[2\left(2x^2+a^2\right)\left(a^2-x^2\right)+\left(a^2-x^2\right)^2\right]dx$$

$$= \frac{\pi}{3\times 2}\int_{0}^{a}\left[2\left(2x^2a^2-2x^4+a^4-a^2x^2\right)+a^4+x^4-2^2x^2\right]dx$$

$$= \frac{\pi}{6}\int_{0}^{a}\left(3a^4-3x^4\right)dx$$

$$= \frac{\pi}{2}\int_{0}^{a}\left(a^4-x^4\right)dx$$

$$= \frac{\pi}{2}\left[a^4x-\frac{x^5}{5}\right]_0^a$$

$$= \frac{\pi}{2}\times\frac{4}{5}a^5 = \frac{2\pi}{5}a^5.$$

Example 11. Using the divergence theorem, evaluate the surface integral $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dy\,dx)$, where $S : x^2 + y^2 + z^2 = 4.$ (*U.P.T.U., 2008*)

Sol. Let $\vec{F} = F_1 i + F_2 j + F_3 k$

From Gauss divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \left[F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy\right] = \iiint_V div\,\vec{F}\,dV \quad ...(i)$$

Comparing L.H.S. of (*i*) with given integral, we get

$$F_1 = yz, \quad F_2 = zx, \quad F_3 = xy$$

So
$$\vec{F} = F_1 i + F_2 j + F_3 k \Rightarrow \vec{F} = (yz)i + (zx)j + (xy)k$$

$$\text{div } \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\{(yz)i + (zx)j + (xy)k\}$$

$$= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$$

Thus

$$\iint_S (yz\, dy\, dz + zx\, dz\, dx + xy\, dy\, dx) = \iiint_V 0.dV = 0.$$

EXERCISE 5.7

1. Use divergence theorem to evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = x^3 i + y^3 j + z^3 k$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. $\left[\textbf{Ans. } \frac{12\pi a^5}{5}\right]$

2. Use divergence theorem to show that $\iint_S \nabla\left(x^2 + y^2 + z^2\right) dS = 6V$ Where S is any closed surface enclosing volume V.

3. Apply divergence theorem to evaluate $\iint_S \overline{F}\hat{n}\, dS$, where $\overline{F} = 4x^3\hat{i} - x^2y\hat{j} + x^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = a^2$ bounded by the planes $z = 0$ and $z = b$. $[\textbf{Ans. } 3ba^4\pi]$

4. Use the divergence theorem to evaluate $\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$, where S is the portion of the plane $x + 2y + 3z = 6$ which lies in the first octant. (*U.P.T.U., 2003*) $[\textbf{Ans. } 18]$

5. The vector field $\vec{F} = x^2 i + zj + yzk$ is defined over the volume of the cuboid given by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ enclosing the surface S. Evaluate the surface integral $\iint_S \vec{F}\cdot d\vec{S}$. (*U.P.T.U., 2001*) $\left[\textbf{Ans. } abc\left(a + \frac{b}{2}\right)\right]$

6. Evaluate $\iint_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right) dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant. (*U.P.T.U., 2004*) $[\textbf{Ans. } 0]$

7. Evaluate $\iint_S \left(e^x dy\, dz - ye^x dz\, dx + 3z\, dx\, dy\right)$, where S is the surface of the cylinder $x^2 + y^2 = c^2$, $0 \le z \le h$. $[\textbf{Ans. } 3\pi\, hc^2]$

8. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 2xyi + yz^2j + xzk$, and S is the surface of the region bounded by $x = 0$, $y = 0$, $z = 0$, $y = 3$ and $x + 2z = 6$. $\left[\textbf{Ans. } \frac{351}{2}\right]$

9. $\vec{F} = 4xi - 2y^2j + z^2k$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.
[**Ans.** Common value 8π]

10. $\vec{F} = \left(x^3 - yz\right)i - 2x^2yj + zk$ taken over the entire surface of the cube $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$. $\left[\textbf{Ans. } \text{Common value } \frac{a^5}{3} + a^3\right]$

11. $\vec{F} = 2xyi + yz^2j + xzk$ and S is the total surface of the rectangular parallelopiped bounded by the coordinate planes and $x = 1$, $y = 2$, $z = 3$. [**Ans.** Common value 33]

12. $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ taken over the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

[**Ans.** Common value 0]

13. $\vec{F} = xi + yj$ taken over the upper half on the unit sphere

$$x^2 + y^2 + z^2 = 1.$$

$\left[\textbf{Ans. } \text{Common value } \frac{4\pi}{3}\right]$

14. Prove that $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\vec{r}.\hat{n}}{r^2} ds$.

15. Evaluate $\iint_S \vec{r}.\hat{n}\,ds$, where S : surface of cube bounded by the planes $x = -1$, $y = -1$, $z = -1$, $x = 1$, $y = 1$, $z = 1$. [**Ans.** 24]

OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer of the choices given below:

1. If $\vec{r} = xi + yj + zk$ is position vector, then value of $\nabla(\log r)$ is

(*i*) $\frac{\vec{r}}{r}$ (*ii*) $\frac{\vec{r}}{r^2}$

(*iii*) $-\frac{\vec{r}}{r^3}$ (*iv*) None of these [*U.P.T.U., 2008*]

2. The unit vector normal to the surface $x^2y + 2xz = 4$ at $(2, -2, 3)$ is

(*i*) $\frac{1}{3}(i - 2j + 2k)$ (*ii*) $\frac{1}{3}(i - 2j - 2k)$

(*iii*) $\frac{1}{3}(i + 2j - 2k)$ (*iv*) None of these

3. If $\vec{r}$ is a position vector then the value of ∇r^n is

(*i*) $nr^{n-2}\,\vec{r}$ (*ii*) nr^{n-2}

(*iii*) nr^2 (*iv*) nr^{n-3}

4. If $f(x, y, z) = 3x^2y - y^3z^2$, then $|\nabla f|$ at $(1, -2, -1)$ is

(*i*) 481 (*ii*) $\sqrt{381}$

(*iii*) $\sqrt{581}$ (*iv*) $\sqrt{481}$

5. If $\vec{a}$ is a constant vector, then grad $\left(\vec{r} \cdot \vec{a}\right)$ is equal to

(*i*) $\vec{r}$ (*ii*) $-\vec{a}$

(*iii*) 0 (*iv*) $\vec{a}$

6. The vector $r^n\,\vec{r}$ is solenoidal if n equals

(*i*) 3 (*ii*) -3

(*iii*) 2 (*iv*) 0

7. If $\vec{r}$ is a position vector then div $\vec{r}$ is equal to

(*i*) 3 (*ii*) 0

(*iii*) 5 (*iv*) -1

8. If $\vec{r}$ is a position vector then curl $\vec{r}$ is equal to

(*i*) -5 (*ii*) 0

(*iii*) 3 (*iv*) -1

9. If $\vec{F} = \nabla\phi$, $\nabla^2\phi = -4\pi\rho$ where P is a constant, then the value of $\iint_S \vec{F}.\hat{n}\, dS$ is:

(*i*) 4π (*ii*) $-4\pi\rho$

(*iii*) $-4\pi\rho V$ (*iv*) V

10. If $\hat{n}$ is the unit outward drawn normal to any closed surface S, the value of $\iiint_V \text{div}\, F\, dV$ is

(*i*) V (*ii*) S

(*iii*) 0 (*iv*) 2S

11. If S is any closed surface enclosing a volume V and $\vec{F} = xi + 2yj + 3zk$ then the value of the integral $\iint_S \vec{F}.\hat{n}\, dS$ is

(*i*) $3V$ (*ii*) $6V$

(*iii*) $2V$ (*iv*) $6S$

12. The integral $\iint_S r^5\, \hat{n}\, dS$ is equal

(*i*) 0 (*ii*) $\iiint_V 5r^3.\vec{r}\, dV$

(*iii*) $\iiint_V 5r^{-3}.\vec{r}\, dV$ (*iv*) None of these

13. A vector $\vec{F}$ is always normal to a given closed surface S in closing V the value of the integral $\iiint_V curl\,\vec{F}\, dV$ is :

(*i*) 0 (*ii*) 0

(*iii*) V (*iv*) S

B. Fill in the blanks:

1. If $\vec{f} = (bxy - z^3)i + (b-2)x^2 j + (1-b)xz^2 k$ has its curl identically equal to zero then $b =$ ....

2. $\nabla^2\left(\frac{1}{r}\right) =$ ....

3. div grad $f =$ ....

4. curl grad $f =$ ....

5. grad $r =$ ....

6. grad $\frac{1}{r} =$ ....

7. If $\vec{A} = 3x\, yz^2\, i + 2x\, y^3\, j - x^2\, yz\, k$ and $f = 3x^2 - yz$ then $\vec{A}.\nabla f =$ ....

8. If $|\vec{r}| = r$, then $\nabla f(r) \times \vec{r} =$ ....

9. If $|\vec{r}| = r$, then $\frac{\nabla f(r)}{\nabla r} =$ ....

10. The directional derivative of $\phi = xy + yz + zx$ in the direction of the vector $i + 2j + k$ at (1, 2, 0) is = ...

11. The value of $\int (xdy - ydx)$ around the circle $x^2 + y^2 = 1$ is ....

12. If $\nabla^2\phi = 0$, $\nabla^2\psi = 0$, then $\iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS =$ ....

13. $\iint \vec{F}.\hat{n}\, dS$ is called the .... $d\vec{F}$ over S.

14. If S is a closed surface, then $\iint_S \vec{r}.\hat{n}\, dS =$ ....

15. $\iiint_V \nabla.\vec{F}\, dV = \iint_S \vec{A}\, dS$, then A is equal to ....

C. Indicate True or False for the following statements:

1. (*i*) If $\vec{v}$ is a solenoidal vector then div $\vec{v} = 0$.

(*ii*) div $\vec{v}$ represents the rate of loss of fluid per unit volume.

(*iii*) If $\vec{f}$ is irrotational then curl $\vec{f} \neq 0$.

(*iv*) The gradient of scalar field $f(x, y, z)$ at any point P represents vector normal to the surface f = const.

2. (*i*) The gradient of a scalar is a scalar.

(*ii*) Curl of a vector is a scalar.

(*iii*) Divergence of a vector is a scalar.

(*iv*) ∇f is a vector along the tangent to the surface $f = 0$.

3. (*i*) The directional derivative of f along $\hat{a}$ is $f \cdot \hat{a}$.

(*ii*) The divergence of a constant vector is zero vector.

(*iii*) The family of surfaces $f(x, y, z) = c$ are called level surfaces.

(*iv*) If $\vec{a}$ and $\vec{b}$ are irrotational then div $\left(\vec{a} \times \vec{b}\right) = 0$.

4. (*i*) Any integral which is evaluated along a curve is called surface integral.

(*ii*) Green's theorem in a plane is a special case of stoke theorem.

(*iii*) If the surface S has a unique normal at each of its points and the direction of this normal depends continuously on the points of S, then the surface is called smooth surface.

(*iv*) The integral $\int_S \vec{F} \cdot d\vec{r}$ is called circulation.

5. (*i*) The formula $\int\int_S \left(curl\, \vec{F}\right) \cdot \hat{n}\, ds = \int_C \vec{F} \cdot d\vec{r}$ is governed by Stoke's theorem.

(*ii*) If the initial and terminal points of a curve coincide, the curve is called closed curve.

(*iii*) If $\hat{n}$ is the unit outward drawn normal to any closed surface S, then $\int\int\int_V \nabla \cdot \hat{n}\, dv \neq S$.

(*iv*) The integral $\frac{1}{2}\oint_C (xdy - ydx)$ represents the area.

D. Match the Following:

1. (*i*) $\nabla \cdot \left(\nabla \times \vec{a}\right)$ (*a*) $d\phi$

(*ii*) curl (ϕ grad ϕ) (*b*) 0

(*iii*) div $\left(\vec{a} \times \vec{r}\right)$ (*c*) 0

(*iv*) $\nabla\phi \cdot d\vec{r}$ (*d*) $\vec{r}$ curl $\vec{a}$

2. (*i*) $\nabla^2 r^2$ (*a*) $\vec{a} \cdot (\nabla f)$

(*ii*) df/ds (*b*) grad $f \pm$ grad g

(*iii*) $(\vec{a}\,.\,\nabla)f$	(*c*) $\nabla f\,.\left(\dfrac{\vec{a}}{\lvert a\rvert}\right)$
(*iv*) $\nabla(f \pm g)$	(*d*) 6
3. (*i*) grad of ϕ along $\hat{n}$	(*a*) a (acceleration)
(*ii*) $\dfrac{d^2\vec{r}}{dt^2}$	(*b*) $\dfrac{\partial\phi}{\partial n}\hat{n}$
(*iii*) curl $\vec{v}$	(*c*) curl $\vec{f} = 0$
(*iv*) irrotational	(*d*) $2\vec{\omega}$

ANSWERS TO OBJECTIVE TYPE QUESTIONS

A. Pick the correct answer:

1. (*ii*)	**2.** (*ii*)	**3.** (*i*)	**4.** (*iv*)
5. (*iv*)	**6.** (*ii*)	**7.** (*i*)	**8.** (*ii*)
9. (*iii*)	**10.** (*ii*)	**11.** (*ii*)	**12.** (*ii*)
13. (*i*)			

B. Fill in the blanks:

1. 4	**2.** 0	**3.** $\nabla^2 f$	**4.** 0
5. $\dfrac{\vec{r}}{r}$	**6.** $\dfrac{\vec{r}}{r^3}$	**7.** –15	**8.** 0
9. $f'(r)$	**10.** $\dfrac{10}{3}$	**11.** 2π	**12.** 0
13. flux	**14.** $3V$	**15.** $\vec{F}\,.\,\hat{n}$	

C. True or False:

1. (*i*) *T*	(*ii*) *T*	(*iii*) *F*	(*iv*) *F*
2. (*i*) *F*	(*ii*) *T*	(*iii*) *T*	(*iv*) *F*
3. (*i*) *F*	(*ii*) *F*	(*iii*) *T*	(*iv*) *T*
4. (*i*) *F*	(*ii*) *F*	(*iii*) *T*	(*iv*) *T*
5. (*i*) *T*	(*ii*) *T*	(*iii*) *F*	(*iv*) *T*

C. Match the following:

1. (*i*) $\rightarrow$ (*b*)	(*ii*) $\rightarrow$ (*c*)	(*iii*) $\rightarrow$ (*d*)	(*iv*) $\rightarrow$ (*a*)
2. (*i*) $\rightarrow$ (*d*)	(*ii*) $\rightarrow$ (*c*)	(*iii*) $\rightarrow$ (*a*)	(*iv*) $\rightarrow$ (*b*)
3. (*i*) $\rightarrow$ (*b*)	(*ii*) $\rightarrow$ (*a*)	(*iii*) $\rightarrow$ (*d*)	(*iv*) $\rightarrow$ (*c*)

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UNSOLVED QUESTION PAPERS (2004–2009)

B.Tech.

(Only for the Candidates Admitted/Readmitted in the Session 2008–09)

(SEM. I) Examination, 2008-09

MATHEMATICS–I

Time: 3 Hours] [*Total Marks: 100*

SECTION A

All parts of this question are **compulsory.** **2 × 10 = 20**

1. (*a*) For which value of '*b*' the rank of the martix.

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix} \text{ is 2, } b = \xrightarrow{\text{4 cm}}$$

(*b*) Determine the constants *a* and *b* such that the curl of vector $\overline{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} + (3xy + byz)\hat{k}$ is zero, *a* = ..., *b* = ...

(*c*) The n^{th} derivative (y_n) of the function $y = x^2 \sin x$ at $x = 0$ is ....

(*d*) With usual notations, match the items on right hand side with those on left hand side for properties of maximum and minimum:

(*i*) Maximum	(*p*) $rt - s^2 = 0$
(*ii*) Minimum	(*q*) $rt - s^2 < 0$
(*iii*) Saddle point	(*r*) $rt - s^2 > 0$, $r > 0$
(*iv*) Failure case	(*s*) $rt - s^2 > 0$ and $r < 0$

(*e*) Match the items on the right hand side with those on left hand side for the following special functions : (Full marks is awarded if all matchings are correct).

(*i*) $\beta(p, q)$	(*p*) $\overline{\lvert}(1/2)$
(*ii*) $\dfrac{\overline{\lvert}p\,\overline{\lvert}q}{\overline{\lvert}p+q}$	(*q*) $\displaystyle\int_0^\infty \frac{y^{p-1}}{(1+y)(p+q)}\,dy$
(*iii*) $\sqrt{\pi}$	(*r*) $\beta(p, q)$
(*iv*) $\dfrac{\pi}{\sin p\pi}$	(*s*) $\overline{\lvert}p\,\overline{\lvert}1-p$

Indicate True or False for the following statements:

(*f*) (*i*) If $|A| = 0$, then at least one eigen value is zero. (True/False)

(*ii*) A^{-1} exists iff 0 is an eigen value of *A*. (True/False) (*iii*) If $|A| \neq 0$, then A is known as singular matrix. (True/False)

(*iv*) Two vectors X and Y is said to be orthogonal Y, $X^T Y = Y^T X \neq 0$. (True/False)

(*g*) (*i*) The curve $y^2 = 4ax$ is symmetric about x-axis. (True/False)

(*ii*) The curve $x^3 + y^3 = 3axy$ is symmetric about the line $y = -x$. (True/False)

(*iii*) The curve $x^2 + y^2 = a^2$ is symmetric about both the axis x and y. (True/False)

(*iv*) The curve $x^3 - y^3 = 3axy$ is symmetric about the line $y = x$. (True/False)

Pick the correct answer of the choices given below:

(*h*) If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector, then value of ∇ (log r) is :

(*i*) $\frac{\bar{r}.}{r}$ (*ii*) $\frac{\bar{r}.}{r^2}$

(*iii*) $-\frac{\bar{r}.}{r^3}$ (*iv*) None of these

(*i*) The Jacobian $\frac{\partial(uv)}{\partial(xy)}$ for the function $u = e^x \sin y$, $v = (x + \log \sin y)$ is

(*i*) 1 (*ii*) $\sin x \sin y - xy \cos x \cos y$

(*iii*) 0 (*iv*) $\frac{e^x}{x}$.

(*j*) The volume of the solid under the surface $az = x^2 + y^2$ and whose base R is the circle $x^2 + y^2 = a^2$ is given as

(*i*) $\pi | 2a$ (*ii*) $\pi a^3 | 2$

(*iii*) $\frac{4}{3}\pi a^3$ (*iv*) none of these.

SECTION B

Attempt any **three** parts of the following: **10 × 3 = 30**

2. (*a*) If $y = (\sin^{-1} x)^2$ prove that $y_n(0) = 0$ for n odd and $y_n(0) = 2, 2^2, 4^2, 6^2 \ldots (n-2)^2$, $n \neq 2$ for n is even.

(*b*) Find the dimension of rectangular box of maximum capacity whose surface area is given when (*a*) box is open at the top (*b*) box is closed.

(*c*) Find a matrix P which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, verify $P^{-1}. AP = D$ where D is the diagonal matrix.

(*d*) Find the area and the mass contained m the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1$ where $\alpha > 0, \beta > 0$ given that density at any point $p(xy)$ is $k\sqrt{xy}$.

(*e*) Using the divergence theorem, evaluate the surface integral $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dy\,dx)$ where $S : x^2 + y^2 + z^2 = 4$.

SECTION C

Attempt any **two** parts from each question. All questions are **compulsory.**

3. (*a*) Trace the curve $r^2 = a^2 \cos 2\theta$

(*b*) If $u = \log\left(\frac{(x^2+y^2)}{(x+y)}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.$$

(*c*) If $V = f(2x - 3y, 3y - 4z, 4z - 2x)$, compute the value of $6V_x + 4V_y + 3V_z$.

4. (*a*) The temperature '*T*' at any point (xyz) in space is $T(xyz) = K\,xyz^2$ where K is constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

(*b*) Verify the chain rule for Jacobians if $x = u$, $y = u \tan v$, $z = w$.

(*c*) The time '*T*' of a complete oscillation of a simple pendulum of length '*L*' is governed by the equation $T = 2\pi\sqrt{\frac{\cdot L}{g}}$, g is constant, find the approximate error in the calculated value of T corresponding to an error of 2% in the value of L.

5. (*a*) Determine '*b*' such that the system of homogeneous equation $2x + y + 2z = 0$; $x + y + 3z = 0$; $4x + 3y + bz = 0$ has (*i*) Trivial solution, (*ii*) Non-trivial solution. Find the Non-trivial solution using matrix method.

(*b*) Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ and hence find A^{-1}.

(*c*) Find the eigen value and corresponding eigen vectors of the matrix.

$$I = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}.$$

6. (*a*) Find the directional derivative of $\nabla(\nabla f)$ at the point (1, –2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $f = 2x^3\,y^2z^4$.

(*b*) Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

$$y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}.$$

(*c*) Prove that $(y^2 - z^2 + 3yz)\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\,\hat{k}$ is both solenoidal and irrotational.

7. (*a*) Changing the order of integration of

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy$$

Show that $\int_0^\infty \left(\frac{\sin nx}{x}\right) dx = \frac{\pi}{2}$.

(*b*) Determine the area bounded by the curves $xy = 2$, $4y = x^2$ and $y = 4$.

(*c*) For a β function, show that

$$\beta(p, q) = \beta(p + 1, q) + \beta(p, q + 1)$$

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B.Tech., First Semester Examination, 2007-08

MATHEMATICS–I

Time: 3 Hours] [*Total Marks: 100*

Note: *Attempt* ***all*** *the problems. Internal choices are mentioned in every problem.*

1. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) Define the eigen values, eigen vectors and the characteristic equation of a square matrix. Find the characteristic equation/polynomial, eigen values and eigen vectors of the matrix:

$$\begin{bmatrix} 2 & 5 & 7 \\ 5 & 3 & 1 \\ 7 & 0 & 2 \end{bmatrix}$$

(*b*) Check the consistency of the following system of linear non-homogeneous equations and find the solution, if exists:

$$\begin{aligned} 7x_1 + 2x_2 + 3x_3 &= 16 \\ 2x_1 + 11x_2 + 5x_3 &= 25 \\ x_1 + 3x_2 + 4x_3 &= 13 \end{aligned}$$

(*c*) Find the inverse of the matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{11} \end{bmatrix}$$

2. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) State Leibnitz theorem for nth differential coefficient of the product of two functions.

If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$(x^2 - 1)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$

(*b*) Verify that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$, where $u(x, y) = \log_e\left(\dfrac{x^2 + y^2}{xy}\right)$

(*c*) If $u = x\sin^{-1}\left(\dfrac{x}{y}\right) + y\sin^{-1}\left(\dfrac{y}{x}\right)$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

3. Attempt any **four** parts of the following: **(5 × 4 = 20)**

(*a*) Expand $e^x \cos y$ about the point $\left(1, \dfrac{\pi}{4}\right)$.

(*b*) Calculate the Jacobian $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ of the following:

$$u = x + 2y + z$$
$$v = x + 2y + 3z$$
$$w = 2x + 3y + 5z$$

(*c*) Discuss the maxima and minima of the function:

$$f(x, y) = \cos x \cos y \cos(x + y)$$

(*d*) Find a point on the ellipse $4x^2 + y^2 = 4$ nearest to the point (1, 2).

(*e*) Find the extreme value $x^2 + y^2 + z^2$ subject to the condition

$$xy + yz + zx = p.$$

(*f*) If $f(x, y) = x^2 y^{\frac{1}{10}}$, compute the value of f when $x = 1.99$ and $y = 3.01$.

4. Attempt any **four** of the following: **(5 × 4 = 20)**

(*a*) Evaluate the following by changing into polar coordinates: $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\, dx\, dx$

(*b*) Find the area enclosed between the parabola $y = 4x - x^2$ and the line $y = x$.

(*c*) Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x, y)\, dx\, dy$

(*d*) Find the volume of the solid surrounded by the surface $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} + \left(\dfrac{z}{c}\right)^{\frac{2}{3}} = 1.$

(*e*) Define Gamma and Beta functions. Prove that $B(l, m)\, B(l + m, n)\, B(l + m + n, p)$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n\, \Gamma p}{\Gamma (l+m+n+p)}$$

(*f*) Show that $\int_0^1 x^5\left(1-x^3\right)^{10} dx = \dfrac{1}{396}$

5. Attempt any **two** of the following: **(10 × 2 = 20)**

(*a*) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that

(*i*) $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector.

(*ii*) grad $r = \dfrac{\vec{r}}{r}$

(*iii*) grad $\dfrac{1}{r} = -\dfrac{\vec{r}}{r^3}$, where $\vec{a}$ is a constant vector.

(*iv*) grad $r^n = nr^{n-2}\,\vec{r}$

when $r = |\vec{r}|$

(*b*) Prove that $\vec{a}\times(\nabla\times\vec{r}) = \nabla(\vec{a}\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r}$, where $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

(*c*) State the Green's theorem. Verify it by evaluating $\int_c \left[(x^3 - xy^3)\, dx + (y^2 - 2xy)\, dy\right]$ where C is the square having the vertices at the points (0, 0), (2, 0), (2, 2) and (0, 2).

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B.Tech., First Semester Examination, 2006-07

MATHEMATICS–I

Time: 3 Hours] [*Total Marks: 100*

Note: *(i) Attempt All questions.*
(ii) All questions carry equal marks.
(iii) In case of numerical problems assume data wherever not provided.
(iv) Be precise in your answer.

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1. Attempt any **four** parts of the following: **(5 × 4 = 20)**

(*a*) If $y = x \log (1 + x)$, prove that $y_n = \left[(-1)^{n-2}\underline{\lfloor n-2}(x+n)\right]/(1+x)^n$

(*b*) If $x = \tan y$, prove that $\left(1+x^2\right)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$

(*c*) If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$ prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z} = 2.$

(*d*) State and prove Euler's theorem for partial differentiation of a homogeneous function $f(x, y)$.

(*e*) If $u(x, y) = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

(*f*) Trace the curve $y^2(a - x) = x^3$, $a > 0$

2. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$

show that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{y^2-x^2}{2uv(u-v)}$

(*b*) Find Taylor series expansion of function on $f(x, y) = e^{-x^2-y^2}\cos xy$ about the point $x_0 = 0$, $y_0 = 0$ up to three terms.

(*c*) Find the minimum distance from the point (1, 2, 0) to the cone $z^2 = x^2 + y^2$.

3. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) Define the gradient, divergence and curl.

(*i*) If $f(x, y, z) = 3x^2y - y^3z^2$, find grad f at the point (1, – 2, – 1).

(*ii*) If $\vec{F}(x, y, z) = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find divergence and curl of $\vec{F}(x,y,z)$.

(*b*) State Gauss divergence theorem. Verify this theorem by evaluating the surface integral as a triple integral

$\int_S\int\left(x^3dy\,dz + x^2ydz\,dx + x^2z\,dx\,dy\right)$, where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$, $(0 \le z \le b)$ and the circular discs $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$.

(*c*) State the Stoke's theorem. Verify this theorem for $\vec{F}(x, y, z) = xz\hat{i} - y\hat{j} + x^2y\hat{k}$, where the surface S is the surface of the region bounded by $x = 0$, $y = 0$, $z = 0$, $2x + y + 2z = 8$ which is not included on xz-plane.

4. Attempt any **four** parts of the following: **(5 × 4 = 20)**

(*a*) Find the rank of matrix

$$\begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}$$

(*b*) Solve the system of equations

$$\begin{aligned} 2x + 3x_2 + x_3 &= 9 \\ x + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 + 2x_3 &= 8 \end{aligned}$$

by Gaussian elimination method.

(*c*) Find the value of λ for which the vectors (1, –2, λ), (2, –1, 5) and (3, –5, 7λ) are linearly dependent.

(*d*) Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$. Also, find the eigen values and eigen vectors of this matrix.

(*e*) Verify the Cayley-Hamilton theorem for the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$. Also, find its inverse using this theorem.

(*f*) Diagonalize the matrix $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Note: *Following question no. 5 is for New Syllabus only (TAS–104/MA–101 (New)).*

5. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) Evaluate by changing the variable $\iint_R (x+y)^2 \, dx \, dy$ where R is the region bounded by the parallelogram $x + y = 0$, $x + y = 3x - 2y = 0$ and $3x - 2y = 3$.

(*b*) Find the volume bounded by the elliptical paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$.

(*c*) Using Beta and Gamma functions, evaluate

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx$$

Note: *Following question no. 6 is for Old Syllabus only (MA-101 (old)).*

6. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) In a binomial distribution the sum and product of the mean and variance of the distribution are $\frac{25}{3}$ and $\frac{50}{3}$ respectively. Find the distribution.

(*b*) From the following data which shows the ages X and systolic blood pressure Y of 12 women, find out whether the two variables ages X and blood pressure Y are correlated?

Ages (X)	:	56	42	72	36	63	47	55	49	38	42	68	60
B.P. (Y)	:	147	125	160	118	149	128	150	145	115	140	152	155

(*c*) (*i*) If θ is the acute angle between the two regression lines in case of two variables x and y, show that

$$\tan\theta = \frac{1-r^2}{r}\cdot\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

where r, σ_x and σ_y have their usual meanings. Explain the significance of the formula when $r = 0$ and $r = \pm 1$.

(*ii*) Two variables x and y are correlated by the equation $ax + by + c = 0$. Show that the correlation between them is –1 if signs of a and b are alike and +1, if they are different.

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B.Tech., First Semester Examination, 2005-06

MATHEMATICS–I

Time: 3 Hours] [*Total Marks: 100*

Note: *(i) Attempt All questions.*
(ii) All questions carry equal marks.
(iii) Question no. 1–4 are common to all candidates.
(iv) Be precise in your answer.

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1. Attempt any **four** parts of the following: **(5 × 4 = 20)**

(*a*) Use elementary transformation to reduce following matrix A to triangular form and hence the rank of A.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(*b*) Define unitary matrix. Show that the matrix

$\begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$ is a unitary matrix if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

(*c*) Reduce the matrix A to diagonal form

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

(*d*) Find the eigen values and eigen vectors of matrix A

$$A = \begin{bmatrix} 1 & 7 & 13 \\ 2 & 5 & 7 \\ 3 & 11 & 5 \end{bmatrix}$$

[**Hint:** $\lambda^3 - 11\lambda^2 - 95\lambda - 116 = 0$, which cannot be solve.]

(*e*) Test the consistency of following system of linear equations and hence find the solution

$$\begin{aligned} 4x_1 - x_2 &= 12 \\ -x_1 + 5x_2 - 2x_3 &= 0 \\ -2x_2 + 4x_3 &= -8 \end{aligned}$$

(*f*) State Cayley-Hamilton theorem. Using this theorem find the inverse of the matrix.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

2. Attempt any **four** parts of the following: **(5 × 4 = 20)**

(*a*) Find the directional derivative of $\frac{1}{r^2}$ in the direction of $\vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

(*b*) Find $\iint \vec{F} \cdot \hat{n}\, ds$, where

$\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and s is the surface of sphere having centre (3, –1, 2) and radius 3.

(*c*) Show that the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. Find the scalar potential.

(*d*) If $\vec{A}$ is a vector function and ϕ is a scalar function, then show that $\nabla \cdot (\phi \vec{A}) = \phi \nabla \cdot \vec{A} + \vec{A} \cdot \nabla \phi$.

(*e*) Apply Green's theorem to evaluate $\oint_C 2y^2 dx + 3x\, dy$, where C is the boundary of closed region bounded between $y = x$ and $y = x^2$.

(*f*) Suppose $\vec{F}(x, y, z) = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field. Find the work done by $\vec{F}$ along the line from the (1, 2, 3) to (3, 5, 7).

3. Attempt any **four** parts of the following: **(5 × 4 = 20)**

(*a*) If $y = (\sin^{-1} x)^2$, prove that

$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0$

Hence find the value of y_n at $x = 0$.

(*b*) If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

(*c*) Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(*d*) Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in the neighbourhood of (1, 1).

(*e*) If $u = x \log(xy)$, where $x^3 + y^3 + 3xy = 1$. Find $\frac{du}{dx}$.

(*f*) State Euler's theorem of differential calculus and verify the theorem for the function

$$u = \log\left(\frac{x^4 + y^4}{x + y}\right).$$

4. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) If J be the Jacobian of the system u, v with respect to x, y and J' the Jacobian of the system x, y with respect to u, v then prove that $JJ' = 1$.

(*b*) A rectangular box open at top is to have capacity of 32 c.c. Find the dimensions of the box least material.

(*c*) A balloon in the form of right circular cylinder of radius 1.5 m and length 4.0 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of the balloon.

For New Syllabus Only

5. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) Evaluate the integral $\int_0^{\infty}\int_0^{x} x \exp\left(-\frac{x^2}{y}\right) dy\, dx$

changing the order of integration.

(*b*) Find the triple integration, the volume paraboloid of revolution $x^4 + y^2 = 4z$ cut off plane $z = 4$.

(*c*) State the Dirichlet's theorem for three variables. Hence evaluate the integral

$\iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$.

where x, y, z are all positive but limited condition $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \leq 1$.

For Old Syllabus Only

6. Attempt any **two** parts of the following: **(10 × 2 = 20)**

(*a*) The following data regarding the heights (y) and weights (x) of 100 college students are given $\Sigma x = 15000$, $\Sigma x^2 = 2272500$, $\Sigma y = 6800$, $\Sigma y = 463025$ and $\Sigma xy = 1022250$.

Find the correlation coefficient between height and weight and equation of regression line of height on weight.

(*b*) Fit a Poisson distribution to the following data and calculate the theoretical frequencies:

x	0	1	2	3	4
f	192	100	24	3	1

(*c*) Assume the mean height of soldiers to be 68.22 inches with a variance of 10.8 inches square. How many soldiers in a regiment of 10,000 would you expect to be over 6 feet tall, given that the area under the standard normal curve between $x = 0$ and $x = 0.31$ is 0.1368 and between $x = 0$ and $x = 1.15$ is 0.3746.

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B.Tech., First Semester Examination, 2004-05

MATHEMATICS–I

Time: 3 Hours] [*Total Marks: 100*

Note: *(1) There are five questions in all.*
(2) Each question has three parts.
(3) Attempt any two parts in each question.

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1. (*a*) Find the eigen values and eigen vectors of the matrix **(5 × 4 = 20)**

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

(*b*) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Hence compute A^{-1}.

(*c*) Reduce the matrix A to its normal form when

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Hence find the rank of A.

2. (*a*) If $y = \sin(m \sin^{-1}x)$, prove that **(10 × 2 = 20)**

$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} + (m^2 + n^2)y_n = 0$, and hence find y_n at $x = 0$.

(*b*) If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

(*c*) Trace the curve $y^2 (2a - x) = x^3$.

3. (*a*) If $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}$ and $y_3 = \frac{x_1 x_2}{x_3}$, show that Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4. **(10 × 2 = 20)**

(*b*) In estimating the cost of a pile of bricks measured as 6 m × 50 m × 4 m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in 1 m^3 and bricks cost Rs. 100 per 1000, find the approximate error in the cost.

(*c*) Find the extreme values of $x^3 + y^3 - 3axy$.

4. (*a*) Calculate the volume of the solid bounded by the surface $x = 0$, $y = 0$, $x + y + z = 1$ and $z = 0$, where D is the domain $x \geq 0$, $y \geq 0$ and $x + y \leq h$. **(10 × 2 = 20)**

(*b*) Prove that $\int_D x^{l-1} y^{m-1} dx\, dy = \frac{\Gamma l \Gamma m}{\Gamma l+m+1} h^{l+m}$

(*c*) Change the order of integration in $l = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate the same.

5. (*a*) Prove div(grad r^n) $= n(n+1)r^{n-2}$, where $r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$. **(10 × 2 = 20)**

(*b*) Apply Green's theorem to evaluate $\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$, where C is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2 + y^2 = a^2$.

(*c*) Evaluate $\int\int_S \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right) ds$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

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